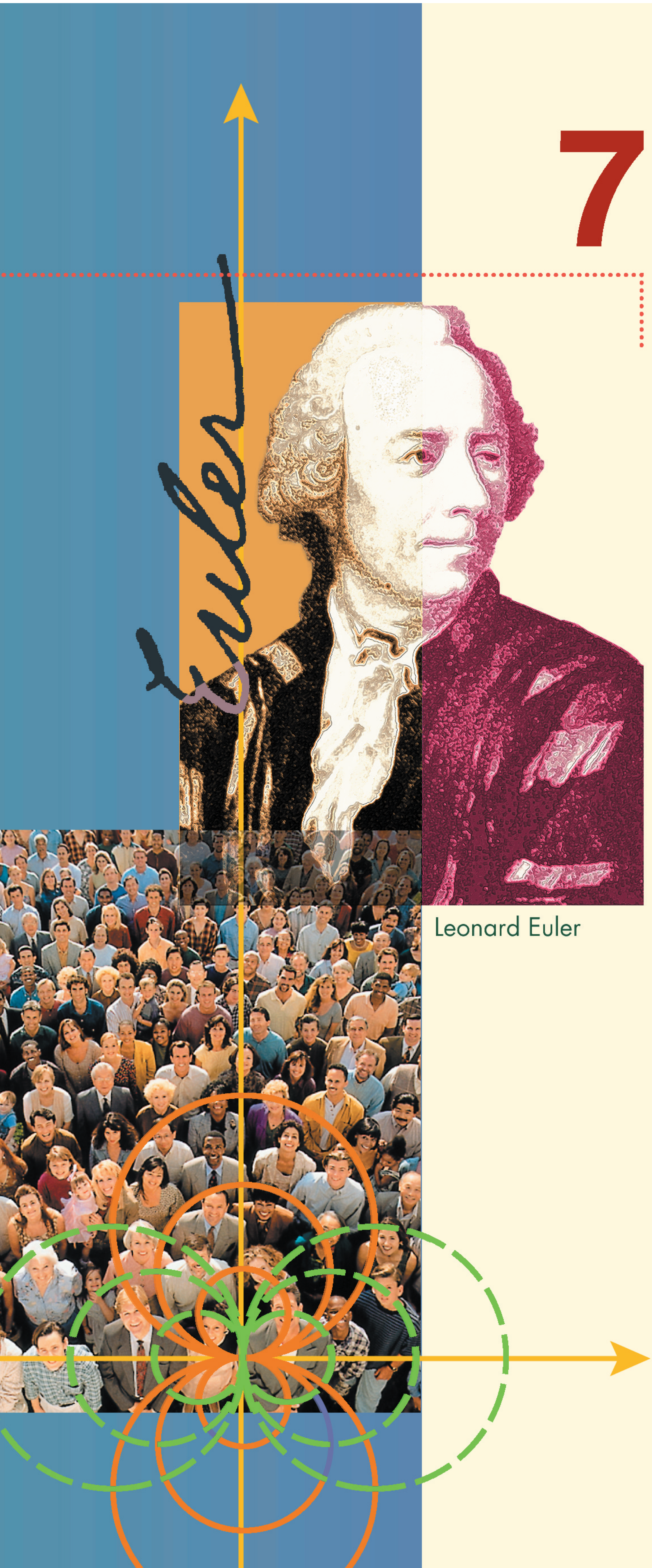


Leonard Euler

7

EXPONENTIAL, LOGARITHMIC, AND INVERSE TRIGONOMETRIC FUNCTIONS

In this chapter we will expand our collection of "elementary" functions to include the exponential, logarithmic, and inverse trigonometric functions. The heart of the chapter is Section 7.1 on inverse functions, in which we develop fundamental ideas that link a function and its inverse numerically, algebraically, and graphically. Our focus will be on those aspects of inverse functions that relate to calculus. In particular, we will see that there is an important connection between the derivative of a function and the derivative of its inverse. This connection will allow us to develop a number of derivative and integral formulas that involve the exponential, logarithmic, and inverse trigonometric functions. With the aid of these formulas, we will discuss a powerful tool for evaluating limits known as L'Hôpital's rule. The chapter concludes with an introduction to some analogs of the trigonometric functions, known as the hyperbolic functions.

7.1 INVERSE FUNCTIONS

In everyday language the term "inversion" conveys the idea of a reversal. For example, in meteorology a temperature inversion is a reversal in the usual temperature properties of air layers; in music, a melodic inversion reverses an ascending interval to the corresponding descending interval; and in grammar an inversion is a reversal of the normal order of words. In mathematics the term ***inverse*** *is used to describe functions that are reverses of one another in the sense that each undoes the effect of the other. The purpose of this section is to discuss this fundamental mathematical idea.*

INVERSE FUNCTIONS

The idea of solving an equation $y = f(x)$ for x as a function of y, say $x = g(y)$, is one of the most important ideas in mathematics. Sometimes, solving an equation is a simple process; for example, using basic algebra the equation

$$y = x^3 + 1 \qquad \boxed{y = f(x)}$$

can be solved for x as a function of y:

$$x = \sqrt[3]{y-1} \qquad \boxed{x = g(y)}$$

The first equation is better for computing y if x is known, and the second is better for computing x if y is known (Figure 7.1.1).

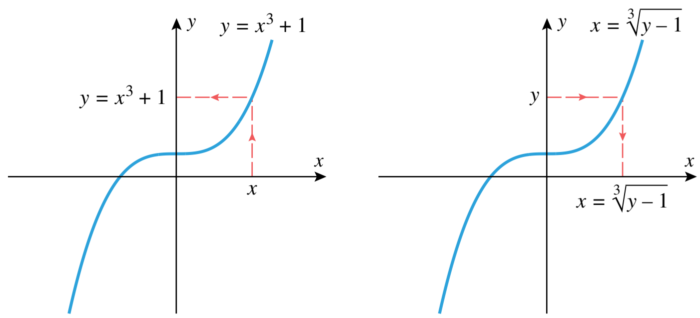


Figure 7.1.1

Our primary interest in this section is to identify relationships that may exist between the functions f and g when an equation $y = f(x)$ is expressed as $x = g(y)$, or conversely. For example, consider the functions $f(x) = x^3 + 1$ and $g(y) = \sqrt[3]{y-1}$ discussed above. When these functions are composed in either order they cancel out the effect of one another in the sense that

$$\begin{aligned} g(f(x)) &= \sqrt[3]{f(x)-1} = \sqrt[3]{(x^3+1)-1} = x \\ f(g(y)) &= [g(y)]^3 + 1 = (\sqrt[3]{y-1}\,)^3 + 1 = y \end{aligned} \tag{1}$$

The first of these equations states that each output of the composition $g(f(x))$ is the same as the input, and the second states that each output of the composition $f(g(y))$ is the same as the input. Pairs of functions with these two properties are so important that there is some terminology for them.

7.1.1 DEFINITION. If the functions f and g satisfy the two conditions

$g(f(x)) = x$ for every x in the domain of f

$f(g(y)) = y$ for every y in the domain of g

then we say that f and g are ***inverses***. Moreover, we call ***f an inverse function for g*** and ***g an inverse function for f***.

Example 1 It follows from (1) that $f(x) = x^3+1$ and $g(y) = \sqrt[3]{y-1}$ are inverses. ◀

It can be shown that a function cannot have two different inverse functions. Thus, if a function f has an inverse function, then the inverse is unique, and we are entitled to talk about *the* inverse of f. The inverse of a function f is commonly denoted by f^{-1} (read "f inverse"). Thus, instead of using g in Example 1, the inverse of $f(x) = x^3$ could have been expressed as $f^{-1}(y) = \sqrt[3]{y-1}$.

WARNING. The symbol f^{-1} should always be interpreted as the inverse of f and *never* as the reciprocal $1/f$.

It is important to understand that a function is determined by the relationship that it establishes between its inputs and outputs and not by the letter used for the independent variable. Thus, even though the formulas $f(x) = 3x$ and $f(y) = 3y$ use different independent variables, they define the *same* function f, since the two formulas have the same "form" and hence assign the same value to each input; for example, in either notation $f(2) = 6$. As we progress through this text, there will be certain occasions on which we will want the independent variables for f and f^{-1} to be the same, and other occasions on which we will want them to be different. Thus, in Example 1 we could have expressed the inverse of $f(x) = x^3 + 1$ as $f^{-1}(x) = \sqrt[3]{x-1}$ had we wanted f and f^{-1} to have the same independent variable.

If we use the notation f^{-1} (rather than g) in Definition 7.1.1, and if we use x as the independent variable in the formulas for both f and f^{-1}, then the defining equations relating these functions are

$$\begin{aligned} f^{-1}(f(x)) &= x \quad \text{for every } x \text{ in the domain of } f \\ f(f^{-1}(x)) &= x \quad \text{for every } x \text{ in the domain of } f^{-1} \end{aligned} \tag{2}$$

Example 2 Confirm each of the following.

(a) The inverse of $f(x) = 2x$ is $f^{-1}(x) = \frac{1}{2}x$.

(b) The inverse of $f(x) = x^3$ is $f^{-1}(x) = x^{1/3}$.

Solution (a).

$$f^{-1}(f(x)) = f^{-1}(2x) = \tfrac{1}{2}(2x) = x$$

$$f(f^{-1}(x)) = f\left(\tfrac{1}{2}x\right) = 2\left(\tfrac{1}{2}x\right) = x$$

Solution (b).

$$f^{-1}(f(x)) = f^{-1}(x^3) = \left(x^3\right)^{1/3} = x$$

$$f(f^{-1}(x)) = f(x^{1/3}) = \left(x^{1/3}\right)^3 = x$$

◀

REMARK. The results in Example 2 should make sense to you intuitively, since the operations of multiplying by 2 and multiplying by $\frac{1}{2}$ in either order cancel the effect of one another, as do the operations of cubing and taking a cube root.

DOMAIN AND RANGE OF INVERSE FUNCTIONS

The equations in (2) imply certain relationships between the domains and ranges of f and f^{-1}. For example, in the first equation the quantity $f(x)$ is an input of f^{-1}, so points in the range of f lie in the domain of f^{-1}; and in the second equation the quantity $f^{-1}(x)$ is an input of f, so points in the range of f^{-1} lie in the domain of f. All of this suggests the following relationships, which we state without formal proof:

$$\begin{aligned} \text{domain of } f^{-1} &= \text{range of } f \\ \text{range of } f^{-1} &= \text{domain of } f \end{aligned} \tag{3}$$

At the beginning of this section we solved the equation $y = f(x) = x^3 + 1$ for x as a function of y to obtain $x = g(y) = \sqrt[3]{y-1}$, and we observed in Example 1 that g is the inverse of f. This was not accidental—whenever an equation $y = f(x)$ is solved for x as a function of y, say $x = g(y)$, then f and g will be inverses. We can see why this is so by making two substitutions:

- Substitute $y = f(x)$ into $x = g(y)$. This yields $x = g(f(x))$, which is the first equation in Definition 7.1.1.
- Substitute $x = g(y)$ into $y = f(x)$. This yields $y = f(g(y))$, which is the second equation in Definition 7.1.1.

Since f and g satisfy the two conditions in Definition 7.1.1, we conclude that they are inverses. In summary:

If an equation $y = f(x)$ can be solved for x as a function of y, then f has an inverse function and the resulting equation is $x = f^{-1}(y)$.

A METHOD FOR FINDING INVERSES

Example 3 Find the inverse of $f(x) = \sqrt{3x-2}$.

Solution. From the discussion above we can find a formula for $f^{-1}(y)$ by solving the equation

$$y = \sqrt{3x-2}$$

for x as a function of y. The computations are

$$y^2 = 3x - 2$$
$$x = \tfrac{1}{3}(y^2 + 2)$$

from which it follows that

$$f^{-1}(y) = \tfrac{1}{3}(y^2 + 2)$$

At this point we have successfully produced a formula for f^{-1}; however, we are not quite done, since there is no guarantee that the natural domain associated with this formula is the correct domain for f^{-1}. To determine whether this is so, we will examine the range of $y = f(x) = \sqrt{3x-2}$. The range consists of all y in the interval $[0, +\infty)$, so from (3) this interval is also the domain of $f^{-1}(y)$; thus, the inverse of f is given by the formula

$$f^{-1}(y) = \tfrac{1}{3}(y^2 + 2), \quad y \geq 0$$ ◀

REMARK. When a formula for f^{-1} is obtained by solving the equation $y = f(x)$ for x as a function of y, the resulting formula has y as the independent variable. If it is preferable to have x as the independent variable for f^{-1}, then there are two ways to proceed: you can solve $y = f(x)$ for x as a function of y, and then replace y by x in the *final* formula for f^{-1}, or you can interchange x and y in the *original* equation and solve the equation $x = f(y)$ for y in terms of x, in which case the final equation will be $y = f^{-1}(x)$. In Example 3, either of these procedures will produce $f^{-1}(x) = \frac{1}{3}(x^2 + 2)$, $x \geq 0$.

Solving $y = f(x)$ for x as a function of y not only provides a method for finding the inverse of a function f, but it also provides an interpretation of what the values of f^{-1} represent. This tells us that for a given y, the quantity $f^{-1}(y)$ is that number x with the property that $f(x) = y$. For example, if $f^{-1}(1) = 4$, then you know that $f(4) = 1$; and similarly, if $f(3) = 7$, then you know that $f^{-1}(7) = 3$.

EXISTENCE OF INVERSE FUNCTIONS

Not every function has an inverse function. In general, in order for a function f to have an inverse function it must assign distinct outputs to distinct inputs. To see why this is so, consider the function $f(x) = x^2$. Since $f(2) = f(-2) = 4$, the function f assigns the same output to two distinct inputs. If f were to have an inverse function, then the equation $f(2) = 4$ would imply that $f^{-1}(4) = 2$, and the equation $f(-2) = 4$ would imply that $f^{-1}(4) = -2$. This is obviously impossible, since f^{-1} cannot be a function and have two different values for $f^{-1}(4)$. Thus, $f(x) = x^2$ has no inverse. Another way to see that $f(x) = x^2$ has no inverse is to attempt to find the inverse by solving the equation $y = x^2$ for x in terms of y. We run into trouble immediately because the resulting equation, $x = \pm\sqrt{y}$, does not express x as a *single* function of y.

Functions that assign distinct outputs to distinct inputs are sufficiently important that there is a name for them—they are said to be ***one-to-one*** or ***invertible***. Stated algebraically, a function f is one-to-one if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$; and stated geometrically, a function f is one-to-one if the graph of $y = f(x)$ is cut at most once by any horizontal line (Figure 7.1.2).

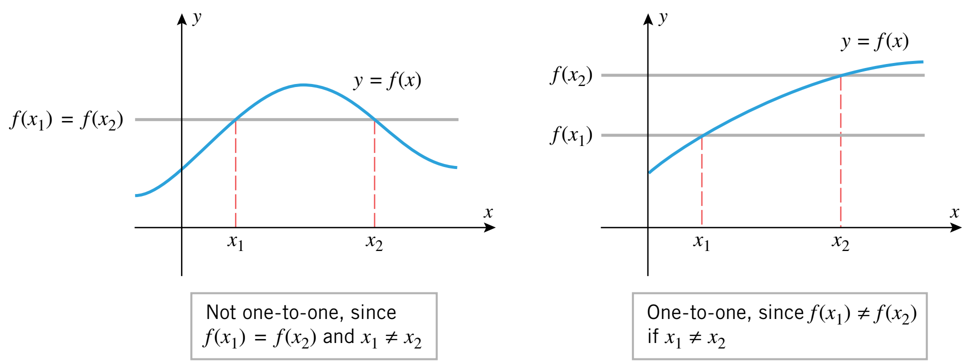


Figure 7.1.2

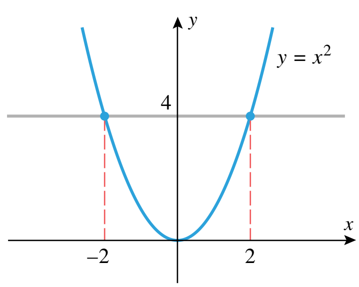


Figure 7.1.3

One can prove that a function f has an inverse function if and only if it is one-to-one, and this provides us with the following geometric test for determining whether a function has an inverse function.

> **7.1.2 THEOREM (*The Horizontal Line Test*).** *A function f has an inverse function if and only if its graph is cut at most once by any horizontal line.*

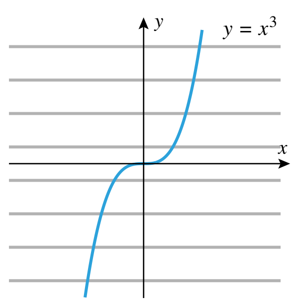


Figure 7.1.4

Example 4 We observed above that the function $f(x) = x^2$ does not have an inverse function. This is confirmed by the horizontal line test, since the graph of $y = x^2$ is cut more than once by certain horizontal lines (Figure 7.1.3). ◀

Example 5 We saw in Example 2(b) that the function $f(x) = x^3$ has an inverse [namely, $f^{-1}(x) = x^{1/3}$]. The existence of an inverse is confirmed by the horizontal line test, since the graph of $y = x^3$ is cut at most once by any horizontal line (Figure 7.1.4). ◀

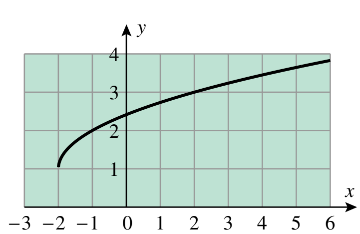


Figure 7.1.5

Example 6 Explain why the function f that is graphed in Figure 7.1.5 has an inverse function, and find $f^{-1}(3)$.

Solution. The function f has an inverse function since its graph passes the horizontal line test. To evaluate $f^{-1}(3)$, we view $f^{-1}(3)$ as that number x for which $f(x) = 3$. From the graph we see that $f(2) = 3$, so $f^{-1}(3) = 2$. ◀

GRAPHS OF INVERSE FUNCTIONS

Our next objective is to explore the relationship between the graphs of f and f^{-1}. For this purpose, it will be desirable to use x as the independent variable for both functions, which means that we will be comparing the graphs of $y = f(x)$ and $y = f^{-1}(x)$.

If (a, b) is a point on the graph $y = f(x)$, then $b = f(a)$. This is equivalent to the statement that $a = f^{-1}(b)$, which means that (b, a) is a point on the graph of $y = f^{-1}(x)$. In short, reversing the coordinates of a point on the graph of f produces a point on the graph of f^{-1}. Similarly, reversing the coordinates of a point on the graph of f^{-1} produces a point on the graph of f (verify). However, the geometric effect of reversing the coordinates of a point is to reflect that point about the line $y = x$ (Figure 7.1.6), and hence the graphs of $y = f(x)$ and $y = f^{-1}(x)$ are reflections of one another about this line (Figure 7.1.7). In summary, we have the following result.

7.1.3 THEOREM. *If f has an inverse function f^{-1}, then the graphs of $y = f(x)$ and $y = f^{-1}(x)$ are reflections of one another about the line $y = x$; that is, each is the mirror image of the other with respect to that line.*

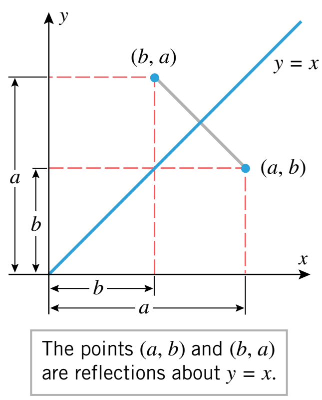


Figure 7.1.6

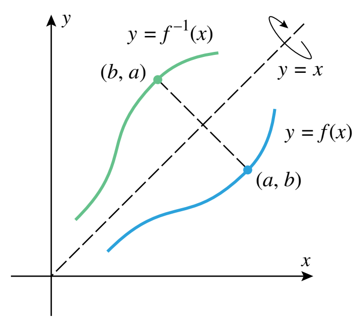


Figure 7.1.7

Example 7 Figure 7.1.8 shows the graphs of the inverse functions discussed in Examples 2 and 3. ◀

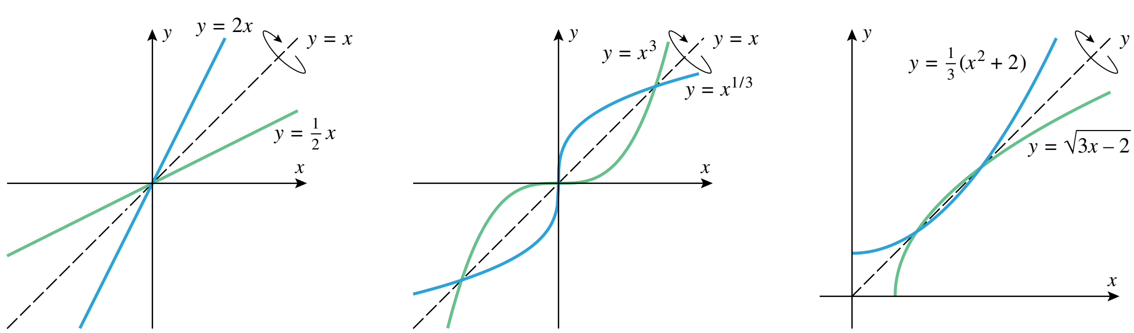


Figure 7.1.8

INCREASING OR DECREASING FUNCTIONS ARE INVERTIBLE

If the graph of a function f is always increasing or always decreasing over the domain of f, then a horizontal line will cut the graph of f at most once (Figure 7.1.9), so f must have an inverse function. In Theorem 4.1.2 we saw that f must be increasing on any interval on which $f'(x) > 0$ and must be decreasing on any interval on which $f'(x) < 0$. Thus, we have the following result.

