FACULDADE DE CIÊNCIAS E TECNOLOGIA
UNIVERSIDADE NOVA DE LISBOA
Departamento de Matemática

Derivadas

1. Calcule a derivada das seguintes funções:

(a) $\log(x+\sqrt{1+x^2})$

(b) $\dfrac{x^2}{\sqrt{9-x^2}}$

(c) $\log\left(\dfrac{1}{1+x}\right)$

(d) $\dfrac{e^{3x}}{e^x-2}$

(e) $\dfrac{1}{\sqrt{4\,x-x^2}}$

(f) $\operatorname{sen}(3x)\cos(2x)$

(g) $\log^2(x)$

(h) $\dfrac{\operatorname{tg}(x)}{3+\operatorname{sen}^2(x)}$

(i) $\dfrac{\operatorname{sen}^2(x)}{1+\cos^2(x)}$

(j) $\dfrac{2\,x\,\log(x)}{(1+x^2)^2}$

(k) $\cos\left(\log\left(\dfrac{1}{x}\right)\right)$

(l) $(x\operatorname{sen}(x))^2$

(m) $\dfrac{x^3}{(1+x^2)^2}$

(n) $\operatorname{sen}(\sqrt{x+1})$

(o) $\dfrac{1}{\sqrt[4]{x}+\sqrt{x}}$

(p) $\dfrac{\operatorname{sen}(x)}{\cos(x)(\cos(x)-1)}$

(q) $x^3\,\cos(x^2)$

(r) $\dfrac{\log(\log(x))}{x}$

(s) $\dfrac{e^{2x}+1}{e^x\,(2+e^{2x})}$

(t) $\log(1-x^2)$

(u) $e^{(e^x+x)}$

(v) $\dfrac{x}{\cos^2(x)}$

(w) $\dfrac{\cos(x)}{1+\cos(x)}$

2. Calcule a derivada das seguintes funções:

(a) $x\,\log(x^2-1)$

(b) $\dfrac{e^x+2}{e^{2\,x}+e^x}$

(c) $\dfrac{x}{(x^2+1)\,\log(\sqrt{x^2+1})}$

(d) $\dfrac{x-1}{(x^2+1)(x-2)^2}$

(e) $\dfrac{e^x}{(e^{2x}+2)\,(e^x-1)}$

(f) $\dfrac{7e^x+8}{(e^{2x}+4)(e^x-1)}$

(g) $\log\left(\dfrac{x}{(x+3)^2}\right)$

(h) $\dfrac{5}{2}(\cos(x))^{2/5}$.

(i) $x\,\operatorname{sen}(x^2)$.

(j) $\dfrac{x}{(x-3)(2+x^2)}$.

(k) $\dfrac{x^2}{\sqrt{4-x^2}}$.

(l) $\cos\left(\left(x+\dfrac{\pi}{2}\right)^2\right)$.

(m) $\dfrac{1}{(x-2)^2(1+x^2)}$.

(n) $\log\left(\dfrac{x}{x+3}\right)$.

(o) $\dfrac{1}{x^4-1}$.

(p) $\dfrac{1}{3+2\,\cos(x)}$.

(q) $x^2\,e^x$.

(r) $\dfrac{1}{2-\operatorname{sen}^2(x)}$.

(s) $(\log(x)+3)^2$.

Resolução

1. (a) A derivada de $\log(f(x))$ é $\dfrac{f'(x)}{f(x)}$, portanto, temos

$$\left(\log(x+\sqrt{1+x^2})\right)' = \frac{(x+\sqrt{1+x^2})'}{x+\sqrt{1+x^2}} = \frac{1+(\sqrt{1+x^2})'}{x+\sqrt{1+x^2}} = \frac{1+\dfrac{2x}{2\sqrt{1+x^2}}}{x+\sqrt{1+x^2}}.$$

Simplificando obtemos

$$\frac{1+\dfrac{2x}{2\sqrt{1+x^2}}}{x+\sqrt{1+x^2}} = \frac{2x+2\sqrt{1+x^2}}{2\sqrt{1+x^2}(x+\sqrt{1+x^2})} = \frac{1}{\sqrt{1+x^2}}.$$