7.1.4 THEOREM. *If the domain of a function f is an interval on which $f'(x) > 0$ or on which $f'(x) < 0$, then f has an inverse function.*

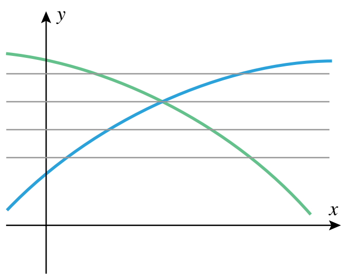


Figure 7.1.9

Example 8 The graph of $f(x) = x^5 + x + 1$ is always increasing on $(-\infty, +\infty)$ since

$$f'(x) = 5x^4 + 1 > 0$$

for all x. However, there is no easy way to solve the equation $y = x^5 + x + 1$ for x in terms of y (try it), so even though we know that f has an inverse function f^{-1}, we cannot produce a formula for $f^{-1}(x)$. ◀

REMARK. What is important to understand here is that our inability to find an explicit formula for the inverse function does not negate the existence of the inverse. In this case the inverse function $x = f^{-1}(y)$ is implicitly defined by the equation $y = x^5 + x + 1$, so we can use implicit differentiation (Section 3.6) to investigate properties of the inverse function determined by its derivative.

RESTRICTING DOMAINS FOR INVERTIBILITY

Frequently, the domain of a function that is not one-to-one can be partitioned into intervals so that the "piece" of the function defined on each interval in the partition is one-to-one. Thus, the function may be viewed as piecewise defined in terms of one-to-one functions. For example, the function $f(x) = x^2$ is not one-to-one on its natural domain, $-\infty < x < +\infty$, but consider

$$f(x) = \begin{cases} x^2, & x < 0 \\ x^2, & x \geq 0 \end{cases}$$

(Figure 7.1.10). The "piece" of $f(x)$ given by

$$g(x) = x^2, \quad x \geq 0$$

is increasing, and so is one-to-one, on its specified domain. Thus, g has an inverse function g^{-1}. Solving

$$y = x^2, \quad x \geq 0$$

for x yields $x = \sqrt{y}$, so $g^{-1}(y) = \sqrt{y}$. Similarly, if

$$h(x) = x^2, \quad x \leq 0$$

then h has an inverse function, $h^{-1}(y) = -\sqrt{y}$. Geometrically, the graphs of $g(x) = x^2$, $x \geq 0$ and $g^{-1}(x) = \sqrt{x}$ are reflections of one another about the line $y = x$, as are the graphs of $h(x) = x^2$, $x \leq 0$ and $h^{-1}x = -\sqrt{x}$ (Figure 7.1.11).

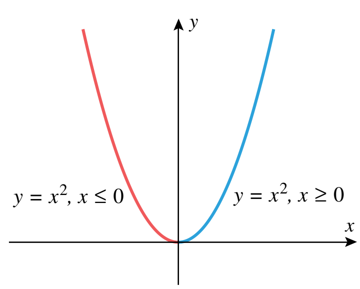


Figure 7.1.10

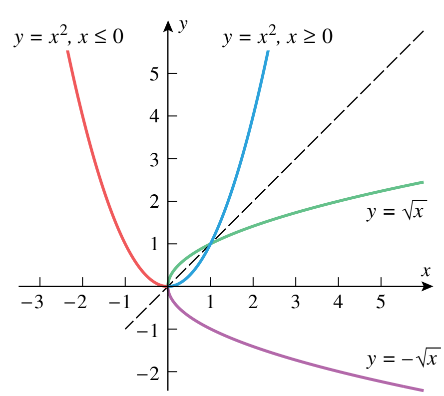


Figure 7.1.11

The functions $g(x)$ and $h(x)$ in the last paragraph are called ***restrictions*** of the function $f(x)$ because each is obtained from $f(x)$ merely by placing a restriction on its domain. In particular, we say that $g(x)$ is the restriction of $f(x)$ to the interval $[0, +\infty)$ and that $h(x)$ is the restriction of $f(x)$ to the interval $(-\infty, 0]$.

CONTINUITY OF INVERSE FUNCTIONS

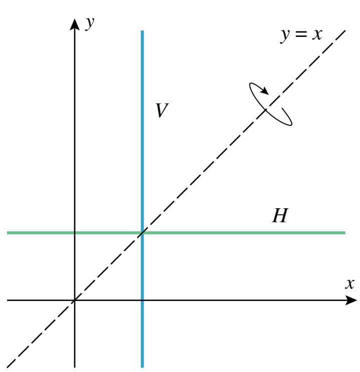
Since the graphs of a one-to-one function f and its inverse function f^{-1} are reflections of one another about the line $y = x$, it is intuitively clear that if the graph of f has no breaks, then neither will the graph of f^{-1}. This suggests the following result, which we state without proof.

> **7.1.5 THEOREM.** *Suppose that f is a function with domain D and range R. If D is an interval and f is continuous and one-to-one on D, then R is an interval and the inverse of f is continuous on R.*

For example, the function $f(x) = x^5+x+1$ in Example 8 has domain and range $(-\infty, +\infty)$, and f is continuous and one-to-one on $(-\infty, +\infty)$. Thus, we can conclude that f^{-1} is continuous on $(-\infty, +\infty)$, despite our inability to find a formula for $f^{-1}(x)$.

DIFFERENTIABILITY OF INVERSE FUNCTIONS

Suppose that f is a function whose domain D is an open interval and that f is continuous and one-to-one on D. Informally, the places where f fails to be differentiable occur where the graph of f has a corner or a vertical tangent line. Similarly, f^{-1} will be differentiable on its domain except where the graph of f^{-1} has a corner or a vertical tangent line. Note that a corner in the graph of f will reflect about the line $y = x$ to a corner in the graph of f^{-1}, and vice versa. However, since a vertical line is the reflection of a horizontal line about the graph of $y = x$ (Figure 7.1.12), a point of vertical tangency on the graph of f^{-1} will correspond to a point of horizontal tangency on the graph of f. Thus, f^{-1} will fail to be differentiable at a point $(x, f^{-1}(x))$ on its graph if $f'(f^{-1}(x)) = 0$.



The vertical line V reflects into the horizontal line H and conversely.

Figure 7.1.12

Now suppose that f is differentiable at a point (a, b) and that $f'(a) \neq 0$. Then

$$y - b = f'(a)(x - a)$$

is the equation of the tangent line to the graph of f at (a, b). The reflection of this line about the graph of $y = x$ should carry it to a tangent line L to the graph of $y = f^{-1}(x)$ at the point (b, a). The equation of L is

$$x - b = f'(a)(y - a) \quad \text{or} \quad y - a = \frac{1}{f'(a)}(x - b)$$

which tells us that the slope of the curve $y = f^{-1}(x)$ at (b, a) and the slope of the curve $y = f(x)$ at (a, b) are reciprocals (Figure 7.1.13). Using $a = f^{-1}(b)$ we obtain

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$$

In summary, we have the following result.

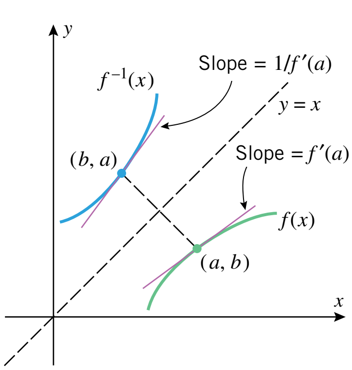


Figure 7.1.13

> **7.1.6 THEOREM (*Differentiability of Inverse Functions*).** *Suppose that f is a function whose domain D is an open interval, and let R be the range of f. If f is differentiable and one-to-one on D, then f^{-1} is differentiable at any value x in R for which $f'(f^{-1}(x)) \neq 0$. Furthermore, if x is in R with $f'(f^{-1}(x)) \neq 0$, then*
>
> $$\frac{d}{dx}[f^{-1}(x)] = \frac{1}{f'(f^{-1}(x))} \tag{4}$$

As an immediate consequence of Theorems 7.1.4 and 7.1.5 we have the following result.

> **7.1.7 COROLLARY.** *If the domain of a function f is an interval on which $f'(x) > 0$ or on which $f'(x) < 0$, then f has an inverse function f^{-1} and $f^{-1}(x)$ is differentiable at any value x in the range of f. The derivative of f^{-1} is given by Formula (4).*

REMARK. A careful proof of Theorem 7.1.6 would involve the definition of the derivative of $f^{-1}(x)$. As a sketch of this argument, consider the special case where f is differentiable and increasing on D and set $g(x) = f^{-1}(x)$. Then g is increasing and continuous on R. Now, $g(x)$ is differentiable at those values of x for which

$$\lim_{w \to x} \frac{g(w) - g(x)}{w - x}$$

exists. For w and x in R with $w \neq x$, set $r = g(w)$ and $s = g(x)$, so $f(r) = w$, $f(s) = x$, and $r \neq s$. Then

$$\frac{g(w) - g(x)}{w - x} = \frac{r - s}{f(r) - f(s)} = \frac{1}{\dfrac{f(r) - f(s)}{r - s}}$$

Using the facts that f and g are continuous and increasing on their domains and that f and g are inverse functions, we can argue that $w \to x$ if and only if $r \to s$. Thus,

$$\lim_{w \to x} \frac{g(w) - g(x)}{w - x}$$

exists provided

$$\lim_{r \to s} \frac{f(r) - f(s)}{r - s}$$

exists and is not zero. That is, $y = f^{-1}(x)$ is differentiable at x if f is differentiable at y and $f'(y) \neq 0$.

Formula (4) can be expressed in a less forbidding form by setting

$$y = f^{-1}(x) \quad \text{so that} \quad x = f(y)$$

Thus,

$$\frac{dy}{dx} = (f^{-1})'(x) \quad \text{and} \quad \frac{dx}{dy} = f'(y) = f'(f^{-1}(x))$$

Substituting these expressions into Formula (4) yields the following alternative version of that formula:

$$\boxed{\frac{dy}{dx} = \frac{1}{dx/dy}} \tag{5}$$

If an explicit formula can be obtained for the inverse of a function, then the differentiability of the inverse function can generally be deduced from that formula. However, if no explicit formula for the inverse can be obtained, then Theorem 7.1.6 is the primary tool for establishing differentiability of the inverse function. Once the differentiability has been established, a derivative function for the inverse function can be obtained either by implicit differentiation or by using Formula (4) or (5).

Example 9 We saw in Example 8 that the function $f(x) = x^5 + x + 1$ is invertible.

(a) Show that f^{-1} is differentiable on the interval $(-\infty, +\infty)$.

(b) Find a formula for the derivative of f^{-1} using Formula (5).

(c) Find a formula for the derivative of f^{-1} using implicit differentiation.

Solution (a). Both the range and domain of f are $(-\infty, +\infty)$. Since

$$f'(x) = 5x^4 + 1 > 0$$

for all x, f^{-1} is differentiable at every x in its domain, $(-\infty, +\infty)$.

Solution (b). If we let $y = f^{-1}(x)$, then

$$x = f(y) = y^5 + y + 1 \tag{6}$$

from which it follows that $dx/dy = 5y^4 + 1$. Then, from Formula (5),

$$\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{5y^4 + 1} \tag{7}$$

Since we were unable to solve (6) for y in terms of x, we must leave (7) in terms of y.

Solution (c). Differentiating (6) implicitly with respect to x yields

$$\frac{d}{dx}[x] = \frac{d}{dx}[y^5 + y + 1]$$

$$1 = (5y^4 + 1)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{5y^4 + 1}$$

which agrees with (7). ◀

GRAPHING INVERSE FUNCTIONS WITH GRAPHING UTILITIES

Most graphing utilities cannot graph inverse functions directly. However, there is a way of graphing inverse functions by expressing the graphs parametrically. To see how this can be done, suppose that we are interested in graphing the inverse of a one-to-one function f. We observed in Section 1.8 that the equation $y = f(x)$ can be expressed parametrically as

$$x = t, \quad y = f(t) \tag{8}$$

Moreover, we know that the graph of f^{-1} can be obtained by interchanging x and y, since this reflects the graph of f about the line $y = x$. Thus, from (8) the graph of f^{-1} can be represented parametrically as

$$x = f(t), \quad y = t \tag{9}$$

For example, Figure 7.1.14 shows the graph of $f(x) = x^5 + x + 1$ and its inverse generated with a graphing utility. The graph of f was generated from the parametric equations

$$x = t, \quad y = t^5 + t + 1$$

and the graph of f^{-1} was generated from the parametric equations

$$x = t^5 + t + 1, \quad y = t$$

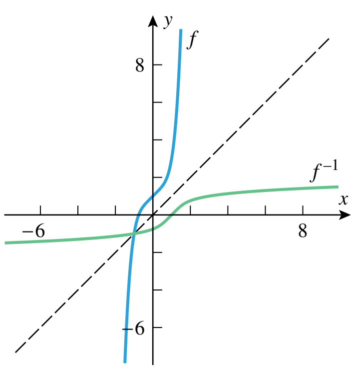


Figure 7.1.14

EXERCISE SET 7.1 Graphing Utility

1. In (a)–(d), determine whether f and g are inverse functions.
   (a) $f(x) = 4x,\ g(x) = \frac{1}{4}x$
   (b) $f(x) = 3x + 1,\ g(x) = 3x - 1$
   (c) $f(x) = \sqrt[3]{x - 2},\ g(x) = x^3 + 2$
   (d) $f(x) = x^4,\ g(x) = \sqrt[4]{x}$

2. Check your answers to Exercise 1 with a graphing utility by determining whether the graphs of f and g are reflections of one another about the line $y = x$.

3. In each part, determine whether the function f defined by the table is one-to-one.

(a)

x	1	2	3	4	5	6
$f(x)$	−2	−1	0	1	2	3

(b)

x	1	2	3	4	5	6
$f(x)$	4	−7	6	−3	1	4

4. In each part, determine whether the function f is one-to-one, and justify your answer.
   (a) $f(t)$ is the number of people in line at a movie theater at time t.
   (b) $f(x)$ is your weight on your xth birthday.
   (c) $f(v)$ is the weight of v cubic inches of lead.

5. In each part, use the horizontal line test to determine whether the function f is one-to-one.
   (a) $f(x) = 3x + 2$ (b) $f(x) = \sqrt{x - 1}$
   (c) $f(x) = |x|$ (d) $f(x) = x^3$
   (e) $f(x) = x^2 - 2x + 2$ (f) $f(x) = \sin x$

6. In each part, generate the graph of the function f with a graphing utility, and determine whether f is one-to-one.
   (a) $f(x) = x^3 - 3x + 2$ (b) $f(x) = x^3 - 3x^2 + 3x - 1$

7. In each part, determine whether f is one-to-one.
   (a) $f(x) = \tan x$
   (b) $f(x) = \tan x,\quad -\pi < x < \pi, x \neq \pm\pi/2$
   (c) $f(x) = \tan x,\quad -\pi/2 < x < \pi/2$

8. In each part, determine whether f is one-to-one.
(a) $f(x) = \cos x$
(b) $f(x) = \cos x, \quad -\pi/2 \le x \le \pi/2$
(c) $f(x) = \cos x, \quad 0 \le x \le \pi$

9. (a) The accompanying figure shows the graph of a function f over its domain $-8 \le x \le 8$. Explain why f has an inverse, and use the graph to find $f^{-1}(2)$, $f^{-1}(-1)$, and $f^{-1}(0)$.
(b) Find the domain and range of f^{-1}.
(c) Sketch the graph of f^{-1}.

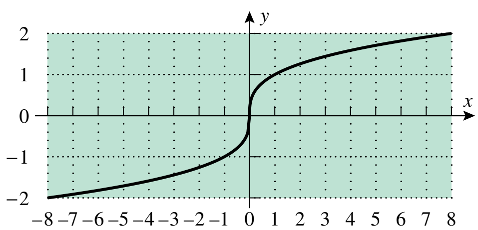


Figure Ex-9

10. (a) Explain why the function f graphed in the accompanying figure has no inverse function on its domain $-3 \le x \le 4$.
(b) Subdivide the domain into three adjacent intervals on each of which the function f has an inverse.

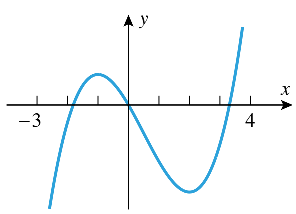


Figure Ex-10

In Exercises 11 and 12, determine whether the function f is one-to-one by examining the sign of $f'(x)$.

11. (a) $f(x) = x^2 + 8x + 1$
(b) $f(x) = 2x^5 + x^3 + 3x + 2$
(c) $f(x) = 2x + \sin x$

12. (a) $f(x) = x^3 + 3x^2 - 8$
(b) $f(x) = x^5 + 8x^3 + 2x - 1$
(c) $f(x) = \dfrac{x}{x+1}$

In Exercises 13–23, find a formula for $f^{-1}(x)$.

13. $f(x) = x^5$

14. $f(x) = 6x$

15. $f(x) = 7x - 6$

16. $f(x) = \dfrac{x+1}{x-1}$

17. $f(x) = 3x^3 - 5$

18. $f(x) = \sqrt[5]{4x+2}$

19. $f(x) = \sqrt[3]{2x-1}$

20. $f(x) = 5/(x^2+1), \quad x \ge 0$

21. $f(x) = 3/x^2, \quad x < 0$

22. $f(x) = \begin{cases} 2x, & x \le 0 \\ x^2, & x > 0 \end{cases}$

23. $f(x) = \begin{cases} 5/2 - x, & x < 2 \\ 1/x, & x \ge 2 \end{cases}$

24. Find a formula for $p^{-1}(x)$, given that
$$p(x) = x^3 - 3x^2 + 3x - 1$$

In Exercises 25–29, find a formula for $f^{-1}(x)$, and state the domain of f^{-1}.

25. $f(x) = (x+2)^4, \quad x \ge 0$

26. $f(x) = \sqrt{x+3}$

27. $f(x) = -\sqrt{3-2x}$

28. $f(x) = 3x^2 + 5x - 2, \quad x \ge 0$

29. $f(x) = x - 5x^2, \quad x \ge 1$

30. The formula $F = \frac{9}{5}C + 32$, where $C \ge -273.15$ expresses the Fahrenheit temperature F as a function of the Celsius temperature C.
(a) Find a formula for the inverse function.
(b) In words, what does the inverse function tell you?
(c) Find the domain and range of the inverse function.

31. (a) One meter is about 6.214×10^{-4} miles. Find a formula $y = f(x)$ that expresses a length x in meters as a function of the same length y in miles.
(b) Find a formula for the inverse of f.
(c) In practical terms, what does the formula $x = f^{-1}(y)$ tell you?

32. Suppose that f is a one-to-one, continuous function such that $\lim_{x\to 3} f(x) = 7$. Find $\lim_{x\to 7} f^{-1}(x)$, and justify your reasoning.

33. Let $f(x) = x^2$, $x > 1$, and $g(x) = \sqrt{x}$.
(a) Show that $f(g(x)) = x$, $x > 1$, and $g(f(x)) = x$, $x > 1$.
(b) Show that f and g are *not* inverses by showing that the graphs of $y = f(x)$ and $y = g(x)$ are not reflections of one another about $y = x$.
(c) Do parts (a) and (b) contradict one another? Explain.

34. Let $f(x) = ax^2 + bx + c$, $a > 0$. Find f^{-1} if the domain of f is restricted to
(a) $x \ge -b/(2a)$
(b) $x \le -b/(2a)$.

35. (a) Show that $f(x) = (3-x)/(1-x)$ is its own inverse.
(b) What does the result in part (a) tell you about the graph of f?

36. Suppose that a line of nonzero slope m intersects the x-axis at $(x_0, 0)$. Find an equation for the reflection of this line about $y = x$.

37. (a) Show that $f(x) = x^3 - 3x^2 + 2x$ is not one-to-one on $(-\infty, +\infty)$.
(b) Find the largest value of k such that f is one-to-one on the interval $(-k, k)$.

38. (a) Show that the function $f(x) = x^4 - 2x^3$ is not one-to-one on $(-\infty, +\infty)$.
(b) Find the smallest value of k such that f is one-to-one on the interval $[k, +\infty)$.

39. Let $f(x) = 2x^3 + 5x + 3$. Find x if $f^{-1}(x) = 1$.

40. Let $f(x) = \dfrac{x^3}{x^2+1}$. Find x if $f^{-1}(x) = 2$.

In Exercises 41–44, use a graphing utility and parametric equations to display the graphs of f and f^{-1} on the same screen.

41. $f(x) = x^3 + 0.2x - 1, \quad -1 \le x \le 2$

42. $f(x) = \sqrt{x^2+2} + x, \quad -5 \le x \le 5$

43. $f(x) = \cos(\cos 0.5x), \quad 0 \le x \le 3$

44. $f(x) = x + \sin x, \quad 0 \le x \le 6$

In Exercises 45–48, find the derivative of f^{-1} by using Formula (5), and check your result by differentiating implicitly.

45. $f(x) = 5x^3 + x - 7$

46. $f(x) = 1/x^2, \ x > 0$

47. $f(x) = 2x^5 + x^3 + 1$

48. $f(x) = 5x - \sin 2x, \quad -\dfrac{\pi}{4} < x < \dfrac{\pi}{4}$

49. Prove that if $a^2 + bc \ne 0$, then the graph of

$$f(x) = \frac{ax+b}{cx-a}$$

is symmetric about the line $y = x$.

50. (a) Prove: If f and g are one-to-one, then so is the composition $f \circ g$.
(b) Prove: If f and g are one-to-one, then

$$(f \circ g)^{-1} = g^{-1} \circ f^{-1}$$

51. Sketch the graph of a function that is one-to-one on $(-\infty, +\infty)$, yet not increasing on $(-\infty, +\infty)$ and not decreasing on $(-\infty, +\infty)$.

52. Prove: A one-to-one function f cannot have two different inverse functions.

53. Let $F(x) = f(2g(x))$ where $f(x) = x^4 + x^3 + 1$ for $0 \le x \le 2$, and $g(x) = f^{-1}(x)$. Find $F(3)$.

7.2 EXPONENTIAL AND LOGARITHMIC FUNCTIONS

When logarithms were introduced in the seventeenth century as a computational tool, they provided scientists of that period computing power that was previously unimaginable. Although computers and calculators have largely replaced logarithms for numerical calculations, the logarithmic functions and their relatives have wide-ranging applications in mathematics and science. Some of these will be introduced in this section.

IRRATIONAL EXPONENTS

In algebra, integer and rational powers of a number b are defined by

$$b^n = b \times b \times \cdots \times b \quad (n \text{ factors}), \qquad b^{-n} = \frac{1}{b^n}, \qquad b^0 = 1,$$

$$b^{p/q} = \sqrt[q]{b^p} = (\sqrt[q]{b})^p, \qquad b^{-p/q} = \frac{1}{b^{p/q}}$$

If b is negative, then some of the fractional powers of b will have imaginary values; for example, $(-2)^{1/2} = \sqrt{-2}$. To avoid this complication we will assume throughout this section that $b \ge 0$, even if it is not stated explicitly.

Observe that the preceding definitions do not include *irrational* powers of b such as

$$2^\pi, \quad 3^{\sqrt{2}}, \quad \text{and} \quad \pi^{-\sqrt{7}}$$

There are various methods for defining irrational powers. One approach is to define irrational powers of b as limits of rational powers of b. For example, to define 2^π we can start with the decimal representation of π, namely,

$$3.1415926\ldots$$

From this decimal we can form a sequence of rational numbers that gets closer and closer to π, namely,

$$3.1, \quad 3.14, \quad 3.141, \quad 3.1415, \quad 3.14159$$

and from these we can form a sequence of *rational* powers of 2:

$$2^{3.1}, \quad 2^{3.14}, \quad 2^{3.141}, \quad 2^{3.1415}, \quad 2^{3.14159}$$

Table 7.2.1

x	2^x
3	8.000000
3.1	8.574188
3.14	8.815241
3.141	8.821353
3.1415	8.824411
3.14159	8.824962
3.141592	8.824974

Since the exponents of the terms in this sequence approach a limit of π, it seems plausible that the terms themselves approach a limit, and it would seem reasonable to *define* 2^π to be this limit. Table 7.2.1 provides numerical evidence that the sequence does, in fact, have a limit and that to four decimal places the value of this limit is $2^\pi \approx 8.8250$. More generally, for any irrational exponent p and positive number b, we can define b^p as the limit of the rational powers of b created from the decimal expansion of p.

FOR THE READER. Confirm the approximation $2^\pi \approx 8.8250$ by computing 2^π directly using your calculating utility.

Although our definition of b^p for irrational p certainly seems reasonable, there is a lot of tedious mathematical detail required to make the definition precise. We will not be concerned with such matters here and will accept without proof that the following familiar laws hold for all real exponents:

$$b^p b^q = b^{p+q}, \quad \frac{b^p}{b^q} = b^{p-q}, \quad (b^p)^q = b^{pq}$$

THE FAMILY OF EXPONENTIAL FUNCTIONS

A function of the form $f(x) = b^x$, where $b > 0$ and $b \neq 1$, is called an ***exponential function with base b***. Some examples are

$$f(x) = 2^x, \quad f(x) = \left(\tfrac{1}{2}\right)^x, \quad f(x) = \pi^x$$

Note that an exponential function has a constant base and variable exponent. Thus, functions such as $f(x) = x^2$ and $f(x) = x^\pi$ would not be classified as exponential functions, since they have a variable base and a constant exponent.

It can be shown that exponential functions are continuous and have one of the basic two shapes shown in Figure 7.2.1a, depending on whether $0 < b < 1$ or $b > 1$. Figure 7.2.1b shows the graphs of some specific exponential functions.

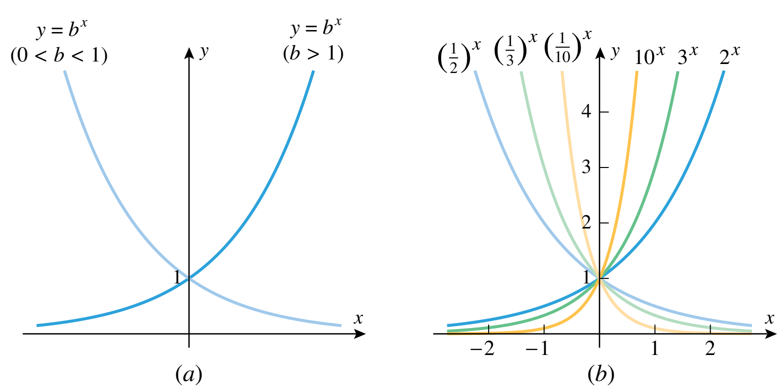


Figure 7.2.1

REMARK. If $b = 1$, then the function b^x is constant, since $b^x = 1^x = 1$. This case is of no interest to us here, so we have excluded it from the family of exponential functions.

FOR THE READER. Use your graphing utility to confirm that the graphs of $y = \left(\frac{1}{2}\right)^x$ and $y = 2^x$ agree with Figure 7.2.1b, and explain why the two graphs are reflections of one another about the y-axis.

Since it is not our objective in this section to develop the properties of exponential functions in rigorous mathematical detail, we will simply observe without proof that the following properties of exponential functions are consistent with the graphs shown in Figure 7.2.1.

7.2.1 THEOREM. *If $b > 0$ and $b \neq 1$, then:*

(*a*) *The function $f(x) = b^x$ is defined for all real values of x, so its natural domain is $(-\infty, +\infty)$.*

(*b*) *The function $f(x) = b^x$ is continuous on the interval $(-\infty, +\infty)$, and its range is $(0, +\infty)$.*

LOGARITHMS

Recall from algebra that a logarithm is an exponent. More precisely, if $b > 0$ and $b \neq 1$, then for positive values of x the ***logarithm to the base b of x*** is denoted by

$$\log_b x$$

and is defined to be that exponent to which b must be raised to produce x. For example,

$$\log_{10} 100 = 2, \quad \log_{10}(1/1000) = -3, \quad \log_2 16 = 4, \quad \log_b 1 = 0, \quad \log_b b = 1$$

$10^2 = 100$ $10^{-3} = 1/1000$ $2^4 = 16$ $b^0 = 1$ $b^1 = b$

Historically, the first logarithms ever studied were the logarithms with base 10, called ***common logarithms***. For such logarithms it is usual to suppress explicit reference to the base and write $\log x$ rather than $\log_{10} x$. More recently, logarithms with base 2 have played a role in computer science, since they arise naturally in the binary number system. However, the most widely used logarithms in applications are the ***natural logarithms***, which have an irrational base denoted by the letter e in honor of the Swiss mathematician Leonhard Euler (p. 11), who first suggested its application to logarithms in an unpublished paper written in 1728. This constant, whose value to six decimal places is

$$e \approx 2.718282 \tag{1}$$

arises as the horizontal asymptote of the graph of the equation

$$y = \left(1 + \frac{1}{x}\right)^x \tag{2}$$

(Figure 7.2.2).

THE VALUES OF $(1 + 1/x)^x$ APPROACH e AS $x \to +\infty$

x	$1 + \frac{1}{x}$	$\left(1 + \frac{1}{x}\right)^x$
1	2	≈ 2.000000
10	1.1	2.593742
100	1.01	2.704814
1000	1.001	2.716924
10,000	1.0001	2.718146
100,000	1.00001	2.718268
1,000,000	1.000001	2.718280

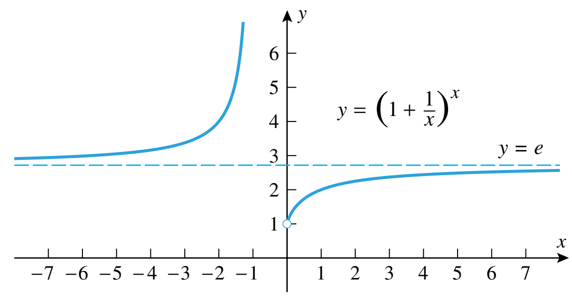


Figure 7.2.2

The fact that $y = e$ is a horizontal asymptote of (2) as $x \to +\infty$ and as $x \to -\infty$ is expressed by the limits

$$e = \lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x \quad \text{and} \quad e = \lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^x \tag{3–4}$$

Later, we will show that these limits can be derived from the limit

$$e = \lim_{x \to 0} (1 + x)^{1/x} \tag{5}$$

which is sometimes taken as the definition of the number e.

It is standard to denote the natural logarithm of x by $\ln x$ (read "ell en of x"), rather than $\log_e x$. Thus, $\ln x$ can be viewed as that power to which e must be raised to produce x. For example,

$$\ln 1 = 0, \qquad \ln e = 1, \qquad \ln 1/e = -1, \qquad \ln(e^2) = 2$$

$$\boxed{\text{Since } e^0 = 1} \quad \boxed{\text{Since } e^1 = e} \quad \boxed{\text{Since } e^{-1} = 1/e} \quad \boxed{\text{Since } e^2 = e^2}$$

In general, the statements

$$y = \ln x \quad \text{and} \quad x = e^y$$

are equivalent.

The exponential function $f(x) = e^x$ is called the ***natural exponential function***. To simplify typography, this function is sometimes written as $\exp x$. Thus, for example, you might see the relationship $e^{x_1+x_2} = e^{x_1}e^{x_2}$ expressed as

$$\exp(x_1 + x_2) = \exp(x_1)\exp(x_2)$$

This notation is also used by graphing and calculating utilities, and it is typical to access the function e^x with some variation of the command EXP.

FOR THE READER. Most scientific calculating utilities provide some way of evaluating common logarithms, natural logarithms, and powers of e. Check your documentation to see how this is done, and then confirm the approximation $e \approx 2.718282$ and the values that appear in the table in Figure 7.2.2.

LOGARITHMIC FUNCTIONS

Figure 7.2.1*a* suggests that if $b > 0$ and $b \neq 1$, then the graph of $y = b^x$ passes the horizontal line test, and this implies that the function $f(x) = b^x$ has an inverse function. To find a formula for this inverse (with x as the independent variable), we can solve the equation $x = b^y$ for y as a function of x. This can be done by taking the logarithm to the base b of both sides of this equation. This yields

$$\log_b x = \log_b(b^y) \tag{6}$$

However, if we think of $\log_b(b^y)$ as that exponent to which b must be raised to produce b^y, then it becomes evident that $\log_b(b^y) = y$. Thus, (6) can be rewritten as

$$y = \log_b x$$

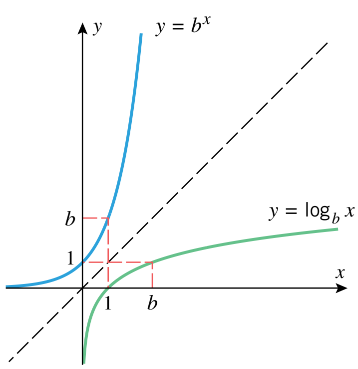


Figure 7.2.3

from which we conclude that the inverse of $f(x) = b^x$ is $f^{-1}(x) = \log_b x$. This implies that the graphs of $y = b^x$ and $y = \log_b x$ are reflections of one another about the line $y = x$ (Figure 7.2.3). We call $\log_b x$ the ***logarithmic function with base b***.

Recall from Section 7.1 that a one-to-one function f and its inverse satisfy the equations

$$f^{-1}(f(x)) = x \quad \text{for every } x \text{ in the domain of } f$$

$$f(f^{-1}(x)) = x \quad \text{for every } x \text{ in the domain of } f^{-1}$$

In particular, if we take $f(x) = b^x$ and $f^{-1}(x) = \log_b x$, and if we keep in mind that the domain of f^{-1} is the same as the range of f, then we obtain

$$\begin{aligned} \log_b(b^x) &= x \quad \text{for all real values of } x \\ b^{\log_b x} &= x \quad \text{for } x > 0 \end{aligned} \tag{7}$$

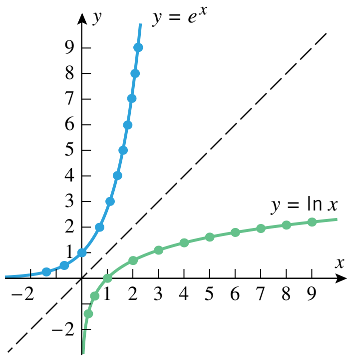
In the special case where $b = e$, these equations become

$$\begin{aligned} \ln(e^x) &= x \quad \text{for all real values of } x \\ e^{\ln x} &= x \quad \text{for } x > 0 \end{aligned} \tag{8}$$

In words, the equations in (7) tell us that the functions b^x and $\log_b x$ cancel out the effect of one another when composed in either order; for example,

$$\log 10^x = x, \quad 10^{\log x} = x, \quad \ln e^x = x, \quad e^{\ln x} = x, \quad \ln e^5 = 5, \quad e^{\ln \pi} = \pi$$

REMARK. Figure 7.2.4 shows computer-generated tables and graphs of $y = e^x$ and $y = \ln x$. The values of $y = e^x$ and $y = \ln x$ have been rounded to the second decimal place in the tables. This explains why the column under $y = e^x$ in the second table is not identical to the column under x in the first table.



x	$y = \ln x$
0.25	−1.39
0.50	−0.69
1	0
2	0.69
3	1.10
4	1.39
5	1.61
6	1.79
7	1.95
8	2.08
9	2.20

x	$y = e^x$
−1.39	0.25
−0.69	0.50
0	1.00
0.69	1.99
1.10	3.00
1.39	4.01
1.61	5.00
1.79	5.99
1.95	7.03
2.08	8.00
2.20	9.03

Figure 7.2.4

The inverse relationship between b^x and $\log_b x$ allows us to translate properties of exponential functions into properties of logarithmic functions, and vice versa.

7.2.2 THEOREM (***Comparison of Exponential and Logarithmic Functions***). *If $b > 0$ and $b \neq 1$, then:*

$b^0 = 1$	$\log_b 1 = 0$
$b^1 = b$	$\log_b b = 1$
range $b^x = (0, +\infty)$	domain $\log_b x = (0, +\infty)$
domain $b^x = (-\infty, +\infty)$	range $\log_b x = (-\infty, +\infty)$
$y = b^x$ *is continuous on* $(-\infty, +\infty)$	$y = \log_b x$ *is continuous on* $(0, +\infty)$

You should recall the following algebraic properties of logarithms from your earlier studies.

7.2.3 THEOREM (***Algebraic Properties of Logarithms***). *If $b > 0$, $b \neq 1$, $a > 0$, $c > 0$, and r is any real number, then:*

$\log_b(ac) = \log_b a + \log_b c$	Product property
$\log_b(a/c) = \log_b a - \log_b c$	Quotient property
$\log_b(a^r) = r \log_b a$	Power property
$\log_b(1/c) = -\log_b c$	Reciprocal property

These properties are often used to expand a single logarithm into sums, differences, and multiples of other logarithms and, conversely, to condense sums, differences, and multiples of logarithms into a single logarithm. For example,

$$\log \frac{xy^5}{\sqrt{z}} = \log xy^5 - \log \sqrt{z} = \log x + \log y^5 - \log z^{1/2} = \log x + 5\log y - \tfrac{1}{2}\log z$$

$$5\log 2 + \log 3 - \log 8 = \log 32 + \log 3 - \log 8 = \log \frac{32 \cdot 3}{8} = \log 12$$

$$\tfrac{1}{3}\ln x - \ln(x^2-1) + 2\ln(x+3) = \ln x^{1/3} - \ln(x^2-1) + \ln(x+3)^2 = \ln \frac{\sqrt[3]{x}(x+3)^2}{x^2-1}$$

REMARK. Expressions of the form $\log_b(u+v)$ and $\log_b(u-v)$ have no useful simplifications in terms of $\log_b u$ and $\log_b v$. In particular,

$$\log_b(u+v) \neq \log_b u + \log_b v$$
$$\log_b(u-v) \neq \log_b u - \log_b v$$

SOLVING EQUATIONS INVOLVING EXPONENTIALS AND LOGARITHMS

The equation $y = e^x$ can be solved for x in terms of y as $x = \ln y$, provided (of course) that y is in the domain of the natural logarithm function and x is in the domain of the natural exponential function; that is, $y > 0$ and x is any real number. Thus,

$$y = e^x \text{ is equivalent to } x = \ln y \text{ if } y > 0 \text{ and } x \text{ is any real number}$$

More generally, if $b > 0$ and $b \neq 1$, then

> $y = b^x$ is equivalent to $x = \log_b y$ if $y > 0$ and x is any real number

Equations of the form $\log_b x = k$ can be solved by converting them to the exponential form $x = b^k$, and equations of the form $b^x = k$ can be solved by taking a logarithm of both sides (usually log or ln).

Example 1 Find x such that

(a) $\log x = \sqrt{2}$ (b) $\ln(x+1) = 5$ (c) $5^x = 7$

Solution (a). Converting the equation to exponential form yields

$$x = 10^{\sqrt{2}} \approx 25.95$$

Solution (b). Converting the equation to exponential form yields

$$x + 1 = e^5 \quad \text{or} \quad x = e^5 - 1 \approx 147.41$$

Solution (c). Taking the natural logarithm of both sides and using the power property of logarithms yields

$$x \ln 5 = \ln 7 \quad \text{or} \quad x = \frac{\ln 7}{\ln 5} \approx 1.21$$

◀

Example 2 A satellite that requires 7 watts of power to operate at full capacity is equipped with a radioisotope power supply whose power output P in watts is given by the equation

$$P = 75e^{-t/125}$$

where t is the time in days that the supply is used. How long can the satellite operate at full capacity?