(b) $$\left(\frac{x^2}{\sqrt{9-x^2}}\right)' = \frac{2x\sqrt{9-x^2} - x^2\cdot\dfrac{-2x}{2\sqrt{9-x^2}}}{(\sqrt{9-x^2})^2} = \frac{2x\sqrt{9-x^2} + x^2\cdot\dfrac{x}{\sqrt{9-x^2}}}{(\sqrt{9-x^2})^2}.$$

Reduzindo ao mesmo denominador e simplificando, obtemos

$$\left(\frac{x^2}{\sqrt{9-x^2}}\right)' = \frac{2x(9-x^2)+x^3}{(9-x^2)\sqrt{9-x^2}} = \frac{x(18-x^2)}{\sqrt{(9-x^2)^3}}.$$

(c) $$\left(\log\left(\frac{1}{1+x}\right)\right)' = \frac{\left(\dfrac{1}{1+x}\right)'}{\dfrac{1}{1+x}} = \frac{-\dfrac{1}{(1+x)^2}}{\dfrac{1}{1+x}} = -\frac{1}{1+x}.$$

(d) $$\left(\frac{e^{3x}}{e^x-2}\right)' = \frac{3\,e^{3x}(e^x-2)-e^{3x}e^x}{(e^x-2)^2} = \frac{e^{3x}(3\,e^x-6-e^x)}{(e^x-2)^2} = \frac{2\,e^{3x}(e^x-3)}{(e^x-2)^2}.$$

(e) $$\left(\frac{1}{\sqrt{4\,x-x^2}}\right)' = -\frac{(\sqrt{4\,x-x^2})'}{(\sqrt{4\,x-x^2})^2} = -\frac{\dfrac{4-2x}{2\sqrt{4\,x-x^2}}}{4\,x-x^2} = -\frac{2-x}{(4\,x-x^2)\sqrt{4\,x-x^2}} = \frac{x-2}{\sqrt{(4\,x-x^2)^3}}.$$

(f) $(\mathrm{sen}(3x)\cos(2x))' = 3\,\cos(3x)\cos(2x) - 2\,\mathrm{sen}(3x)\,\mathrm{sen}(2x)$.

(g) $\left(\log^2(x)\right)' = 2\,\log(x)\cdot\dfrac{1}{x}$

(h) $$\begin{aligned}\left(\frac{\mathrm{tg}(x)}{3+\mathrm{sen}^2(x)}\right)' &= \frac{\sec^2(x)\,(3+\mathrm{sen}^2(x)) - 2\,\mathrm{tg}(x)\,\mathrm{sen}(x)\cos(x)}{(3+\mathrm{sen}^2(x))^2}\\
&= \frac{\sec^2(x)\,(3+\mathrm{sen}^2(x)) - 2\,\dfrac{\mathrm{sen}(x)}{\cos(x)}\cdot\mathrm{sen}(x)\cos(x)}{(3+\mathrm{sen}^2(x))^2}\\
&= \frac{\sec^2(x)\,(3+\mathrm{sen}^2(x)) - 2\,\mathrm{sen}^2(x)}{(3+\mathrm{sen}^2(x))^2}\\
&= \frac{\dfrac{1}{\cos^2(x)}\cdot(3+\mathrm{sen}^2(x)) - 2\,\mathrm{sen}^2(x)}{(3+\mathrm{sen}^2(x))^2}\\
&= \frac{3+\mathrm{sen}^2(x) - 2\,\mathrm{sen}^2(x)\cos^2(x)}{\cos^2(x)(3+\mathrm{sen}^2(x))^2}\\
&= \frac{3+\mathrm{sen}^2(x)(1-2\,\cos^2(x))}{\cos^2(x)(3+\mathrm{sen}^2(x))^2}.\end{aligned}$$

(i) $$\left(\frac{\operatorname{sen}^2(x)}{1+\cos^2(x)}\right)' = \frac{2\operatorname{sen}(x)\cos(x)(1+\cos^2(x))+2\operatorname{sen}^2(x)\operatorname{sen}(x)\cos(x)}{(1+\cos^2(x))^2}$$
$$= \frac{2\operatorname{sen}(x)\cos(x)(1+\cos^2(x)+\operatorname{sen}^2(x))}{(1+\cos^2(x))^2} = \frac{2\operatorname{sen}(2x)}{(1+\cos^2(x))^2}.$$