Solution. The power P will fall to 7 watts when

$$7 = 75e^{-t/125}$$

The solution for t is as follows:

$$7/75 = e^{-t/125}$$
$$\ln(7/75) = \ln(e^{-t/125})$$
$$\ln(7/75) = -t/125$$
$$t = -125\ln(7/75) \approx 296.4$$

so the satellite can operate at full capacity for about 296 days. ◀

Here is a more complicated example.

Example 3 Solve $\dfrac{e^x - e^{-x}}{2} = 1$ for x.

Solution. Multiplying both sides of the given equation by 2 yields

$$e^x - e^{-x} = 2$$

or equivalently,

$$e^x - \frac{1}{e^x} = 2$$

Multiplying through by e^x yields

$$e^{2x} - 1 = 2e^x \quad \text{or} \quad e^{2x} - 2e^x - 1 = 0$$

This is really a quadratic equation in disguise, as can be seen by rewriting it in the form

$$(e^x)^2 - 2e^x - 1 = 0$$

and letting $u = e^x$ to obtain

$$u^2 - 2u - 1 = 0$$

Solving for u by the quadratic formula yields

$$u = \frac{2 \pm \sqrt{4+4}}{2} = \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}$$

or, since $u = e^x$,

$$e^x = 1 \pm \sqrt{2}$$

But e^x cannot be negative, so we discard the negative value $1 - \sqrt{2}$; thus,

$$\begin{aligned} e^x &= 1 + \sqrt{2} \\ \ln e^x &= \ln(1 + \sqrt{2}) \\ x &= \ln(1 + \sqrt{2}) \approx 0.881 \end{aligned}$$

◀

CHANGE OF BASE FORMULA FOR LOGARITHMS

Scientific calculators generally provide keys for evaluating common logarithms and natural logarithms but have no keys for evaluating logarithms with other bases. However, this is not a serious deficiency because it is possible to express a logarithm with any base in terms of logarithms with any other base (see Exercise 40). For example, the following formula expresses a logarithm with base b in terms of natural logarithms:

$$\log_b x = \frac{\ln x}{\ln b} \tag{9}$$

We can derive this result by letting $y = \log_b x$, from which it follows that $b^y = x$. Taking the natural logarithm of both sides of this equation we obtain $y \ln b = \ln x$, from which (9) follows.

Example 4 Use a calculating utility to evaluate $\log_2 5$ by expressing this logarithm in terms of natural logarithms.

Solution. From (9) we obtain

$$\log_2 5 = \frac{\ln 5}{\ln 2} \approx 2.321928$$

◀

LOGARITHMIC SCALES IN SCIENCE AND ENGINEERING

Logarithms are used in science and engineering to deal with quantities whose units vary over an excessively wide range of values. For example, the "loudness" of a sound can be measured by its ***intensity*** I (in watts per square meter), which is related to the energy transmitted by the sound wave—the greater the intensity, the greater the transmitted energy, and the louder the sound is perceived by the human ear. However, intensity units are unwieldy because they vary over an enormous range. For example, a sound at the threshold of human hearing has an intensity of about 10^{-12} W/m^2, a close whisper has an intensity that is about 100 times the hearing threshold, and a jet engine at 50 meters has an intensity that is about $1{,}000{,}000{,}000{,}000 = 10^{12}$ times the hearing threshold. To see how logarithms can be used

Table 7.2.2

β (dB)	I/I_0
0	$10^0 = 1$
10	$10^1 = 10$
20	$10^2 = 100$
30	$10^3 = 1{,}000$
40	$10^4 = 10{,}000$
50	$10^5 = 100{,}000$
⋮	⋮
120	$10^{12} = 1{,}000{,}000{,}000{,}000$

to reduce this wide spread, observe that if

$$y = \log x$$

then increasing x by a *factor* of 10 *adds* 1 unit to y since

$$\log 10x = \log 10 + \log x = 1 + y$$

Physicists and engineers take advantage of this property by measuring loudness in terms of the ***sound level*** β, which is defined by

$$\beta = 10 \log(I/I_0)$$

where $I_0 = 10^{-12}$ W/m^2 is a reference intensity close to the threshold of human hearing. The units of β are ***decibels*** (dB), named in honor of the telephone inventor Alexander Graham Bell. With this scale of measurement, *multiplying* the intensity I by a factor of 10 *adds* 10 dB to the sound level β (verify). This results in a more tractable scale than intensity for measuring sound loudness (Table 7.2.2). Some other familiar logarithmic scales are the ***Richter scale*** used to measure earthquake intensity and the **pH** ***scale*** used to measure acidity in chemistry, both of which are discussed in the exercises.

Peter Townsend of the Who sustained permanent hearing reduction due to the high decibel level of his band's music.

Example 5 In 1976 the rock group The Who set the record for the loudest concert: 120 dB. By comparison, a jackhammer positioned at the same spot as The Who would have produced a sound level of 92 dB. What is the ratio of the sound intensity of The Who to the sound intensity of a jackhammer?

Solution. Let I_1 and $\beta_1 (= 120$ dB) denote the intensity and sound level of The Who, and let I_2 and β_2 ($= 92$ dB) denote the intensity and sound level of the jackhammer. Then

$$I_1/I_2 = (I_1/I_0)/(I_2/I_0)$$
$$\log(I_1/I_2) = \log(I_1/I_0) - \log(I_2/I_0)$$
$$10\log(I_1/I_2) = 10\log(I_1/I_0) - 10\log(I_2/I_0)$$
$$10\log(I_1/I_2) = \beta_1 - \beta_2 = 120 - 92 = 28$$
$$\log(I_1/I_2) = 2.8$$

Thus, $I_1/I_2 = 10^{2.8} \approx 630$, which tells us that the sound intensity of The Who was 631 times greater than a jackhammer! ◀

EXPONENTIAL AND LOGARITHMIC GROWTH

Table 7.2.3

x	e^x	$\ln x$
1	2.72	0.00
2	7.39	0.69
3	20.09	1.10
4	54.60	1.39
5	148.41	1.61
6	403.43	1.79
7	1096.63	1.95
8	2980.96	2.08
9	8103.08	2.20
10	22026.47	2.30
100	2.69×10^{43}	4.61
1000	1.97×10^{434}	6.91

The growth patterns of e^x and $\ln x$ illustrated by Table 7.2.3 are worth noting. Both functions increase as x increases, but they increase in dramatically different ways—e^x increases extremely rapidly and $\ln x$ increases extremely slowly. For example, at $x = 10$ the value of e^x is over 22,000, but at $x = 1000$ the value of $\ln x$ has not even reached 7.

The table strongly suggests that $e^x \to +\infty$ as $x \to +\infty$. However, the growth of $\ln x$ is so slow that its limiting behavior as $x \to +\infty$ is not clear from the table. In spite of its slow growth, it is still true that $\ln x \to +\infty$ as $x \to +\infty$. To see that this is so, choose any positive number M (as large as you like). The value of $\ln x$ will reach M when $x = e^M$, since

$$\ln x = \ln(e^M) = M$$

Since $\ln x$ increases as x increases, we can conclude that $\ln x > M$ for $x > e^M$; hence, $\ln x \to +\infty$ as $x \to +\infty$ since the values of $\ln x$ eventually exceed any positive number M (Figure 7.2.5).

In summary,

$$\lim_{x \to +\infty} e^x = +\infty \qquad \lim_{x \to +\infty} \ln x = +\infty \tag{10–11}$$

The following limits, which are consistent with Figure 7.2.5, can be deduced numerically

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by constructing appropriate tables of values (verify):

$$\lim_{x\to-\infty} e^x = 0 \qquad \lim_{x\to 0^+} \ln x = -\infty \tag{12–13}$$

The following limits can be deduced numerically, but they can be seen more readily by noting that the graph of $y = e^{-x}$ is the reflection about the y-axis of the graph of $y = e^x$ (Figure 7.2.6):

$$\lim_{x\to+\infty} e^{-x} = 0 \qquad \lim_{x\to-\infty} e^{-x} = +\infty \tag{14–15}$$

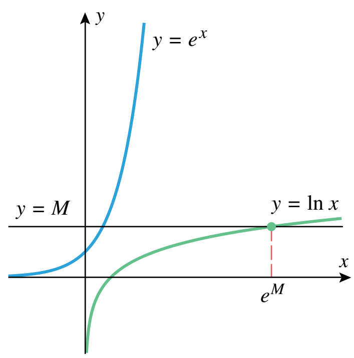


Figure 7.2.5

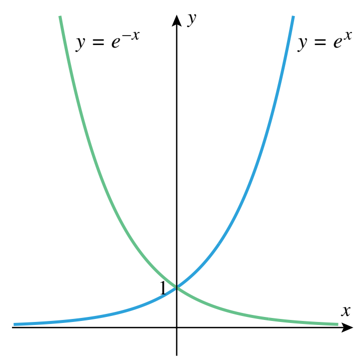


Figure 7.2.6

EXERCISE SET 7.2 Graphing Utility

In Exercises 1 and 2, simplify the expression without using a calculating utility.

1. (a) $-8^{2/3}$ (b) $(-8)^{2/3}$ (c) $8^{-2/3}$

2. (a) 2^{-4} (b) $4^{1.5}$ (c) $9^{-0.5}$

In Exercises 3 and 4, use a calculating utility to approximate the expression. Round your answer to four decimal places.

3. (a) $2^{1.57}$ (b) $5^{-2.1}$

4. (a) $\sqrt[5]{24}$ (b) $\sqrt[8]{0.6}$

In Exercises 5 and 6, find the exact value of the expression without using a calculating utility.

5. (a) $\log_2 16$ (b) $\log_2\left(\frac{1}{32}\right)$
(c) $\log_4 4$ (d) $\log_9 3$

6. (a) $\log_{10}(0.001)$ (b) $\log_{10}(10^4)$
(c) $\ln(e^3)$ (d) $\ln(\sqrt{e}\,)$

In Exercises 7 and 8, use a calculating utility to approximate the expression. Round your answer to four decimal places.

7. (a) $\log 23.2$ (b) $\ln 0.74$

8. (a) $\log 0.3$ (b) $\ln \pi$

In Exercises 9 and 10, use the logarithm properties in Theorem 7.2.3 to rewrite the expression in terms of r, s, and t, where $r = \ln a$, $s = \ln b$, and $t = \ln c$.

9. (a) $\ln a^2\sqrt{bc}$ (b) $\ln \dfrac{b}{a^3c}$

10. (a) $\ln \dfrac{\sqrt[3]{c}}{ab}$ (b) $\ln \sqrt{\dfrac{ab^3}{c^2}}$

In Exercises 11 and 12, expand the logarithm in terms of sums, differences, and multiples of simpler logarithms.

11. (a) $\log(10x\sqrt{x-3}\,)$ (b) $\ln \dfrac{x^2\sin^3 x}{\sqrt{x^2+1}}$

12. (a) $\log \dfrac{\sqrt[3]{x+2}}{\cos 5x}$ (b) $\ln \sqrt{\dfrac{x^2+1}{x^3+5}}$

In Exercises 13–15, rewrite the expression as a single logarithm.

13. $4\log 2 - \log 3 + \log 16$

14. $\frac{1}{2}\log x - 3\log(\sin 2x) + 2$

15. $2\ln(x+1) + \frac{1}{3}\ln x - \ln(\cos x)$

In Exercises 16–25, solve for x without using a calculating utility.

16. $\log_{10}(1+x) = 3$

17. $\log_{10}(\sqrt{x}) = -1$

18. $\ln(x^2) = 4$

19. $\ln(1/x) = -2$

20. $\log_3(3^x) = 7$

21. $\log_5(5^{2x}) = 8$

22. $\log_{10} x^2 + \log_{10} x = 30$

23. $\log_{10} x^{3/2} - \log_{10}\sqrt{x} = 5$

24. $\ln 4x - 3\ln(x^2) = \ln 2$

25. $\ln(1/x) + \ln(2x^3) = \ln 3$

In Exercises 26–31, solve for x without using a calculating utility. Use the natural logarithm anywhere that logarithms are needed.

26. $3^x = 2$

27. $5^{-2x} = 3$

28. $3e^{-2x} = 5$

29. $2e^{3x} = 7$

30. $e^x - 2xe^x = 0$

31. $xe^{-x} + 2e^{-x} = 0$

In Exercises 32 and 33, rewrite the given equation as a quadratic equation in u, where $u = e^x$; then solve for x.

32. $e^{2x} - e^x = 6$

33. $e^{-2x} - 3e^{-x} = -2$

In Exercises 34–36, sketch the graph of the equation without using a graphing utility.

34. (a) $y = 1 + \ln(x-2)$ (b) $y = 3 + e^{x-2}$

35. (a) $y = \left(\frac{1}{2}\right)^{x-1} - 1$ (b) $y = \ln|x|$

36. (a) $y = 1 - e^{-x+1}$ (b) $y = 3\ln\sqrt[3]{x-1}$

37. Use a calculating utility and the change of base formula (9) to find the values of $\log_2 7.35$ and $\log_5 0.6$, rounded to four decimal places.

In Exercises 38 and 39, graph the functions on the same screen of a graphing utility. [Use the change of base formula (9), where needed.]

38. $y = \ln x,\ y = e^x,\ \log x,\ 10^x$

39. $y = \log_2 x,\ \ln x,\ \log_5 x,\ \log x$

40. (a) Derive the general change of base formula

$$\log_b x = \frac{\log_a x}{\log_a b}$$

(b) Use the result in part (a) to find the exact value of $(\log_2 81)(\log_3 32)$ without using a calculating utility.

41. Use a graphing utility to estimate the two points of intersection of the graphs of $y = x^{0.2}$ and $y = \ln x$.

42. The United States public debt D, in billions of dollars, has been modeled as $D = 0.051517(1.1306727)^x$, where x is the number of years since 1900. Based on this model, when did the debt first reach one trillion dollars?

43. (a) Is the curve in the accompanying figure the graph of an exponential function? Explain your reasoning.
(b) Find the equation of an exponential function that passes through the point $(4, 2)$.
(c) Find the equation of an exponential function that passes through the point $\left(2, \frac{1}{4}\right)$.
(d) Use a graphing utility to generate the graph of an exponential function that passes through the point $(2, 5)$.

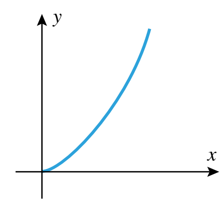


Figure Ex-43

44. (a) Make a conjecture about the general shape of the graph of $y = \log(\log x)$, and sketch the graph of this equation and $y = \log x$ in the same coordinate system.
(b) Check your work in part (a) with a graphing utility.

45. Find the fallacy in the following "proof" that $\frac{1}{8} > \frac{1}{4}$. Multiply both sides of the inequality $3 > 2$ by $\log\frac{1}{2}$ to get

$$\begin{aligned} 3\log\tfrac{1}{2} &> 2\log\tfrac{1}{2} \\ \log\left(\tfrac{1}{2}\right)^3 &> \log\left(\tfrac{1}{2}\right)^2 \\ \log\tfrac{1}{8} &> \log\tfrac{1}{4} \\ \tfrac{1}{8} &> \tfrac{1}{4} \end{aligned}$$

46. Prove the four algebraic properties of logarithms in Theorem 7.2.3.

47. If equipment in the satellite of Example 2 requires 15 watts to operate correctly, what is the operational lifetime of the power supply?

48. The equation $Q = 12e^{-0.055t}$ gives the mass Q in grams of radioactive potassium-42 that will remain from some initial quantity after t hours of radioactive decay.
(a) How many grams were there initially?
(b) How many grams remain after 4 hours?
(c) How long will it take to reduce the amount of radioactive potassium-42 to half of the initial amount?

49. The acidity of a substance is measured by its pH value, which is defined by the formula

$$\text{pH} = -\log[H^+]$$

where the symbol $[H^+]$ denotes the concentration of hydrogen ions measured in moles per liter. Distilled water has a pH of 7; a substance is called *acidic* if it has pH < 7 and *basic* if it has pH > 7. Find the pH of each of the following substances and state whether it is acidic or basic.

	SUBSTANCE	$[H^+]$
(a)	Arterial blood	3.9×10^{-8} mol/L
(b)	Tomatoes	6.3×10^{-5} mol/L
(c)	Milk	4.0×10^{-7} mol/L
(d)	Coffee	1.2×10^{-6} mol/L

50. Use the definition of pH in Exercise 49 to find $[H^+]$ in a solution having a pH equal to
(a) 2.44 (b) 8.06

51. The perceived loudness β of a sound in decibels (dB) is related to its intensity I in watts/square meter (W/m^2) by the equation

$$\beta = 10\log(I/I_0)$$

where $I_0 = 10^{-12}$ W/m^2. Damage to the average ear occurs at 90 dB or greater. Find the decibel level of each of the following sounds and state whether it will cause ear damage.

	SOUND	I
(a)	Jet aircraft (from 500 ft)	1.0×10^2 W/m^2
(b)	Amplified rock music	1.0 W/m^2
(c)	Garbage disposal	1.0×10^{-4} W/m^2
(d)	TV (mid volume from 10 ft)	3.2×10^{-5} W/m^2

In Exercises 52–54, use the definition of the decibel level of a sound (see Exercise 51).

52. If one sound is three times as intense as another, how much greater is its decibel level?

53. According to one source, the noise inside a moving automobile is about 70 dB, whereas an electric blender generates 93 dB. Find the ratio of the intensity of the noise of the blender to that of the automobile.

54. Suppose that the decibel level of an echo is $\frac{2}{3}$ the decibel level of the original sound. If each echo results in another echo, how many echoes will be heard from a 120-dB sound given that the average human ear can hear a sound as low as 10 dB?

55. On the ***Richter scale***, the magnitude M of an earthquake is related to the released energy E in joules (J) by the equation

$$\log E = 4.4 + 1.5M$$

(a) Find the energy E of the 1906 San Francisco earthquake that registered $M = 8.2$ on the Richter scale.
(b) If the released energy of one earthquake is 10 times that of another, how much greater is its magnitude on the Richter scale?

56. Suppose that the magnitudes of two earthquakes differ by 1 on the Richter scale. Find the ratio of the released energy of the larger earthquake to that of the smaller earthquake. [*Note:* See Exercise 55 for terminology.]

In Exercises 57 and 58, use Formula (3) or (5), as appropriate, to find the limit.

57. Find $\lim\limits_{x \to 0} (1 - 2x)^{1/x}$. [*Hint:* Let $t = -2x$.]

58. Find $\lim\limits_{x \to +\infty} (1 + 3/x)^x$. [*Hint:* Let $t = 3/x$.]

7.3 DERIVATIVES AND INTEGRALS INVOLVING LOGARITHMIC AND EXPONENTIAL FUNCTIONS

In this section we will obtain derivative formulas for logarithmic and exponential functions, and we will discuss the general relationship between the derivative of a one-to-one function and its inverse function.

DERIVATIVES OF LOGARITHMIC FUNCTIONS

The natural logarithm plays a special role in calculus that can be motivated by differentiating $\log_b x$, where b is an arbitrary base. For this purpose, recall that $\log_b x$ is continuous for $x > 0$. We will also need the limit

$$\lim_{v \to 0} (1 + v)^{1/v} = e$$

that was given in Formula (5) of Section 7.2 (with x rather than v as the variable).

Using the definition of a derivative, we obtain

$$
\begin{aligned}
\frac{d}{dx}[\log_b x] &= \lim_{w \to x} \frac{\log_b w - \log_b x}{w - x} \\
&= \lim_{w \to x} \left[\frac{1}{w-x} \log_b \left(\frac{w}{x}\right)\right] && \text{The quotient property of logarithms in Theorem 7.2.3} \\
&= \lim_{w \to x} \left[\frac{1}{w-x} \log_b \left(\frac{x + (w-x)}{x}\right)\right] \\
&= \lim_{w \to x} \left[\frac{1}{w-x} \log_b \left(1 + \frac{w-x}{x}\right)\right] \\
&= \lim_{w \to x} \left[\frac{1}{x}\frac{x}{w-x} \log_b \left(1 + \frac{w-x}{x}\right)\right] \\
&= \lim_{v \to 0} \left[\frac{1}{x}\frac{1}{v} \log_b (1+v)\right] && \text{Let } v = x/(w-x) \text{ and note that } v \to 0 \text{ if and only if } w \to x. \\
&= \frac{1}{x} \lim_{v \to 0} \left[\frac{1}{v} \log_b (1+v)\right] && x \text{ is fixed for this limit computation, so } 1/x \text{ can be moved through the limit sign.} \\
&= \frac{1}{x} \lim_{v \to 0} \left[\log_b (1+v)^{1/v}\right] && \text{The power property of logarithms in Theorem 7.2.3} \\
&= \frac{1}{x} \log_b \left[\lim_{v \to 0} (1+v)^{1/v}\right] && \log_b x \text{ is continuous on } (0, +\infty)\text{, so we can move the limit through the function symbol.} \\
&= \frac{1}{x} \log_b e && \text{Formula (5) of Section 7.2}
\end{aligned}
$$

Thus,

$$\frac{d}{dx}[\log_b x] = \frac{1}{x} \log_b e, \qquad x > 0$$

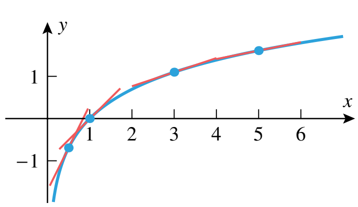
But from Formula (9) of Section 7.2 we have that $\log_b e = 1/\ln b$, so we can rewrite this derivative formula as

$$\frac{d}{dx}[\log_b x] = \frac{1}{x \ln b}, \qquad x > 0 \tag{1}$$

In the special case where $b = e$, we have that $\ln e = 1$, so this formula becomes

$$\frac{d}{dx}[\ln x] = \frac{1}{x}, \qquad x > 0 \tag{2}$$

Thus, among all possible bases, the base $b = e$ produces the simplest formula for the derivative of $\log_b x$. This is one of the reasons why the natural logarithm function is preferred over other logarithms in calculus.



$y = \ln x$ with tangent lines

Figure 7.3.1

Example 1

(a) Figure 7.3.1 shows the graph of $y = \ln x$ and its tangent lines at the points $x = \frac{1}{2}$, 1, 3, and 5. Find the slopes of those tangent lines.

(b) Does the graph of $y = \ln x$ have any horizontal tangent lines? Use the derivative of $\ln x$ to justify your answer.

Solution (a). From (2), the slopes of the tangent lines at the points $x = \frac{1}{2}$, 1, 3, and 5 are $1/x = 2, 1, \frac{1}{3}$, and $\frac{1}{5}$, which is consistent with Figure 7.3.1.

Solution (b). From the graph of $y = \ln x$, it does not appear that there are any horizontal tangent lines. This is confirmed by the fact that $dy/dx = 1/x$ is not equal to zero for any real value of x. ◀

If u is a differentiable function of x, and if $u(x) > 0$, then applying the chain rule to (1) and (2) produces the following generalized derivative formulas:

$$\frac{d}{dx}[\log_b u] = \frac{1}{u \ln b} \cdot \frac{du}{dx} \quad \text{and} \quad \frac{d}{dx}[\ln u] = \frac{1}{u} \cdot \frac{du}{dx} \tag{3–4}$$

Example 2 Find $\dfrac{d}{dx}[\ln(x^2+1)]$.

Solution. From (4) with $u = x^2 + 1$,

$$\frac{d}{dx}[\ln(x^2+1)] = \frac{1}{x^2+1} \cdot \frac{d}{dx}[x^2+1] = \frac{1}{x^2+1} \cdot 2x = \frac{2x}{x^2+1}$$ ◀

When possible, the properties of logarithms in Theorem 7.2.3 should be used to convert products, quotients, and exponents into sums, differences, and constant multiples *before* differentiating a function involving logarithms.

Example 3

$$\begin{aligned}\frac{d}{dx}\left[\ln\left(\frac{x^2 \sin x}{\sqrt{1+x}}\right)\right] &= \frac{d}{dx}\left[2\ln x + \ln(\sin x) - \frac{1}{2}\ln(1+x)\right] \\ &= \frac{2}{x} + \frac{\cos x}{\sin x} - \frac{1}{2(1+x)} \\ &= \frac{2}{x} + \cot x - \frac{1}{2+2x}\end{aligned}$$ ◀

Example 4 Find $\dfrac{d}{dx}[\ln|x|]$.

Solution. The function $\ln|x|$ is defined for all x, except $x = 0$; we will consider the cases $x > 0$ and $x < 0$ separately.

If $x > 0$, then $|x| = x$, so

$$\frac{d}{dx}[\ln|x|] = \frac{d}{dx}[\ln x] = \frac{1}{x}$$

If $x < 0$, then $|x| = -x$, so from (4) we have

$$\frac{d}{dx}[\ln|x|] = \frac{d}{dx}[\ln(-x)] = \frac{1}{(-x)} \cdot \frac{d}{dx}[-x] = \frac{1}{x}$$

Since the same formula results in both cases, we have shown that

$$\frac{d}{dx}[\ln|x|] = \frac{1}{x} \quad \text{if } x \neq 0 \tag{5}$$

Example 5 From (5) and the chain rule,

$$\frac{d}{dx}[\ln|\sin x|] = \frac{1}{\sin x} \cdot \frac{d}{dx}[\sin x] = \frac{\cos x}{\sin x} = \cot x$$ ◀

LOGARITHMIC DIFFERENTIATION

We now consider a technique called ***logarithmic differentiation*** that is useful for differentiating functions that are composed of products, quotients, and powers.

Example 6 The derivative of

$$y = \frac{x^2\sqrt[3]{7x-14}}{(1+x^2)^4} \tag{6}$$

is messy to calculate directly. However, if we first take the natural logarithm of both sides and then use its properties, we can write

$$\ln y = 2\ln x + \tfrac{1}{3}\ln(7x-14) - 4\ln(1+x^2)$$

Differentiating both sides with respect to x yields

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x} + \frac{7/3}{7x-14} - \frac{8x}{1+x^2} \tag{7}$$

Thus, on solving for dy/dx and using (6) we obtain

$$\frac{dy}{dx} = \frac{x^2\sqrt[3]{7x-14}}{(1+x^2)^4}\left[\frac{2}{x} + \frac{1}{3x-6} - \frac{8x}{1+x^2}\right] \tag{8}$$

REMARK. Since $\ln y$ is defined only for $y > 0$, logarithmic differentiation of $y = f(x)$ is valid only on intervals where $f(x)$ is positive. Thus, the derivative obtained in the preceding example is valid on the interval $(2, +\infty)$, since the given function is positive for $x > 2$. However, the formula is actually valid on the interval $(-\infty, 2)$ as well. This can be seen by taking absolute values before proceeding with the logarithmic differentiation and noting that $\ln|y|$ is defined for all y except $y = 0$. If we do this and simplify using properties of logarithms and absolute values, we obtain

$$\ln|y| = 2\ln|x| + \tfrac{1}{3}\ln|7x-14| - 4\ln|1+x^2|$$

Differentiating both sides with respect to x yields (7), and hence results in (8).

In general, if the derivative of $y = f(x)$ is to be obtained by logarithmic differentiation, then the same formula for dy/dx will result regardless of whether one first takes absolute values or not. Thus, a derivative formula obtained by logarithmic differentiation will be valid except perhaps at points where $f(x)$ is zero. The formula may, in fact, be valid at those points as well, but it is not guaranteed.

INTEGRALS INVOLVING ln x

Formula (2) states that the function $\ln x$ is an antiderivative of $1/x$ on the interval $(0, +\infty)$, whereas Formula (5) states that the function $\ln|x|$ is an antiderivative of $1/x$ on each of the intervals $(-\infty, 0)$ and $(0, +\infty)$. Thus we have the companion integration formula to (5),

$$\int \frac{1}{u}\,du = \ln|u| + C \tag{9}$$

with implicit understanding that the formula is applicable only across an interval that does not contain 0.

Example 7 Applying Formula (9),

$$\int_1^e \frac{1}{x}\,dx = \ln|x|\Big]_1^e = \ln|e| - \ln|1| = 1 - 0 = 1$$

$$\int_{-e}^{-1} \frac{1}{x}\,dx = \ln|x|\Big]_{-e}^{-1} = \ln|-1| - \ln|-e| = 0 - 1 = -1$$ ◀

Example 8 Evaluate $\displaystyle\int \frac{3x^2}{x^3+5}\,dx$.

Solution. Make the substitution

$$u = x^3 + 5, \quad du = 3x^2\,dx$$

so that

$$\int \frac{3x^2}{x^3+5}\,dx = \underbrace{\int \frac{1}{u}\,du}_{\text{Formula (9)}} = \ln|u| + C = \ln|x^3+5| + C$$

Example 9 Evaluate $\int \tan x\,dx$.

Solution.

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = \underbrace{-\int \frac{1}{u}\,du}_{\substack{u=\cos x\\ du=-\sin x\,dx}} = -\ln|u| + C = -\ln|\cos x| + C$$

REMARK. The last two examples illustrate an important point: Any integral of the form

$$\int \frac{g'(x)}{g(x)}\,dx$$

(where the numerator of the integrand is the derivative of the denominator) can be evaluated by the u-substitution $u = g(x)$, $du = g'(x)\,dx$, since this substitution yields

$$\int \frac{g'(x)}{g(x)}\,dx = \int \frac{du}{u} = \ln|u| + C = \ln|g(x)| + C$$

DERIVATIVES OF IRRATIONAL POWERS OF x

We know from Formula (15) of Section 3.6 that the differentiation formula

$$\frac{d}{dx}[x^r] = rx^{r-1} \tag{10}$$

holds for rational values of r. We will now use logarithmic differentiation to show that this formula holds if r is *any* real number (rational or irrational). In our computations we will assume that x^r is a differentiable function and that the familiar laws of exponents hold for real exponents.

Let $y = x^r$, where r is a real number. The derivative dy/dx can be obtained by logarithmic differentiation as follows:

$$\ln|y| = \ln|x^r| = r\ln|x|$$

$$\frac{d}{dx}[\ln|y|] = \frac{d}{dx}[r\ln|x|]$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{r}{x}$$

$$\frac{dy}{dx} = \frac{r}{x}y = \frac{r}{x}x^r = rx^{r-1}$$

This establishes (10) for real values of r. Thus, for example,

$$\frac{d}{dx}[x^\pi] = \pi x^{\pi-1} \quad\text{and}\quad \frac{d}{dx}\left[x^{\sqrt{2}}\right] = \sqrt{2}x^{\sqrt{2}-1} \tag{11}$$

Note that Formula (10) justifies the integration formula

$$\int x^r\,dx = \left[\frac{x^{r+1}}{r+1}\right] + C \quad (r \neq -1)$$

(Table 5.2.1) for *any* real number r other than -1.

DERIVATIVES OF EXPONENTIAL FUNCTIONS

By (1) we know that

$$\frac{d}{dx}[\log_b x]$$

is a nonzero function, so Theorem 7.1.6 establishes that the inverse function for $\log_b x$ is differentiable on $(-\infty, +\infty)$.

To obtain a derivative formula for the exponential function with base b, we rewrite $y = b^x$ as

$$x = \log_b y$$

and differentiate implicitly using (3) to obtain

$$1 = \frac{1}{y \ln b} \cdot \frac{dy}{dx}$$

Solving for dy/dx and replacing y by b^x we have

$$\frac{dy}{dx} = y \ln b = b^x \ln b$$

Thus, we have shown that

$$\frac{d}{dx}[b^x] = b^x \ln b \tag{12}$$

In the special case where $b = e$ we have $\ln e = 1$, so that (12) becomes

$$\frac{d}{dx}[e^x] = e^x \tag{13}$$

Moreover, if u is a differentiable function of x, then it follows from (12) and (13) that

$$\frac{d}{dx}[b^u] = b^u \ln b \cdot \frac{du}{dx} \quad \text{and} \quad \frac{d}{dx}[e^u] = e^u \cdot \frac{du}{dx} \tag{14–15}$$

REMARK. It is important to distinguish between differentiating the exponential function b^x (variable exponent and constant base) and the power function x^b (variable base and constant exponent). For example, compare the derivative of x^π in (11) to the following derivative of π^x, which is obtained from (12):

$$\frac{d}{dx}[\pi^x] = \pi^x \ln \pi$$

Example 10 The following computations use Formulas (14) and (15).

$$\frac{d}{dx}[2^{\sin x}] = (2^{\sin x})(\ln 2) \cdot \frac{d}{dx}[\sin x] = (2^{\sin x})(\ln 2)(\cos x)$$

$$\frac{d}{dx}[e^{-2x}] = e^{-2x} \cdot \frac{d}{dx}[-2x] = -2e^{-2x}$$

$$\frac{d}{dx}\left[e^{x^3}\right] = e^{x^3} \cdot \frac{d}{dx}[x^3] = 3x^2 e^{x^3}$$

$$\frac{d}{dx}[e^{\cos x}] = e^{\cos x} \cdot \frac{d}{dx}[\cos x] = -(\sin x)e^{\cos x}$$ ◀

The rules

$$\frac{d}{dx}(u^n) = n \cdot u^{n-1}\frac{du}{dx} \quad \text{if } n \text{ is a real number}$$

$$\frac{d}{dx}(b^u) = b^u \ln b \cdot \frac{du}{dx} \quad \text{if } b > 0,\ b \neq 1$$

deal with derivatives of exponential expressions in which either the base or the exponent of the expression is a number. The following example illustrates the application of logarithmic differentiation for finding dy/dx when y is an expression of the form $y = u^v$ where *both* u and v are nonconstant functions of x.

Example 11 Use logarithmic differentiation to find $\frac{d}{dx}[(x^2+1)^{\sin x}]$.

Solution. Setting $y = (x^2+1)^{\sin x}$ we have

$$\ln y = \ln[(x^2+1)^{\sin x}] = (\sin x)\ln(x^2+1)$$

Then

$$\frac{d}{dx}(\ln y) = \frac{1}{y}\cdot\frac{dy}{dx}$$
$$= \frac{d}{dx}[(\sin x)\ln(x^2+1)] = (\sin x)\frac{1}{x^2+1}(2x) + (\cos x)\ln(x^2+1)$$

Thus,

$$\frac{dy}{dx} = y\left[\frac{2x\sin x}{x^2+1} + (\cos x)\ln(x^2+1)\right]$$
$$= (x^2+1)^{\sin x}\left[\frac{2x\sin x}{x^2+1} + (\cos x)\ln(x^2+1)\right] \quad \blacktriangleleft$$

INTEGRALS INVOLVING EXPONENTIAL FUNCTIONS

Associated with derivatives (14) and (15) are the companion integration formulas

$$\int b^u\,du = \frac{b^u}{\ln b} + C \quad \text{and} \quad \int e^u\,du = e^u + C \tag{16–17}$$

Example 12

$$\int 2^x\,dx = \frac{2^x}{\ln 2} + C \quad \blacktriangleleft$$

Example 13 Evaluate $\int e^{5x}\,dx$.

Solution. Let $u = 5x$ so that $du = 5\,dx$ or $dx = \frac{1}{5}\,du$, which yields

$$\int e^{5x}\,dx = \frac{1}{5}\int e^u\,du = \frac{1}{5}e^u + C = \frac{1}{5}e^{5x} + C \quad \blacktriangleleft$$

Example 14

$$\int e^{-x}\,dx = -\int e^u\,du = -e^u + C = -e^{-x} + C$$

$$\boxed{\begin{array}{l} u = -x \\ du = -dx \end{array}}$$

$$\int x^2 e^{x^3}\,dx = \frac{1}{3}\int e^u\,du = \frac{1}{3}e^u + C = \frac{1}{3}e^{x^3} + C$$

$$\boxed{\begin{array}{l} u = x^3 \\ du = 3x^2\,dx \end{array}}$$

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int e^u\,du = 2e^u + C = 2e^{\sqrt{x}} + C \quad \blacktriangleleft$$

$$\boxed{\begin{array}{l} u = \sqrt{x} \\ du = \frac{1}{2\sqrt{x}}\,dx \end{array}}$$

Example 15 Evaluate $\int_0^{\ln 3} e^x(1+e^x)^{1/2}\,dx$.

Solution. Make the u-substitution

$$u = 1 + e^x, \quad du = e^x\,dx$$

and change the x-limits of integration ($x = 0$, $x = \ln 3$) to u-limits ($u = 1 + e^0 = 2$, $u = 1 + e^{\ln 3} = 1 + 3 = 4$).

$$\int_0^{\ln 3} e^x(1+e^x)^{1/2}\,dx = \int_2^4 u^{1/2}\,du = \frac{2}{3}u^{3/2}\Big]_2^4 = \frac{2}{3}[4^{3/2} - 2^{3/2}] = \frac{16 - 4\sqrt{2}}{3} \quad ◀$$

EXERCISE SET 7.3 Graphing Utility

In Exercises 1–30, find dy/dx.

1. $y = \ln 2x$
2. $y = \ln(x^3)$
3. $y = (\ln x)^2$
4. $y = \ln(\sin x)$
5. $y = \ln|\tan x|$
6. $y = \ln(2 + \sqrt{x})$
7. $y = \ln\left(\dfrac{x}{1+x^2}\right)$
8. $y = \ln(\ln x)$
9. $y = \ln|x^3 - 7x^2 - 3|$
10. $y = x^3 \ln x$
11. $y = \sqrt{\ln x}$
12. $y = \sqrt{1 + \ln^2 x}$
13. $y = \cos(\ln x)$
14. $y = \sin^2(\ln x)$
15. $y = x^3 \log_2(3 - 2x)$
16. $y = x\left[\log_2(x^2 - 2x)\right]^3$
17. $y = \dfrac{x^2}{1 + \log x}$
18. $y = \dfrac{\log x}{1 + \log x}$
19. $y = e^{7x}$
20. $y = e^{-5x^2}$
21. $y = x^3 e^x$
22. $y = e^{1/x}$
23. $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$
24. $y = \sin(e^x)$
25. $y = e^{x \tan x}$
26. $y = \dfrac{e^x}{\ln x}$
27. $y = e^{(x - e^{3x})}$
28. $y = \exp(\sqrt{1 + 5x^3})$
29. $y = \ln(1 - xe^{-x})$
30. $y = \ln(\cos e^x)$

In Exercises 31 and 32, find dy/dx by implicit differentiation.

31. $y + \ln xy = 1$
32. $y = \ln(x \tan y)$

In Exercises 33 and 34, use the method of Example 3 to help perform the indicated differentiation.

33. $\dfrac{d}{dx}\left[\ln \dfrac{\cos x}{\sqrt{4 - 3x^2}}\right]$
34. $\dfrac{d}{dx}\left[\ln\sqrt{\dfrac{x-1}{x+1}}\right]$

In Exercises 35–38, find dy/dx using the method of logarithmic differentiation.

35. $y = x\sqrt[3]{1 + x^2}$
36. $y = \sqrt[5]{\dfrac{x-1}{x+1}}$
37. $y = \dfrac{(x^2 - 8)^{1/3}\sqrt{x^3 + 1}}{x^6 - 7x + 5}$
38. $y = \dfrac{\sin x \cos x \tan^3 x}{\sqrt{x}}$

In Exercises 39–42, find $f'(x)$ by Formula (14) and then by logarithmic differentiation.

39. $f(x) = 2^x$
40. $f(x) = 3^{-x}$
41. $f(x) = \pi^{\sin x}$
42. $f(x) = \pi^{x \tan x}$

In Exercises 43–46, find dy/dx using the method of logarithmic differentiation.

43. $y = \left(x^3 - 2x\right)^{\ln x}$
44. $y = x^{\sin x}$
45. $y = (\ln x)^{\tan x}$
46. $y = \left(x^2 + 3\right)^{\ln x}$

47. Find $f'(x)$ if $f(x) = x^e$.
48. (a) Explain why Formula (12) cannot be used to find $(d/dx)[x^x]$.
    (b) Find this derivative by logarithmic differentiation.
49. Find
    (a) $\dfrac{d}{dx}[\log_x e]$ (b) $\dfrac{d}{dx}[\log_x 2]$.
50. Use Part 2 of the Fundamental Theorem of Calculus (5.6.3) to find the derivative.
    (a) $\dfrac{d}{dx}\displaystyle\int_0^x e^{t^2}\,dt$ (b) $\dfrac{d}{dx}\displaystyle\int_1^x \ln t\,dt$
51. Let $f(x) = e^{kx}$ and $g(x) = e^{-kx}$. Find
    (a) $f^{(n)}(x)$ (b) $g^{(n)}(x)$.
52. Find dy/dt if $y = e^{-\lambda t}(A \sin \omega t + B \cos \omega t)$, where A, B, λ, and ω are constants.
53. Find $f'(x)$ if
    $$f(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$$
    where μ and σ are constants and $\sigma \neq 0$.
54. Show that for any constants A and k, the function $y = Ae^{kt}$ satisfies the equation $dy/dt = ky$.
55. Show that for any constants A and B, the function
    $$y = Ae^{2x} + Be^{-4x}$$
    satisfies the equation
    $$y'' + 2y' - 8y = 0$$
56. Show that
    (a) $y = xe^{-x}$ satisfies the equation $xy' = (1 - x)y$
    (b) $y = xe^{-x^2/2}$ satisfies the equation $xy' = (1 - x^2)y$.