(j) $$\left(\frac{2\,x\,\log(x)}{(1+x^2)^2}\right)' = \frac{\left(2\,\log(x)+2x\cdot\dfrac{1}{x}\right)(1+x^2)^2-2\,x\,\log(x)\,2(1+x^2)2x}{(1+x^2)^4}$$
$$= \frac{(2\,\log(x)+2)\,(1+x^2)-8\,x^2\,\log(x)}{(1+x^2)^3}$$
$$= \frac{2\,\log(x)+2+2\,x^2\,\log(x)+2x^2-8\,x^2\,\log(x)}{(1+x^2)^3}$$
$$= \frac{2\,\log(x)+2-6\,x^2\,\log(x)+2x^2}{(1+x^2)^3}$$
$$= \frac{2\left(\log(x)-3\,x^2\,\log(x)+x^2+1\right)}{(1+x^2)^3}$$

(k) $$\left(\cos\left(\log\left(\frac{1}{x}\right)\right)\right)' = -\operatorname{sen}\left(\log\left(\frac{1}{x}\right)\right)\left(\log\left(\frac{1}{x}\right)\right)' = \frac{1}{x}\cdot\operatorname{sen}\left(\log\left(\frac{1}{x}\right)\right).$$

(l) $$\left((x\operatorname{sen}(x))^2\right)' = 2\,x\operatorname{sen}(x)\,(x\operatorname{sen}(x))' = 2\,x\operatorname{sen}(x)\,(\operatorname{sen}(x)+x\,\cos(x))\,.$$

(m) $$\left(\frac{x^3}{(1+x^2)^2}\right)' = \frac{3x^2(1+x^2)^2-4x^4(1+x^2)}{(1+x^2)^4} = \frac{3x^2(1+x^2)-4x^4}{(1+x^2)^3} = \frac{3x^2-x^4}{(1+x^2)^3} = \frac{x^2(3-x^2)}{(1+x^2)^3}.$$

(n) $$\left(\operatorname{sen}(\sqrt{x+1})\right)' = \cos(\sqrt{x+1})\,\left(\sqrt{x+1}\right)' = \frac{\cos(\sqrt{x+1})}{2\,\sqrt{x+1}}.$$

(o) $$\left(\frac{1}{\sqrt[4]{x}+\sqrt{x}}\right)' = -\frac{\dfrac{1}{4\sqrt[4]{x^3}}+\dfrac{1}{2\sqrt{x}}}{\left(\sqrt[4]{x}+\sqrt{x}\right)^2} = -\frac{\dfrac{1+2\sqrt[4]{x}}{4\sqrt[4]{x^3}}}{\left(\sqrt[4]{x}+\sqrt{x}\right)^2} = -\frac{1+2\sqrt[4]{x}}{4\sqrt[4]{x^3}\left(\sqrt[4]{x}+\sqrt{x}\right)^2}.$$

(p) $$\left(\frac{\operatorname{sen}(x)}{\cos(x)(\cos(x)-1)}\right)' = \frac{\cos^2(x)(\cos(x)-1)-\operatorname{sen}(x)\big(\cos(x)(\cos(x)-1)\big)'}{\cos^2(x)(\cos(x)-1)^2}$$
$$= \frac{\cos^2(x)(\cos(x)-1)-\operatorname{sen}(x)\big(\cos^2(x)-\cos(x)\big)'}{\cos^2(x)(\cos(x)-1)^2}$$
$$= \frac{\cos^2(x)(\cos(x)-1)-\operatorname{sen}(x)\big(-2\operatorname{sen}(x)\cos(x)+\operatorname{sen}(x)\big)}{\cos^2(x)(\cos(x)-1)^2}$$
$$= \frac{\cos^2(x)(\cos(x)-1)+\operatorname{sen}^2(x)(2\cos(x)-1)}{\cos^2(x)(\cos(x)-1)^2}$$
$$= \frac{\cos(x)-1+\operatorname{sen}^2(x)\cos(x)}{\cos^2(x)(\cos(x)-1)^2}$$
$$= \frac{-\cos^3(x)+2\cos(x)-1}{\cos^2(x)(\cos(x)-1)^2}.$$