In Exercises 57 and 58, find the limit by interpreting the expression as an appropriate derivative.

57. (a) $\displaystyle\lim_{w \to 1} \frac{\ln w}{w - 1}$ (b) $\displaystyle\lim_{w \to 0} \frac{10^w - 1}{w}$

58. (a) $\displaystyle\lim_{\Delta x \to 0} \frac{\ln(e^2 + \Delta x) - 2}{\Delta x}$ (b) $\displaystyle\lim_{w \to 1} \frac{2^w - 2}{w - 1}$

In Exercises 59 and 60, evaluate the integral, and check your answer by differentiating.

59. $\displaystyle\int \left[\frac{2}{x} + 3e^x\right] dx$ **60.** $\displaystyle\int \left[\frac{1}{2t} - \sqrt{2}e^t\right] dt$

In Exercises 61 and 62, evaluate the integrals by making the indicated substitutions.

61. (a) $\displaystyle\int \frac{dx}{x \ln x}$; $u = \ln x$ (b) $\displaystyle\int e^{-5x}\, dx$; $u = -5x$

62. (a) $\displaystyle\int \frac{\sin 3\theta}{1 + \cos 3\theta}\, d\theta$; $u = 1 + \cos 3\theta$

(b) $\displaystyle\int \frac{e^x}{1 + e^x}\, dx$; $u = 1 + e^x$

In Exercises 63–72, evaluate the integrals by making appropriate substitutions.

63. $\displaystyle\int e^{2x}\, dx$ **64.** $\displaystyle\int \frac{dx}{2x}$

65. $\displaystyle\int e^{\sin x} \cos x\, dx$ **66.** $\displaystyle\int x^3 e^{x^4}\, dx$

67. $\displaystyle\int x^2 e^{-2x^3}\, dx$ **68.** $\displaystyle\int \frac{e^x + e^{-x}}{e^x - e^{-x}}\, dx$

69. $\displaystyle\int \frac{dx}{e^x}$ **70.** $\displaystyle\int \sqrt{e^x}\, dx$

71. $\displaystyle\int \frac{e^{\sqrt{y+1}}}{\sqrt{y+1}}\, dy$ **72.** $\displaystyle\int \frac{dy}{\sqrt{y}e^{\sqrt{y}}}$

In Exercises 73–76, evaluate each integral by first modifying the form of the integrand and then making an appropriate substitution.

73. $\displaystyle\int \frac{t+1}{t}\, dt$ **74.** $\displaystyle\int e^{2\ln x}\, dx$

75. $\displaystyle\int [\ln(e^x) + \ln(e^{-x})]\, dx$ **76.** $\displaystyle\int \cot x\, dx$

In Exercises 77 and 78, evaluate the integrals using Part 1 of the Fundamental Theorem of Calculus (5.6.1).

77. $\displaystyle\int_{\ln 2}^{3} 5e^x\, dx$ **78.** $\displaystyle\int_{1/2}^{1} \frac{1}{2x}\, dx$

79. Evaluate the definite integrals by making the indicated u-substitutions.

(a) $\displaystyle\int_0^1 e^{2x-1}\, dx$; $u = 2x - 1$

(b) $\displaystyle\int_e^{e^2} \frac{\ln x}{x}\, dx$; $u = \ln x$

80. Evaluate the definite integral by making the indicated u-substitution and then applying a formula from geometry.

$$\int_{e^{-6}}^{e^6} \frac{\sqrt{36 - (\ln x)^2}}{x}\, dx; \quad u = \ln x$$

In Exercises 81 and 82, evaluate the definite integral two ways: first by a u-substitution in the definite integral and then by a u-substitution in the corresponding indefinite integral.

81. $\displaystyle\int_{-\ln 3}^{\ln 3} \frac{e^x}{e^x + 4}\, dx$ **82.** $\displaystyle\int_0^{\ln 5} e^x(3 - 4e^x)\, dx$

In Exercises 83–86, evaluate the definite integrals by any method.

83. $\displaystyle\int_0^e \frac{dx}{x + e}$ **84.** $\displaystyle\int_1^{\sqrt{2}} xe^{-x^2}\, dx$

85. $\displaystyle\int_0^{\ln 2} e^{-3x}\, dx$ **86.** $\displaystyle\int_{-1}^{1} |e^x - 1|\, dx$

87. (a) Graph some representative integral curves of the function $f(x) = e^x/2$.

(b) Find an equation for the integral curve that passes through the point $(0, 1)$.

88. Use a graphing utility to generate some typical integral curves of $f(x) = x/(x^2 + 1)$ over the interval $(-5, 5)$.

89. Solve the initial-value problems.

(a) $\dfrac{dy}{dt} = 2e^{-t}$, $y(1) = 3 - \dfrac{2}{e}$

(b) $\dfrac{dy}{dt} = \dfrac{1}{t}$, $y(-1) = 5$

7.4 GRAPHS AND APPLICATIONS INVOLVING LOGARITHMIC AND EXPONENTIAL FUNCTIONS

In this section we will apply the techniques developed in Chapter 4 to graphing functions involving logarithmic or exponential functions. We will also look at applications of differentiation and integration in some contexts that entail logarithmic or exponential functions.

SOME PROPERTIES OF e^x AND $\ln x$

In Section 7.2 we presented computer-generated graphs of $y = e^x$ and $y = \ln x$ (Figure 7.2.4). For reference, these curves are shown in Figure 7.4.1. Since $f(x) = e^x$ and $g(x) = \ln x$ are inverses, their graphs are reflections of one another about the line $y = x$. These graphs suggest that e^x and $\ln x$ have the properties listed in Table 7.4.1.

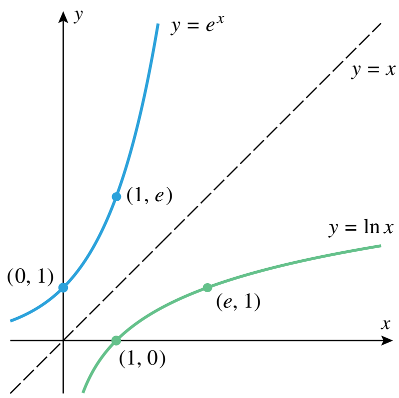


Figure 7.4.1

Table 7.4.1

PROPERTIES OF e^x	PROPERTIES OF $\ln x$
$e^x > 0$ for all x	$\ln x > 0$ if $x > 1$ $\ln x < 0$ if $0 < x < 1$ $\ln x = 0$ if $x = 1$
e^x is increasing on $(-\infty, +\infty)$	$\ln x$ is increasing on $(0, +\infty)$
The graph of e^x is concave up on $(-\infty, +\infty)$	The graph of $\ln x$ is concave down on $(0, +\infty)$

We can verify that $y = e^x$ is increasing and its graph is concave up from its first and second derivatives. For all x in $(-\infty, +\infty)$ we have

$$\frac{d}{dx}[e^x] = e^x > 0 \quad \text{and} \quad \frac{d^2}{dx^2}[e^x] = \frac{d}{dx}[e^x] = e^x > 0$$

The first of these inequalities demonstrates that e^x is increasing on $(-\infty, +\infty)$, and the second inequality shows that the graph of e^x is concave up on $(-\infty, +\infty)$.

Similarly, for all x in $(0, +\infty)$ we have

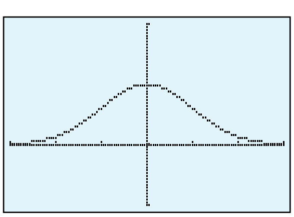
$$\frac{d}{dx}[\ln x] = \frac{1}{x} > 0 \quad \text{and} \quad \frac{d^2}{dx^2}[\ln x] = \frac{d}{dx}\left[\frac{1}{x}\right] = -\frac{1}{x^2} < 0$$

The first of these inequalities demonstrates that $\ln x$ is increasing on $(0, +\infty)$, and the second inequality shows that the graph of $\ln x$ is concave down on $(0, +\infty)$.

GRAPHING EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Example 1 Generate or sketch a graph of $y = e^{-x^2/2}$ and identify the exact locations of all relative extrema and inflection points.

Solution. Figure 7.4.2 shows a calculator-generated graph of $y = e^{-x^2/2}$ in the window $[-3, 3] \times [-1, 2]$. This figure suggests that the graph is symmetric about the y-axis and has a relative maximum at $x = 0$, a horizontal asymptote $y = 0$, and two inflection points. The following analysis will identify the exact locations of these features.



$[-3, 3] \times [-1, 2]$
xScl = 1, yScl = 1

Figure 7.4.2

- *Symmetries:* Replacing x by $-x$ does not change the equation, so the graph is symmetric about the y-axis.
- *x-intercepts:* Setting $y = 0$ leads to the equation $e^{-x^2/2} = 0$, which has no solutions since all powers of e have positive values. Thus, there are no x-intercepts.
- *y-intercepts:* Setting $x = 0$ yields the y-intercept $y = 1$.
- *Vertical asymptotes:* There are no vertical asymptotes since $e^{-x^2/2}$ is defined and continuous on $(-\infty, +\infty)$.
- *Horizontal asymptotes:* Since $-x^2/2 \to -\infty$ as $x \to -\infty$ or $x \to +\infty$, it follows from Formula (12) of Section 7.2 that

  $$\lim_{x \to -\infty} e^{-x^2/2} = \lim_{x \to +\infty} e^{-x^2/2} = 0$$

  Thus, $e^{-x^2/2}$ is asymptotic to $y = 0$ as $x \to -\infty$ and as $x \to +\infty$.

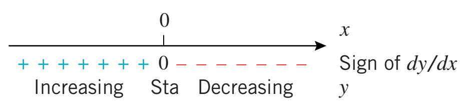
- *Derivatives:*

$$\frac{dy}{dx} = e^{-x^2/2}\frac{d}{dx}\left[-\frac{x^2}{2}\right] = -xe^{-x^2/2}$$

$$\frac{d^2y}{dx^2} = -x\frac{d}{dx}\left[e^{-x^2/2}\right] + e^{-x^2/2}\frac{d}{dx}[-x]$$

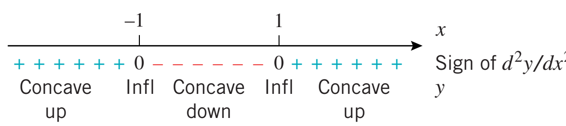
$$= x^2e^{-x^2/2} - e^{-x^2/2} = (x^2-1)e^{-x^2/2}$$

- *Intervals of increase and decrease; relative extrema:* Since $e^{-x^2/2} > 0$ for all x, the sign of $dy/dx = -xe^{-x^2/2}$ is the same as the sign of $-x$.



The analysis reveals a relative maximum $e^0 = 1$ at $x = 0$.

- *Concavity:* Since $e^{-x^2/2} > 0$ for all x, the sign of $d^2y/dx^2 = (x^2-1)e^{-x^2/2}$ is the same as the sign of $x^2 - 1$.



Thus, the inflection points occur at $x = -1$ and at $x = 1$. These inflection points are $(-1, e^{-1/2}) \approx (-1, 0.61)$ and $(1, e^{-1/2}) \approx (1, 0.61)$.

Our analysis confirms that the calculater-generated graph in Figure 7.4.2 reveals all of the essential features of the graph of $y = e^{-x^2/2}$. ◀

Example 2 Generate or sketch a graph of $y = \ln x/x$ and identify the exact locations of all relative extrema and inflection points.

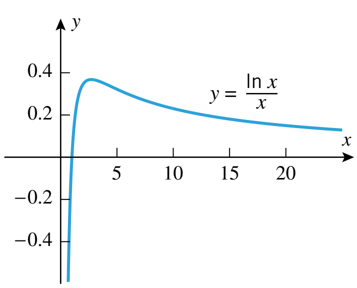


Figure 7.4.3

Solution. Note that since the domain of $\ln x/x$ is $(0, +\infty)$, the graph lies entirely to the right of the y-axis. Figure 7.4.3 shows a graph of $y = \ln x/x$ obtained with a graphing utility. This figure suggests that the graph and has one relative maximum, a horizontal asymptote $y = 0$, a vertical asymptote $x = 0$, and one inflection point. The following analysis will identify the exact locations of these features.

- *Symmetries:* None.
- *x-intercepts:* Setting $y = 0$ leads to the equation $y = \ln x/x = 0$, whose only solution occurs when $\ln x = 0$, or $x = 1$.
- *y-intercepts:* There are no y-intercepts since $\ln x$ is not defined at $x = 0$.
- *Vertical asymptotes:* Since

$$\lim_{x\to 0^+}\frac{1}{x} = +\infty \quad \text{and} \quad \lim_{x\to 0^+}\ln x = -\infty$$

it follows that values of

$$y = \frac{\ln x}{x} = \frac{1}{x}(\ln x)$$

will decrease without bound as $x \to 0^+$, so

$$\lim_{x\to 0^+}\frac{\ln x}{x} = -\infty$$

and the graph has a vertical asymptote $x = 0$.

- *Horizontal asymptotes:* Note that $\ln x/x > 0$ for $x > 1$. We will see below that $y = \ln x/x$ is decreasing for sufficiently large values of x, so $y = \ln x/x$ is decreasing and positive on $(1, \infty)$. We will develop a technique in Section 7.7 that will allow us to conclude that
$$\lim_{x \to +\infty} \frac{\ln x}{x} = 0$$
Thus, $\ln x/x$ is asymptotic to $y = 0$ as $x \to +\infty$.
- *Derivatives:*
$$\frac{dy}{dx} = \frac{x(1/x) - (\ln x)(1)}{x^2} = \frac{1 - \ln x}{x^2}$$
$$\frac{d^2y}{dx^2} = \frac{x^2(-1/x) - (1 - \ln x)(2x)}{x^4} = \frac{2x\ln x - 3x}{x^4} = \frac{2\ln x - 3}{x^3}$$
- *Intervals of increase and decrease; relative extrema:* Since $x^2 > 0$ for all $x > 0$, the sign of
$$\frac{dy}{dx} = \frac{1 - \ln x}{x^2}$$
is the same as the sign of $1 - \ln x$. But $\ln x$ is an increasing function with $\ln e = 1$, so $1 - \ln x$ is positive for $x < e$ and negative for $x > e$. We encapsulate this in the following diagram.

0 e x
$+\infty$ + + + + + + + 0 – – – – – – – Sign of dy/dx
Increasing Sta Decreasing y

The analysis reveals a relative maximum $(\ln e)/e = 1/e \approx 0.37$ at $x = e$.
- *Concavity:* Since $x^3 > 0$ for all $x > 0$, the sign of
$$\frac{d^2y}{dx^2} = \frac{2\ln x - 3}{x^3}$$
is the same as the sign of $2\ln x - 3$. Now, $2\ln x - 3 = 0$ when $\ln x = \frac{3}{2}$, or $x = e^{3/2}$. Again, since $\ln x$ is an increasing function, $2\ln x - 3$ is negative for $x < e^{3/2}$ and positive for $x > e^{3/2}$. We encapsulate this in the following diagram.

0 $e^{3/2}$ x
$-\infty$ – – – – – – – – – – – – – 0 + + + + + + Sign of d^2y/dx^2
Concave down Infl Concave up y

Thus, an inflection point occurs at $\left(e^{3/2}, \frac{3}{2}e^{3/2}\right) \approx (4.48, 0.33)$.

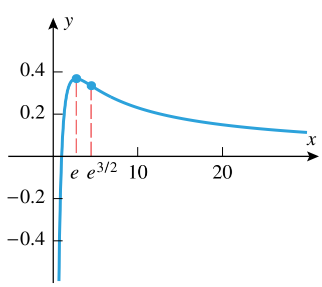


Figure 7.4.4

Figure 7.4.4 shows our earlier graph with the relative maximum and inflection point identified. ◀

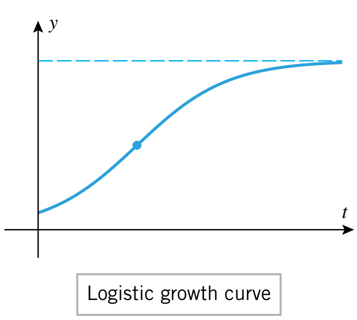


Figure 7.4.5

LOGISTIC CURVES

When a population grows in an environment in which space or food is limited, the graph of population versus time is typically an S-shaped curve of the form shown in Figure 7.4.5. The scenario described by this curve is a population that grows slowly at first and then more and more rapidly as the number of individuals producing offspring increases. However, at a certain point in time (where the inflection point occurs) the environmental factors begin to show their effect, and the growth rate begins a steady decline. Over an extended period of time the population approaches a limiting value that represents the upper limit on the number of individuals that the available space or food can sustain. Population growth curves of this type are called ***logistic growth curves***.

Example 3 We will show in a later chapter that logistic growth curves arise from equations of the form

$$y = \frac{L}{1 + Ae^{-kt}} \tag{1}$$

where y is the population at time t ($t \geq 0$) and A, k, and L are positive constants. Show that Figure 7.4.6 correctly describes the graph of this equation.

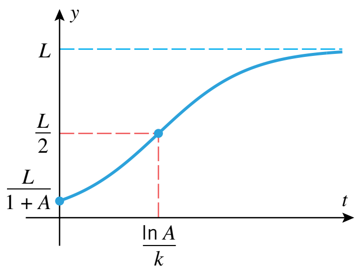


Figure 7.4.6

Solution. We leave it for you to confirm that at time $t = 0$ the value of y is

$$y = \frac{L}{1 + A}$$

and that for $t \geq 0$ the population y satisfies

$$\frac{L}{1 + A} \leq y < L$$

This is consistent with the graph in Figure 7.4.6. The horizontal asymptote at $y = L$ is confirmed by the limit

$$\lim_{t \to +\infty} \frac{L}{1 + Ae^{-kt}} = \frac{L}{1 + 0} = L$$

Physically, L represents the upper limit on the size of the population.

To investigate intervals of increase or decrease, concavity, and inflection points, we need the first and second derivatives of y with respect to t. We leave it for you to confirm that

$$\frac{dy}{dt} = \frac{k}{L}y(L - y) \tag{2}$$

$$\frac{d^2y}{dt^2} = \frac{k^2}{L^2}y(L - y)(L - 2y) \tag{3}$$

Since $k > 0$, $y > 0$, and $L - y > 0$, it follows from (2) that $dy/dt > 0$ for all t. Thus, y is always increasing and there are no stationary points, which is consistent with Figure 7.4.6.

Since $y > 0$ and $L - y > 0$, it follows from (3) that

$$\frac{d^2y}{dt^2} > 0 \quad \text{if} \quad L - 2y > 0$$

$$\frac{d^2y}{dt^2} < 0 \quad \text{if} \quad L - 2y < 0$$

Thus, the graph of y versus t is concave up if $y < L/2$, concave down if $y > L/2$, and has an inflection point where $y = L/2$, all of which is consistent with Figure 7.4.6.

Finally, we leave it as an exercise for you to confirm that the inflection point occurs at time

$$t = \frac{1}{k}\ln A = \frac{\ln A}{k} \tag{4}$$

by solving the equation

$$\frac{L}{2} = \frac{L}{1 + Ae^{-kt}}$$

for t. ◀

NEWTON'S LAW OF COOLING

Example 4 A glass of lemonade with a temperature of 40°F is left to sit in a room whose temperature is a constant 70°F. Using a principle of physics, called ***Newton's Law of Cooling***, one can show that if the temperature of the lemonade reaches 52°F in 1 hour, then the temperature T of the lemonade as a function of the elapsed time t is modeled by the equation

$$T = 70 - 30e^{-0.5t}$$

where T is in °F and t is in hours. The graph of this equation, shown in Figure 7.4.7, conforms

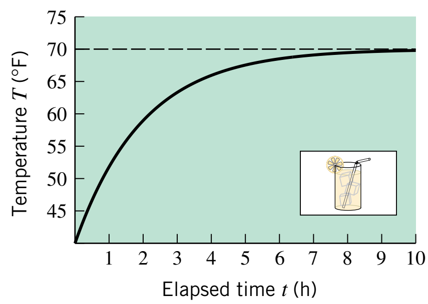


Figure 7.4.7

to our everyday experience that the temperature of the lemonade gradually approaches the temperature of the room.

(a) In words, what happens to the *rate* of temperature rise over time?

(b) Use a derivative to confirm your conclusion in part (a).

(c) Find the average temperature T_{ave} of the lemonade over the first 5 hours.

Solution (a). The rate of change of temperature with respect to time is the slope of the curve $T = 70 - 30e^{-0.5t}$. As t increases, the curve rises to a horizontal asymptote, so the slope of the curve decreases to zero. Thus, the temperature rises at an ever-decreasing rate.

Solution (b). The rate of change of temperature with respect to time is

$$\frac{dT}{dt} = \frac{d}{dt}[70 - 30e^{-0.5t}] = -30(-0.5e^{-0.5t}) = 15e^{-0.5t}$$

As t increases, this derivative decreases, which confirms the conclusion in part (a).

Solution (c). From Definition 5.7.5 the average value of T over the time interval [0, 5] is

$$T_{\text{ave}} = \frac{1}{5}\int_0^5 (70 - 30e^{-0.5t})\,dt \tag{5}$$

To evaluate this integral, we make the substitution

$$u = -0.5t \quad \text{so that} \quad du = -0.5\,dt \quad [\text{or } dt = -2\,du]$$

With this substitution we have

$$u = 0 \quad \text{if} \quad t = 0$$
$$u = (-0.5)5 = -2.5 \quad \text{if} \quad t = 5$$

Thus, (5) can be expressed as

$$\begin{aligned} T_{\text{ave}} &= \frac{1}{5}\int_0^{-2.5} (70 - 30e^{u})(-2)\,du = -\frac{2}{5}\int_0^{-2.5} (70 - 30e^{u})\,du \\ &= -\frac{2}{5}\Big[70u - 30e^{u}\Big]_{u=0}^{-2.5} = -\frac{2}{5}\big[(-175 - 30e^{-2.5}) - (-30)\big] \\ &= 58 + 12e^{-2.5} \approx 58.99^{\circ}\text{F} \end{aligned}$$

◀

EXERCISE SET 7.4 Graphing Utility C CAS

In Exercises 1 and 2, use the given derivative to find all critical numbers of f, and at each critical number determine whether a relative maximum, relative minimum, or neither occurs there.

1. (a) $f'(x) = xe^{-x}$ (b) $f'(x) = (e^x - 2)(e^x + 3)$

2. (a) $f'(x) = \ln\left(\dfrac{2}{1 + x^2}\right)$

(b) $f'(x) = (1 - x)\ln x, \quad x > 0$

In Exercises 3 and 4, use a graphing utility to estimate the absolute maximum and minimum values of f, if any, on the stated interval, and then use calculus methods to find the exact values.

3. $f(x) = x^3e^{-2x}$; $[1, 4]$ **4.** $f(x) = \dfrac{\ln(2x)}{x}$; $[1, e]$

We will develop techniques in Section 7.7 to verify that

$$\lim_{x\to+\infty}\frac{e^x}{x} = +\infty, \quad \lim_{x\to+\infty}\frac{x}{e^x} = 0, \quad \lim_{x\to-\infty} xe^x = 0$$

In Exercises 5–14: (a) Use these results, as necessary, to find the limits of $f(x)$ as $x \to +\infty$ and as $x \to -\infty$. (b) Give a graph of $f(x)$ and identify all relative extrema, inflection points, and asymptotes (as appropriate). Check your work with a graphing utility.

5. $f(x) = xe^x$ **6.** $f(x) = xe^{-2x}$

7. $f(x) = x^2e^{-2x}$ **8.** $f(x) = x^2e^{2x}$

9. $f(x) = xe^{x^2}$ **10.** $f(x) = e^{-1/x^2}$

11. $f(x) = \dfrac{e^x}{x}$ **12.** $f(x) = xe^{-x}$

13. $f(x) = x^2e^{1-x}$ **14.** $f(x) = x^3e^{x-1}$

We will develop techniques in Section 7.7 to verify that

$$\lim_{x\to+\infty}\frac{\ln x}{x^r} = 0, \quad \lim_{x\to+\infty}\frac{x^r}{\ln x} = +\infty, \quad \lim_{x\to 0^+} x^r \ln x = 0$$

for any positive real number r. In Exercises 15–20: (a) Use these results, as necessary, to find the limits of $f(x)$ as $x \to +\infty$ and as $x \to 0^+$. (b) Give a graph of $f(x)$ and identify all relative extrema, inflection points, and asymptotes (as appropriate). Check your work with a graphing utility.

15. $f(x) = x\ln x$ **16.** $f(x) = x^2\ln x$

17. $f(x) = \dfrac{\ln x}{x^2}$ **18.** $f(x) = \dfrac{\ln x}{\sqrt{x}}$

19. $f(x) = x^2\ln(2x)$ **20.** $f(x) = \ln(x^2 + 1)$

21. Consider the family of curves $y = xe^{-bx}$ $(b > 0)$.
(a) Use a graphing utility to generate some members of this family.
(b) Discuss the effect of varying b on the shape of the graph, and discuss the locations of the relative extrema and inflection points.

22. Consider the family of curves $y = e^{-bx^2}$ $(b > 0)$.
(a) Use a graphing utility to generate some members of this family.
(b) Discuss the effect of varying b on the shape of the graph, and discuss the locations of the relative extrema and inflection points.

23. (a) Determine whether the following limits exist, and if so, find them:

$$\lim_{x\to+\infty} e^x\cos x, \quad \lim_{x\to-\infty} e^x\cos x$$

(b) Sketch the graphs of $y = e^x$, $y = -e^x$, and $y = e^x\cos x$ in the same coordinate system, and label any points of intersection.
(c) Use a graphing utility to generate some members of the family $y = e^{ax}\cos bx$ ($a > 0$ and $b > 0$), and discuss the effect of varying a and b on the shape of the curve.

24. Find a point on the graph of $y = e^{3x}$ at which the tangent line passes through the origin.

25. (a) Make a conjecture about the shape of the graph of $y = \frac{1}{2}x - \ln x$, and draw a rough sketch.
(b) Check your conjecture by graphing the equation over the interval $0 < x < 5$ with a graphing utility.
(c) Show that the slopes of the tangent lines to the curve at $x = 1$ and $x = e$ have opposite signs.
(d) What does part (c) imply about the existence of a horizontal tangent line to the curve? Explain your reasoning.
(e) Find the exact x-coordinates of all horizontal tangent lines to the curve.

26. The concentration $C(t)$ of a drug in the bloodstream t hours after it has been injected is commonly modeled by an equation of the form

$$C(t) = \frac{K(e^{-bt} - e^{-at})}{a - b}$$

where $K > 0$ and $a > b > 0$.
(a) At what time does the maximum concentration occur?
(b) Let $K = 1$ for simplicity, and use a graphing utility to check your result in part (a) by graphing $C(t)$ for various values of a and b.

27. Suppose that the population of deer on an island is modeled by the equation

$$P(t) = \frac{95}{5 - 4e^{-t/4}}$$

where $P(t)$ is the number of deer t weeks after an initial observation at time $t = 0$.
(a) Use a graphing utility to graph the function $P(t)$.
(b) In words, explain what happens to the population over time. Check your conclusion by finding $\lim_{t\to+\infty} P(t)$.
(c) In words, what happens to the *rate* of population growth over time? Check your conclusion by graphing $P'(t)$.

28. Suppose that the population of oxygen-dependent bacteria in a pond is modeled by the equation

$$P(t) = \frac{60}{5 + 7e^{-t}}$$

where $P(t)$ is the population (in billions) t days after an initial observation at time $t = 0$.
(a) Use a graphing utility to graph the function $P(t)$.
(b) In words, explain what happens to the population over time. Check your conclusion by finding $\lim_{t\to+\infty} P(t)$.
(c) In words, what happens to the *rate* of population growth over time? Check your conclusion by graphing $P'(t)$.

29. Suppose that the spread of a flu virus on a college campus is modeled by the function

$$y(t) = \frac{1000}{1 + 999e^{-0.9t}}$$

where $y(t)$ is the number of infected students at time t (in days, starting with $t = 0$). Use a graphing utility to estimate the day on which the virus is spreading most rapidly.

30. Suppose that the number of bacteria in a culture at time t is given by $N = 5000(25 + te^{-t/20})$.
(a) Find the largest and smallest number of bacteria in the culture during the time interval $0 \le t \le 100$.
(b) At what time during the time interval in part (a) is the number of bacteria decreasing most rapidly?

31. Suppose that a population y grows according to the logistic model given by Formula (1).
(a) At what rate is y increasing at time $t = 0$?
(b) In words, describe how the rate of growth of y varies with time.
(c) At what time is the population growing most rapidly?

32. Show that the inflection point of the logistic growth curve in Example 3 occurs at the time t given by Formula (4).

33. The equilibrium constant k of a balanced chemical reaction changes with the absolute temperature T according to the law

$$k = k_0 \exp\left(-\frac{q(T - T_0)}{2T_0T}\right)$$

where k_0, q, and T_0 are constants. Find the rate of change of k with respect to T.

34. Recall from Section 7.2 that the loudness β of a sound in decibels (dB) is given by $\beta = 10\log(I/I_0)$, where I is the intensity of the sound in watts per square meter (W/m^2) and I_0 is a constant that is approximately the intensity of a sound at the threshold of human hearing. Find the rate of change of β with respect to I at the point where
(a) $I/I_0 = 10$ (b) $I/I_0 = 100$ (c) $I/I_0 = 1000$

35. A particle is moving along the curve $y = x\ln x$. Find all values of x at which the rate of change of y with respect to time is three times that of x. [Assume that dx/dt is never zero.]

36. Let $s(t) = t/e^t$ be the position function of a particle moving along a coordinate line, where s is in meters and t is in seconds. Use a graphing utility to generate the graphs of $s(t)$, $v(t)$, and $a(t)$ for $t \ge 0$, and use those graphs where needed.
(a) Use the appropriate graph to make a rough estimate of the time at which the particle reverses the direction of its motion; and then find the time exactly.
(b) Find the exact position of the particle when it reverses the direction of its motion.
(c) Use the appropriate graphs to make a rough estimate of the time intervals on which the particle is speeding up and on which it is slowing down; and then find those time intervals exactly.

In Exercises 37 and 38, find the area under the curve $y = f(x)$ over the stated interval.

37. $f(x) = e^x$; $[1, 3]$ **38.** $f(x) = \dfrac{1}{x}$; $[1, 5]$

In Exercises 39 and 40, sketch the region enclosed by the curves, and find its area.

39. $y = e^x$, $y = e^{2x}$, $x = 0$, $x = \ln 2$

40. $x = 1/y$, $x = 0$, $y = 1$, $y = e$

In Exercises 41 and 42, sketch the curve and find the total area between the curve and the given interval on the x-axis.

41. $y = e^x - 1$; $[-1, 1]$ **42.** $y = \dfrac{x-1}{x}$; $\left[\frac{1}{2}, 2\right]$

In Exercises 43–45, find the average value of the function over the given interval.

43. $f(x) = 1/x$; $[1, e]$ **44.** $f(x) = e^x$; $[-1, \ln 5]$

45. $f(x) = e^{-2x}$; $[0, 4]$

46. Find a positive value of k such that the area under the graph of $y = e^{2x}$ over the interval $[0, k]$ is 3 square units.

47. Suppose that at time $t = 0$ there are 750 bacteria in a growth medium and the bacteria population $y(t)$ grows at the rate $y'(t) = 802.137e^{1.528t}$ bacteria per hour. How many bacteria will there be in 12 hours?

48. Suppose that the value of a yacht in dollars after t years of use is $V(t) = 275{,}000e^{-0.17t}$. What is the average value of the yacht over its first 10 years of use?

49. Suppose that a particle moving along a coordinate line has velocity $v(t) = 25 + 10e^{-0.05t}$ ft/s.
(a) What is the distance traveled by the particle from time $t = 0$ to time $t = 10$?
(b) Does the term $10e^{-0.05t}$ have much effect on the distance traveled by the particle over that time interval? Explain your reasoning.

50. A particle moves with velocity $v(t)$ meters per second along an s-axis. Find the displacement and distance traveled by the particle during the given time interval.
(a) $v(t) = e^t - 2$; $0 \le t \le 3$
(b) $v(t) = \frac{1}{2} - 1/t$; $1 \le t \le 3$

C **51.** Let the velocity function for a particle that is at the origin initially and moves along an s-axis be $v(t) = 0.5 - te^{-t}$.
(a) Generate the velocity versus time curve, and use it to make a conjecture about the sign of the displacement over the time interval $0 \le t \le 5$.
(b) Use a CAS to find the displacement.

C **52.** Let the velocity function for a particle that is at the origin initially and moves along an s-axis be $v(t) = t\ln(t + 0.1)$.
(a) Generate the velocity versus time curve, and use it to make a conjecture about the sign of the displacement over the time interval $0 \le t \le 1$.
(b) Use a CAS to find the displacement.

In Exercises 53 and 54, use a graphing utility to determine the number of times the curves intersect, and then apply Newton's Method, where needed, to approximate the x-coordinates of all intersections.

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53. $y = 1$ and $y = e^x \cos x$; $0 < x < \pi$

54. $y = e^{-x}$ and $y = \ln x$

55. For the function $f(x) = e^x/(1+x^2)$, use Newton's Method to approximate the x-coordinates of the inflection points to two decimal places.

56. (a) Show that $e^x \geq 1 + x$ if $x \geq 0$.
(b) Show that $e^x \geq 1 + x + \frac{1}{2}x^2$ if $x \geq 0$.
(c) Confirm the inequalities in parts (a) and (b) with a graphing utility.

In Exercises 57 and 58, find the volume of the solid that results when the region enclosed by the given curves is revolved about the x-axis.

57. $y = e^x,\ y = 0,\ x = 0,\ x = \ln 3$

58. $y = e^{-2x},\ y = 0,\ x = 0,\ x = 1$

In Exercises 59 and 60, use cylindrical shells to find the volume of the solid generated when the region enclosed by the given curves is revolved about the y-axis.

59. $y = \dfrac{1}{x^2+1},\ x = 0,\ x = 1,\ y = 0$

60. $y = e^{x^2},\ x = 1,\ x = \sqrt{3},\ y = 0$

In Exercises 61 and 62, find the exact arc length of the parametric curve without eliminating the parameter.

61. $x = e^t \cos t,\ y = e^t \sin t \quad (0 \leq t \leq \pi/2)$

62. $x = e^t(\sin t + \cos t),\ y = e^t(\cos t - \sin t) \quad (1 \leq t \leq 4)$

In Exercises 63 and 64, express the exact arc length of the curve over the given interval as an integral that has been simplified to eliminate the radical, and then evaluate the integral using a CAS.

C **63.** $y = \ln(\sec x)$ from $x = 0$ to $x = \pi/4$

C **64.** $y = \ln(\sin x)$ from $x = \pi/4$ to $\pi/2$

In Exercises 65 and 66, use a CAS or a calculator with numeric integration capability to approximate the area of the surface generated by revolving the curve about the stated axis. Round your answer to two decimal places.

C **65.** $y = e^x,\ 0 \leq x \leq 1$; x-axis

C **66.** $y = e^x,\ 1 \leq y \leq e$; y-axis

C **67.** Use a CAS to find the area of the surface generated by revolving the parametric curve $x = e^t \cos t,\ y = e^t \sin t$, $0 \leq t \leq \pi/2$ about the x-axis.

7.5 LOGARITHMIC FUNCTIONS FROM THE INTEGRAL POINT OF VIEW

In Section 7.2 we discussed natural logarithms from the viewpoint of exponents; that is, we regarded $y = \ln x$ to mean that $e^y = x$. In this section we will show that $\ln x$ can also be expressed as an integral with a variable upper limit. This integral representation of $\ln x$ is important mathematically because it provides a convenient way of establishing properties such as differentiability and continuity. However, it is also important in applications because it provides a way of recognizing when integral solutions of problems can be expressed as natural logarithms.

EXPONENTS

Our work earlier in this chapter was built on the somewhat shaky foundation of extending our definition of exponential expressions b^x $(b > 0)$ to allow for exponents that could be any real number. The process started by defining integer exponents by

$$b^0 = 1,\ b^1 = b,\ b^2 = b \cdot b,\ b^3 = b \cdot b \cdot b, \ldots,\ b^{-1} = \frac{1}{b},\ b^{-2} = \frac{1}{b^2}, \ldots$$

Rational exponents were defined as solutions to equations involving integer exponents:

$$b^{p/q} \quad \text{is the (positive) solution to} \quad x^q = b^p$$

For example, $2^{3.1}$ is the (positive) solution to $x^{10} = 2^{31}$. We claimed that this could be extended to irrational exponents via approximations using rational exponents. For example, it was argued that 2^π could be defined as the limiting value of the sequence

$$2^3,\quad 2^{3.1},\quad 2^{3.14},\quad 2^{3.141},\quad 2^{3.1415},\quad 2^{3.14159}, \ldots$$

where the exponents are successive terminating decimal approximations of π. We then claimed that the resulting exponential function $y = b^x$ is continuous on $(-\infty, +\infty)$ and has the familiar properties of exponents:

$$b^0 = 1 \quad b^{-p} = \frac{1}{b^p} \quad b^{p+q} = b^p b^q \quad b^{p-q} = \frac{b^p}{b^q} \quad (b^p)^q = b^{pq}$$

We further claimed that (for $b > 0$, $b \neq 1$) $f(x) = b^x$ is a one-to-one function, so it has an inverse function that we named $\log_b x$. We also claimed that

$$\lim_{v\to+\infty}\left(1+\frac{1}{v}\right)^v = e \quad \text{and} \quad \lim_{v\to-\infty}\left(1+\frac{1}{v}\right)^v = e$$

which allowed us to use these limits to define e and to find derivatives of exponential and logarithmic functions:

$$\frac{d}{dx}[b^x] = b^x \log_e b \quad \text{and} \quad \frac{d}{dx}[\log_b x] = \frac{1}{x\log_e b}$$

In particular, defining $\ln x = \log_e x$, we have

$$\frac{d}{dx}[e^x] = e^x \quad \text{and} \quad \frac{d}{dx}[\ln x] = \frac{1}{x}$$

Now, for $x > 0$ we have

$$\int_1^x \frac{1}{t}\,dt = \ln t\Big]_1^x = \ln x - \ln 1 = \ln x \qquad (1)$$

This relates the natural logarithm function $\ln x$ to a definite integral of a continuous function, an expression for which we have developed a precise definition.

FORMAL DEFINITION OF ln *x*

A rigorous approach to logarithmic and exponential functions uses (1) as a starting point to define $\ln x$ and defines the natural exponential function as the inverse function for $\ln x$. The challenge is then to demonstrate the consistency of these definitions with our familiar properties for logarithms and exponents.

7.5.1 DEFINITION. The ***natural logarithm*** of x is denoted by $\ln x$ and is defined by the integral

$$\ln x = \int_1^x \frac{1}{t}\,dt, \quad x > 0 \qquad (2)$$

Geometrically, $\ln x$ is the area under the curve $y = 1/t$ from $t = 1$ to $t = x$ when $x > 1$, and $\ln x$ is the negative of the area under the curve $y = 1/t$ from $t = x$ to $t = 1$ when $0 < x < 1$ (Figure 7.5.1). Since $1/t > 0$ for $t > 0$, $\ln x$ will be an increasing function on $(0, +\infty)$. Moreover, if $x = 1$, then $\ln x = 0$, since the upper and lower limits of (2)

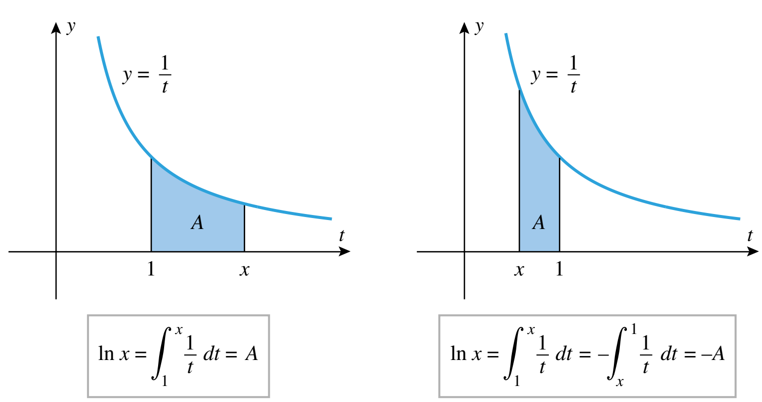

Figure 7.5.1

are the same. All of this is consistent with the computer-generated graph of $y = \ln x$ in Figure 7.2.4.