(q) $$\left(x^3\,\cos(x^2)\right)' = 3x^2\,\cos(x^2)-2\,x^4\operatorname{sen}(x^2).$$

(r) $$\left(\frac{\log(\log(x))}{x}\right)' = \frac{x(\log(\log(x)))' - \log(\log(x))}{x^2} = \frac{x\dfrac{(\log(x))'}{\log(x)} - \log(\log(x))}{x^2}$$
$$= \frac{\dfrac{1}{\log(x)} - \log(\log(x))}{x^2} = \frac{1 - \log(x)\cdot\log(\log(x))}{x^2\,\log(x)}.$$

(s) $$\left(\frac{e^{2x}+1}{e^x\,(2+e^{2x})}\right)' = \frac{2\,e^{2x}\big(e^x\,(2+e^{2x})\big) - (e^{2x}+1)\big(e^x\,(2+e^{2x})\big)'}{\big(e^x\,(2+e^{2x})\big)^2}$$
$$= \frac{2\,e^{3x}\,(2+e^{2x}) - (e^{2x}+1)(2\,e^x+3\,e^{3x})}{\big(e^x\,(2+e^{2x})\big)^2}$$
$$= \frac{e^x\big(2\,e^{2x}\,(2+e^{2x}) - (e^{2x}+1)(2+3\,e^{2x})\big)}{\big(e^x\big)^2\,\big(2+e^{2x}\big)^2}$$
$$= \frac{2\,e^{2x}\,(2+e^{2x}) - (e^{2x}+1)(2+3\,e^{2x})}{e^x\,\big(2+e^{2x}\big)^2}$$
$$= \frac{4\,e^{2x}+2\,e^{4x}-2\,e^{2x}-2-3\,e^{4x}-3\,e^{2x}}{e^x\,\big(2+e^{2x}\big)^2}$$
$$= -\frac{e^{4x}+e^{2x}+2}{e^x\,\big(2+e^{2x}\big)^2}.$$

(t) $\big(\log(1-x^2)\big)' = \dfrac{-2x}{1-x^2} = \dfrac{2x}{x^2-1}.$

(u) $\left(e^{(e^x+x)}\right)' = (e^x+x)'\,e^{(e^x+x)} = (e^x+1)\,e^{(e^x+x)}.$

(v) $\left(\dfrac{x}{\cos^2(x)}\right)' = \dfrac{\cos^2(x)+2\,x\,\cos(x)\,\operatorname{sen}(x)}{\cos^4(x)} = \dfrac{\cos(x)+2\,x\,\operatorname{sen}(x)}{\cos^3(x)}.$

(w) $\left(\dfrac{\cos(x)}{1+\cos(x)}\right)' = \dfrac{-\operatorname{sen}(x)(1+\cos(x))+\cos(x)\,\operatorname{sen}(x)}{(1+\cos(x))^2} = \dfrac{-\operatorname{sen}(x)}{(1+\cos(x))^2}.$

2. (a) $\big(x\,\log(x^2-1)\big)' = \log(x^2-1) + x\cdot\dfrac{2x}{x^2-1} = \log(x^2-1) + \dfrac{2x^2}{x^2-1}.$

(b) $$\left(\frac{e^x+2}{e^{2\,x}+e^x}\right)' = \frac{e^x(e^{2\,x}+e^x) - (e^x+2)(2\,e^{2\,x}+e^x)}{\big(e^{2\,x}+e^x\big)^2}$$
$$= \frac{e^x\big(e^{2\,x}+e^x-(e^x+2)(2\,e^x+1)\big)}{\big(e^x\big)^2\big(e^x+1\big)^2}$$
$$= \frac{e^{2\,x}+e^x-2\,e^{2x}-e^x-4e^x-2}{e^x\big(e^x+1\big)^2}$$
$$= \frac{-e^{2x}-4e^x-2}{e^x\big(e^x+1\big)^2} = -\frac{e^{2x}+4e^x+2}{e^x\big(e^x+1\big)^2}.$$

(c)
$$\left(\frac{x}{(x^2+1)\,\log(\sqrt{x^2+1})}\right)' = \frac{(x^2+1)\,\log(\sqrt{x^2+1}) - x\big((x^2+1)\,\log(\sqrt{x^2+1})\big)'}{\big((x^2+1)\,\log(\sqrt{x^2+1})\big)^2}$$

$$= \frac{(x^2+1)\,\log(\sqrt{x^2+1}) - x\left(2x\,\log(\sqrt{x^2+1}) + (x^2+1)\cdot\dfrac{(\sqrt{x^2+1})'}{\sqrt{x^2+1}}\right)}{\big((x^2+1)\,\log(\sqrt{x^2+1})\big)^2}$$

$$= \frac{(x^2+1)\,\log(\sqrt{x^2+1}) - \left(2x^2\,\log(\sqrt{x^2+1}) + x(x^2+1)\cdot\dfrac{2x}{2(\sqrt{x^2+1})^2}\right)}{\big((x^2+1)\,\log(\sqrt{x^2+1})\big)^2}$$

$$= \frac{(x^2+1)\,\log(\sqrt{x^2+1}) - \left(2x^2\,\log(\sqrt{x^2+1}) + x^2\right)}{\big((x^2+1)\,\log(\sqrt{x^2+1})\big)^2}$$

$$= \frac{(1-x^2)\,\log(\sqrt{x^2+1}) - x^2}{\big((x^2+1)\,\log(\sqrt{x^2+1})\big)^2}.$$

(d)
$$\left(\frac{x-1}{(x^2+1)(x-2)^2}\right)' = \frac{(x^2+1)(x-2)^2 - (x-1)\big(2x(x-2)^2 + 2(x^2+1)(x-2)\big)}{(x^2+1)^2(x-2)^4}$$

$$= \frac{(x^2+1)(x-2) - (x-1)(2x(x-2) + 2(x^2+1))}{(x^2+1)^2(x-2)^3}$$

$$= \frac{x^3 - 2x^2 + x - 2 - (x-1)(2x^2 - 4x + 2x^2 + 2)}{(x^2+1)^2(x-2)^3}$$

$$= \frac{x^3 - 2x^2 + x - 2 - (x-1)(4x^2 - 4x + 2)}{(x^2+1)^2(x-2)^3}$$

$$= \frac{x^3 - 2x^2 + x - 2 - 4x^3 + 4x^2 - 2x + 4x^2 - 4x + 2}{(x^2+1)^2(x-2)^3}$$

$$= \frac{-3x^3 + 6x^2 - 5x}{(x^2+1)^2(x-2)^3} = -\frac{x\big(3x^2 - 6x + 5\big)}{(x^2+1)^2(x-2)^3}.$$

(e)
$$\left(\frac{e^x}{(e^{2x}+2)\,(e^x-1)}\right)' = \frac{e^x(e^{2x}+2)\,(e^x-1) - e^x(2\,e^{2x}(e^x-1) + e^x(e^{2x}+2))}{(e^{2x}+2)^2\,(e^x-1)^2}$$

$$= \frac{e^x\big((e^{2x}+2)\,(e^x-1) - (2\,e^{2x}(e^x-1) + e^x(e^{2x}+2))\big)}{(e^{2x}+2)^2\,(e^x-1)^2}$$

$$= \frac{e^x\big(e^{3x} - e^{2x} + 2e^x - 2 - 2\,e^{3x} + 2e^{2x} - e^{3x} - 2e^x\big)}{(e^{2x}+2)^2\,(e^x-1)^2}$$

$$= \frac{e^x\big(-2e^{3x} + e^{2x} - 2\big)}{(e^{2x}+2)^2\,(e^x-1)^2}$$

(f) $$\left(\frac{7e^x+8}{(e^{2x}+4)(e^x-1)}\right)' = \frac{7e^x(e^{2x}+4)(e^x-1)-(7e^x+8)(2\,e^{2x}(e^x-1)+e^x(e^{2x}+4))}{(e^{2x}+4)^2(e^x-1)^2}$$

$$= \frac{7e^x(e^{3x}-e^{2x}+4e^x-4)-(7e^x+8)(3\,e^{3x}-2\,e^{2x}+4e^x)}{(e^{2x}+4)^2(e^x-1)^2}$$

$$= \frac{e^x(7e^{3x}-7e^{2x}+28e^x-28-(21e^{3x}-14e^{2x}+28e^x+24\,e^{2x}-16\,e^x+32))}{(e^{2x}+4)^2(e^x-1)^2}$$

$$= \frac{e^x(7e^{3x}-7e^{2x}+28e^x-28-21e^{3x}-10e^{2x}-12e^x-32)}{(e^{2x}+4)^2(e^x-1)^2}$$

$$= \frac{e^x(-14e^{3x}-17e^{2x}+16e^x-60)}{(e^{2x}+4)^2(e^x-1)^2}.$$

(g) $$\left(\log\left(\frac{x}{(x+3)^2}\right)\right)' = \frac{\left(\frac{x}{(x+3)^2}\right)'}{\frac{x}{(x+3)^2}} = \frac{\frac{(x+3)^2-2\,x(x+3)}{(x+3)^4}}{\frac{x}{(x+3)^2}} = \frac{3-x}{x(x+3)}.$$

(h) $$\left(\frac{5}{2}(\cos(x))^{2/5}\right)' = -\frac{5}{2}\cdot\frac{2}{5}(\cos(x))^{-3/5}\,\mathrm{sen}(x) = -\frac{\mathrm{sen}(x)}{\big(\cos(x)\big)^{3/5}}.$$

(i) $\big(x\,\mathrm{sen}(x^2)\big)' = \mathrm{sen}(x^2)+2x^2\cos(x^2).$

(j) $$\left(\frac{x}{(x-3)(2+x^2)}\right)' = \frac{(x-3)(2+x^2)-x(2+x^2+2x(x-3))}{(x-3)^2(2+x^2)^2} = \frac{-3x^3+3x^2-6}{(x-3)^2(2+x^2)^2}.$$

(k) $$\left(\frac{x^2}{\sqrt{4-x^2}}\right)' = \frac{2x\sqrt{4-x^2}-x^2\big(\sqrt{4-x^2}\big)'}{4-x^2} = \frac{2x\sqrt{4-x^2}-x^2\cdot\frac{-2x}{2\sqrt{4-x^2}}}{4-x^2}$$

$$= \frac{2x(4-x^2)+x^3}{(4-x^2)\sqrt{4-x^2}} = \frac{x(8-x^2)}{(4-x^2)\sqrt{4-x^2}}.$$

(l) $$\left(\cos\left(\left(x+\frac{\pi}{2}\right)^2\right)\right)' = -2\left(x+\frac{\pi}{2}\right)\mathrm{sen}\left(\left(x+\frac{\pi}{2}\right)^2\right).$$

(m) $$\left(\frac{1}{(x-2)^2(1+x^2)}\right)' = -\frac{2(x-2)(1+x^2)+2x(x-2)^2}{(x-2)^4(1+x^2)^2}$$

$$= -\frac{2(1+x^2)+2x(x-2)}{(x-2)^3(1+x^2)^2} = -\frac{4x^2-4x+2}{(x-2)^3(1+x^2)^2}.$$

(n) $$\left(\log\left(\frac{x}{x+3}\right)\right)' = \frac{\left(\frac{x}{x+3}\right)'}{\frac{x}{x+3}} = \frac{\frac{x+3-x}{(x+3)^2}}{\frac{x}{x+3}} = \frac{3}{x(x+3)}.$$

(o) $$\left(\frac{1}{x^4-1}\right)' = -\frac{4x^3}{(x^4-1)^2}.$$

(p) $$\left(\frac{1}{3+2\,\cos(x)}\right)' = \frac{2\,\mathrm{sen}(x)}{(3+2\,\cos(x))^2}.$$

(q) $\left(x^2\, e^x\right)' = 2x\, e^x + x^2\, e^x = x\, e^x (2 + x).$

(r) $\left(\dfrac{1}{2 - \operatorname{sen}^2(x)}\right)' = \dfrac{2 \operatorname{sen}(x) \cos(x)}{\left(2 - \operatorname{sen}^2(x)\right)^2} = \dfrac{\operatorname{sen}(2x)}{\left(2 - \operatorname{sen}^2(x)\right)^2}.$

(s) $\left((\log(x) + 3)^2\right)' = \dfrac{2}{x} \cdot (\log(x) + 3).$