FOR THE READER. Review Theorem 5.5.8, and then explain why x is required to be positive in Definition 7.5.1.

APPROXIMATING ln x NUMERICALLY

For specific values of x, the value of $\ln x$ can be approximated numerically by approximating the definite integral in (2), say by using the midpoint approximation that was discussed in Section 5.4.

Example 1 Approximate $\ln 2$ using the midpoint approximation with $n = 10$.

Solution. From (2), the exact value of $\ln 2$ is represented by the integral

$$\ln 2 = \int_1^2 \frac{1}{t}\,dt$$

The midpoint rule is given in Formulas (8) and (9) of Section 5.4. Expressed in terms of t, the latter formula is

$$\int_a^b f(t)\,dt \approx \Delta t \sum_{k=1}^{n} f(t_k^*)$$

where Δt is the common width of the subintervals and $t_1^*, t_2^*, \ldots, t_n^*$ are the midpoints. In this case we have 10 subintervals, so $\Delta t = (2-1)/10 = 0.1$. The computations to six decimal places are shown in Table 7.5.1. By comparison, a calculator set to display six decimal places gives $\ln 2 \approx 0.693147$, so the magnitude of the error in the midpoint approximation is about 0.000311. Greater accuracy in the midpoint approximation can be obtained by increasing n. For example, the midpoint approximation with $n = 100$ yields $\ln 2 \approx 0.693144$, which is correct to five decimal places. ◀

PROPERTIES OF ln x

Table 7.5.1

$n = 10$
$\Delta t = (b-a)/n = (2-1)/10 = 0.1$

k	t_k^*	$1/t_k^*$
1	1.05	0.952381
2	1.15	0.869565
3	1.25	0.800000
4	1.35	0.740741
5	1.45	0.689655
6	1.55	0.645161
7	1.65	0.606061
8	1.75	0.571429
9	1.85	0.540541
10	1.95	0.512821
		6.928355

$$\Delta t \sum_{k=1}^{n} f(t_k^*) \approx (0.1)(6.928355) \approx 0.692836$$

Definition 7.5.1 is not only useful for approximating values of $\ln x$; it is the key to establishing many of the fundamental properties of the natural logarithm. For example, by Part 2 of the Fundamental Theorem of Calculus (Theorem 5.6.3), we have

$$\frac{d}{dx}[\ln x] = \frac{1}{x} \quad (x > 0) \tag{3}$$

In particular, the natural logarithm function is differentiable on $(0, +\infty)$, so we also have that $\ln x$ is continuous on $(0, +\infty)$.

We can use (3) to establish that our definition for $\ln x$ satisfies the expected logarithm properties.

7.5.2 THEOREM. *For any positive numbers a and c and any rational number r:*

(*a*) $\ln ac = \ln a + \ln c$ (*b*) $\ln \frac{1}{c} = -\ln c$

(*c*) $\ln \frac{a}{c} = \ln a - \ln c$ (*d*) $\ln a^r = r \ln a$

Proof (a). Treating a as a constant, consider the function $f(x) = \ln(ax)$. Then

$$f'(x) = \frac{1}{ax}\cdot\frac{d}{dx}(ax) = \frac{1}{ax}\cdot a = \frac{1}{x}$$

Thus, $\ln ax$ and $\ln x$ have the same derivative on $(0, +\infty)$, so these functions must differ by a constant on this interval. That is, there is a constant k such that

$$\ln ax - \ln x = k \tag{4}$$

on $(0, +\infty)$. Substituting $x = 1$ into this equation we conclude that $\ln a = k$ (verify). Thus, (4) can be written as

$$\ln ax - \ln x = \ln a$$

Setting $x = c$ establishes that

$$\ln ac - \ln c = \ln a \quad \text{or} \quad \ln ac = \ln a + \ln c$$

Proofs (b) and (c). Part (b) follows immediately from part (a) by substituting $1/c$ for a (verify). Then

$$\ln \frac{a}{c} = \ln \left(a \cdot \frac{1}{c}\right) = \ln a + \ln \frac{1}{c} = \ln a - \ln c$$

Proof (d). Since

$$\frac{d}{dx}[\ln x^r] = \frac{1}{x^r} \cdot \frac{d}{dx}[x^r] = \frac{1}{x^r} \cdot rx^{r-1} = \frac{r}{x}$$

and

$$\frac{d}{dx}[r \ln x] = r \cdot \frac{d}{dx}[\ln x] = \frac{r}{x}$$

the functions $\ln x^r$ and $r \ln x$ have the same derivative on $(0, +\infty)$. Thus, there is a constant k such that

$$\ln x^r - r \ln x = k$$

Substituting $x = 1$ into this equation we conclude that $k = 0$ (verify), so

$$\ln x^r - r \ln x = 0 \quad \text{or} \quad \ln x^r = r \ln x$$

Setting $x = a$ completes the proof. ■

The function $\ln x$ is defined and increasing for x in the interval $(0, +\infty)$. Now, for any integer N, if $x > 2^N$, then

$$\ln x > \ln 2^N = N \ln 2$$

by Theorem 7.5.2(d). Since

$$\ln 2 = \int_1^2 \frac{1}{t}\, dt > 0$$

$N \ln 2$ can be made arbitrarily large by choosing N appropriately, so

$$\lim_{x \to +\infty} \ln x = +\infty$$

Furthermore, by observing that $v = 1/x \to +\infty$ as $x \to 0^+$, we can use the preceding limit and Theorem 7.5.2(b) to conclude that

$$\lim_{x \to 0^+} \ln x = \lim_{v \to +\infty} \ln \frac{1}{v} = \lim_{v \to +\infty} (-\ln v) = -\infty$$

These results are summarized in the following theorem.

7.5.3 THEOREM.

(*a*) *The domain of* $\ln x$ *is* $(0, +\infty)$.

(*b*) $\lim_{x \to 0^+} \ln x = -\infty$ *and* $\lim_{x \to +\infty} \ln x = +\infty$

(*c*) *The range of* $\ln x$ *is* $(-\infty, +\infty)$.

DEFINITION OF e^x

In Section 7.2 we introduced e informally as the value of a limit, although we did not have the mathematical tools to prove the existence of this limit. We now give a precise definition of the number e and confirm that it matches the desired limit.

Since $\ln x$ is increasing and continuous on $(0, +\infty)$ with range $(-\infty, +\infty)$, there is exactly one (positive) solution to the equation $\ln x = 1$. We *define* e to be the unique solution to $\ln x = 1$, so

$$\ln e = 1 \tag{5}$$

Furthermore, if x is any real number, there is a unique positive solution y to $\ln y = x$, so for irrational values of x we *define* e^x to be this solution. That is, when x is irrational, e^x is defined by

$$\ln e^x = x \tag{6}$$

Note that for rational values of x, we also have $\ln e^x = x \ln e = x$ from Theorem 7.5.2(d). Moreover, it follows immediately that $e^{\ln x} = x$ for any $x > 0$. Thus, (6) defines the exponential function for all real values of x as the inverse of the natural logarithm function.

7.5.4 DEFINITION. The inverse of the natural logarithm function $\ln x$ is denoted by e^x and is called the ***natural exponential function***.

We can now establish the differentiability of e^x, confirm that

$$\frac{d}{dx}[e^x] = e^x$$

and verify the limits in Formulas (3)–(5) of Section 7.2.

7.5.5 THEOREM. *The natural exponential function e^x is differentiable on $(-\infty, +\infty)$ and its derivative is*

$$\frac{d}{dx}[e^x] = e^x$$

Proof. Because $\ln x$ is differentiable and

$$\frac{d}{dx}[\ln x] = \frac{1}{x} > 0$$

for all x in $(0, +\infty)$, it follows from Corollary 7.1.7, with $f(x) = \ln x$ and $f^{-1}(x) = e^x$, that e^x is differentiable on $(-\infty, +\infty)$ and its derivative is

$$\frac{d}{dx}\underbrace{[e^x]}_{f^{-1}(x)} = \underbrace{\frac{1}{1/e^x}}_{f'(f^{-1}(x))} = e^x \qquad \blacksquare$$

7.5.6 THEOREM.

$$(a)\ \lim_{x\to 0}(1+x)^{1/x} = e \qquad (b)\ \lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = e \qquad (c)\ \lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = e$$

Proof. We will prove part (a); the proofs of parts (b) and (c) follow from this limit and are left as exercises. We first observe that

$$\frac{d}{dx}[\ln(x+1)]\bigg|_{x=0} = \frac{1}{x+1}\cdot 1\bigg|_{x=0} = 1$$

However, using the definition of the derivative, we obtain

$$\begin{aligned}\frac{d}{dx}[\ln(x+1)]\bigg|_{x=0} &= \lim_{w\to 0}\frac{\ln(w+1)-\ln(0+1)}{w-0}\\ &= \lim_{w\to 0}\left[\frac{1}{w}\cdot\ln(w+1)\right] = \lim_{w\to 0}[\ln(w+1)^{1/w}]\end{aligned}$$

Thus,

$$1 = \lim_{w\to 0}[\ln(w+1)^{1/w}], \quad \text{so} \quad e = e^{(\lim_{w\to 0}[\ln(w+1)^{1/w}])}$$

Since e^x is continuous on $(-\infty, +\infty)$, we can move the limit symbol through the function symbol, and once more using the inverse relationship between e^x and $\ln x$, we obtain

$$e = \lim_{w\to 0} e^{[\ln(w+1)^{1/w}]} = \lim_{w\to 0}(w+1)^{1/w}$$

which establishes the limit in part (a).

IRRATIONAL EXPONENTS

Recall from Theorem 7.5.2(d) that if $a > 0$ and r is a rational number, then $\ln a^r = r \ln a$. Then $a^r = e^{\ln a^r} = e^{r\ln a}$ for any positive value of a and any rational number r. But the expression $e^{r\ln a}$ makes sense for *any* real number r, whether rational or irrational, so it is a good candidate to give meaning to a^r for any real number r.

7.5.7 DEFINITION. If $a > 0$ and r is a real number, a^r is defined by

$$a^r = e^{r\ln a} \tag{7}$$

With this definition it can be shown that the standard algebraic properties of exponents, such as

$$a^p a^q = a^{p+q}, \quad \frac{a^p}{a^q} = a^{p-q}, \quad (a^p)^q = a^{pq}, \quad (a^p)(b^p) = (ab)^p$$

hold for any real values of a, b, p, and q, where a and b are positive. In addition, using (7) for a real exponent r, we can define the power function x^r whose domain consists of all positive real numbers and, for a positive base b, we can define the ***base b exponential function*** $\boldsymbol{b^x}$ whose domain consists of all real numbers.

7.5.8 THEOREM.

(a) *For any real number r, the power function x^r is differentiable on $(0, +\infty)$ and its derivative is*

$$\frac{d}{dx}[x^r] = rx^{r-1}$$

(b) *For $b > 0$ and $b \neq 1$, the base b exponential function b^x is differentiable on $(-\infty, +\infty)$ and its derivative is*

$$\frac{d}{dx}[b^x] = b^x \ln b$$

Proof. The differentiability of $x^r = e^{r\ln x}$ and $b^x = e^{x\ln b}$ on their domains follows from the differentiability of $\ln x$ on $(0, +\infty)$ and of e^x on $(-\infty, +\infty)$:

$$\frac{d}{dx}[x^r] = \frac{d}{dx}[e^{r\ln x}] = e^{r\ln x}\cdot\frac{d}{dx}[r\ln x] = x^r\cdot\frac{r}{x} = rx^{r-1}$$

$$\frac{d}{dx}[b^x] = \frac{d}{dx}[e^{x\ln b}] = e^{x\ln b}\cdot\frac{d}{dx}[x\ln b] = b^x\ln b$$

GENERAL LOGARITHMS

We note that for $b > 0$ and $b \neq 1$, the function b^x is one-to-one, and so has an inverse function. Using the definition of b^x, we can solve $y = b^x$ for x as a function of y:

$$y = b^x = e^{x\ln b}$$

$$\ln y = \ln(e^{x\ln b}) = x\ln b$$

$$\frac{\ln y}{\ln b} = x$$

Thus, the inverse function for b^x is $(\ln x)/(\ln b)$.

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7.5.9 DEFINITION. For $b > 0$ and $b \neq 1$, the ***base b logarithm*** function, denoted $\log_b x$, is defined by

$$\log_b x = \frac{\ln x}{\ln b} \tag{8}$$

It follows immediately from this definition that $\log_b x$ is the inverse function for b^x and satisfies the properties in Theorem 7.2.2. Furthermore, $\log_b x$ is differentiable on $(0, +\infty)$, and its derivative is

$$\frac{d}{dx}[\log_b x] = \frac{1}{x \ln b}$$

As a final note of consistency, we observe that $\log_e x = \ln x$.

FUNCTIONS DEFINED BY INTEGRALS

The functions we have dealt with thus far in this text are called ***elementary functions***; they include polynomial, rational, power, exponential, logarithmic, and trigonometric functions, and all other functions that can be obtained from these by addition, subtraction, multiplication, division, root extraction, and composition.

However, there are many important functions that do not fall into this category. Such functions occur in many ways, but they commonly arise in the course of solving initial-value problems of the form

$$\frac{dy}{dx} = f(x), \quad y(x_0) = y_0 \tag{9}$$

Recall from Example 7 of Section 5.2 and the discussion preceding it that the basic method for solving (9) is to integrate $f(x)$, and then use the initial condition to determine the constant of integration. It can be proved that if f is continuous, then (9) has a unique solution and that this procedure produces it. However, there is another approach: Instead of solving each initial-value problem individually, we can find a general formula for the solution of (9), and then apply that formula to solve specific problems. We will now show that

$$y(x) = y_0 + \int_{x_0}^{x} f(t)\,dt \tag{10}$$

is a formula for the solution of (9). To confirm that this is so we must show that $dy/dx = f(x)$ and that $y(x_0) = y_0$. The computations are as follows:

$$\frac{dy}{dx} = \frac{d}{dx}\left[y_0 + \int_{x_0}^{x} f(t)\,dt\right] = 0 + f(x) = f(x)$$

$$y(x_0) = y_0 + \int_{x_0}^{x_0} f(t)\,dt = y_0 + 0 = y_0$$

Example 2 In Example 7 of Section 5.2 we showed that the solution of the initial-value problem

$$\frac{dy}{dx} = \cos x, \quad y(0) = 1$$

is $y(x) = 1 + \sin x$. This initial-value problem can also be solved by applying Formula (10) with $f(x) = \cos x$, $x_0 = 0$, and $y_0 = 1$. This yields

$$y(x) = 1 + \int_0^x \cos t\,dt = 1 + \big[\sin t\big]_{t=0}^{x} = 1 + \sin x$$ ◀

In the last example we were able to perform the integration in Formula (10) and express the solution of the initial-value problem as an elementary function. However, sometimes this will not be possible, in which case the solution of the initial-value problem must be left in terms of an "unevaluated" integral. For example, from (10), the solution of the initial-value problem

$$\frac{dy}{dx} = e^{-x^2}, \quad y(0) = 1$$

is

$$y(x) = 1 + \int_0^x e^{-t^2}\,dt$$

However, it can be shown that there is no way to express the integral in this solution as an elementary function. Thus, we have encountered a *new* function, which we regard to be *defined* by the integral. A close relative of this function, known as the ***error function***, plays an important role in probability and statistics; it is denoted by $\operatorname{erf}(x)$ and is defined as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt \tag{11}$$

Indeed, many of the most important functions in science and engineering are defined as integrals that have special names and notations associated with them. For example, the functions defined by

$$S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt \quad \text{and} \quad C(x) = \int_0^x \cos\left(\frac{\pi t^2}{2}\right) dt \tag{12–13}$$

are called the ***Fresnel sine and cosine functions***, respectively, in honor of the French physicist Augustin Fresnel (1788–1827), who first encountered them in his study of diffraction of light waves.

EVALUATING AND GRAPHING FUNCTIONS DEFINED BY INTEGRALS

The following values of $S(1)$ and $C(1)$ were produced by a CAS that has a built-in algorithm for approximating definite integrals:

$$S(1) = \int_0^1 \sin\left(\frac{\pi t^2}{2}\right) dt \approx 0.438259, \qquad C(1) = \int_0^1 \cos\left(\frac{\pi t^2}{2}\right) dt \approx 0.779893$$

To generate graphs of functions defined by integrals, computer programs choose a set of x-values in the domain, approximate the integral for each of those values, and then plot the resulting points. Thus, there is a lot of computation involved in generating such graphs, since each plotted point requires the approximation of an integral. The graphs of the Fresnel functions in Figure 7.5.2 were generated in this way using a CAS.

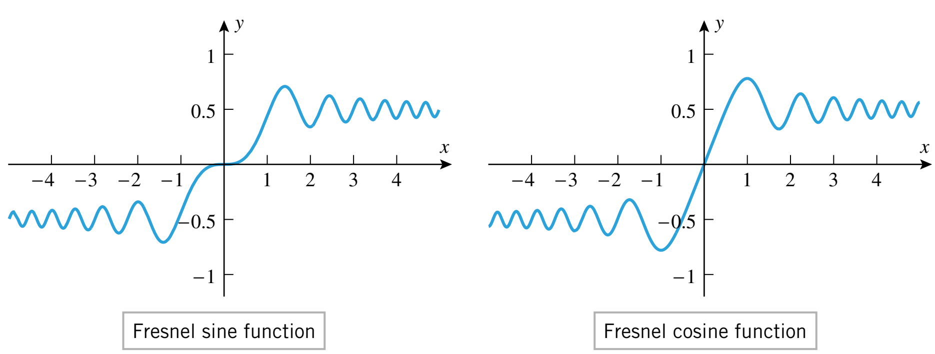

Figure 7.5.2

REMARK. Although it required a considerable amount of computation to generate the graphs of the Fresnel functions, the derivatives of $S(x)$ and $C(x)$ are easy to obtain using Part 2 of the Fundamental Theorem of Calculus (5.6.3); they are

$$S'(x) = \sin\left(\frac{\pi x^2}{2}\right) \quad \text{and} \quad C'(x) = \cos\left(\frac{\pi x^2}{2}\right) \tag{14–15}$$

These derivatives can be used to determine the locations of the relative extrema and inflection points and to investigate other properties of $S(x)$ and $C(x)$.

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INTEGRALS WITH FUNCTIONS AS LIMITS OF INTEGRATION

Various applications can lead to integrals in which one or both of the limits of integration is a function of x. Some examples are

$$\int_x^1 \sqrt{\sin t}\,dt, \qquad \int_{x^2}^{\sin x} \sqrt{t^3+1}\,dt, \qquad \int_{\ln x}^{\pi} \frac{dt}{t^7-8}$$

We will complete this section by showing how to differentiate integrals of the form

$$\int_a^{g(x)} f(t)\,dt \tag{16}$$

where a is constant. Derivatives of other kinds of integrals with functions as limits of integration will be discussed in the exercises.

To differentiate (16) we can view the integral as a composition $F(g(x))$, where

$$F(x) = \int_a^x f(t)\,dt$$

If we now apply the chain rule, we obtain

$$\frac{d}{dx}\left[\int_a^{g(x)} f(t)\,dt\right] = \frac{d}{dx}[F(g(x))] = F'(g(x))g'(x) = \underbrace{f(g(x))g'(x)}_{\text{Theorem 5.6.3}}$$

Thus,

$$\boxed{\frac{d}{dx}\left[\int_a^{g(x)} f(t)\,dt\right] = f(g(x))g'(x)} \tag{17}$$

In words:

To differentiate an integral with a constant lower limit and a function as the upper limit, substitute the upper limit into the integrand, and multiply by the derivative of the upper limit.

Example 3

$$\frac{d}{dx}\left[\int_1^{\sin x} (1-t^2)\,dt\right] = (1-\sin^2 x)\cos x = \cos^3 x$$ ◀

HISTORICAL NOTE

The connection between natural logarithms and integrals was made in the middle of the seventeenth century in the course of investigating areas under the curve $y = 1/t$. The problem being considered was to find values of $t_1, t_2, t_3, \ldots, t_n, \ldots$ for which the areas $A_1, A_2, A_3, \ldots, A_n, \ldots$ in Figure 7.5.3*a* would be equal. Through the combined work of

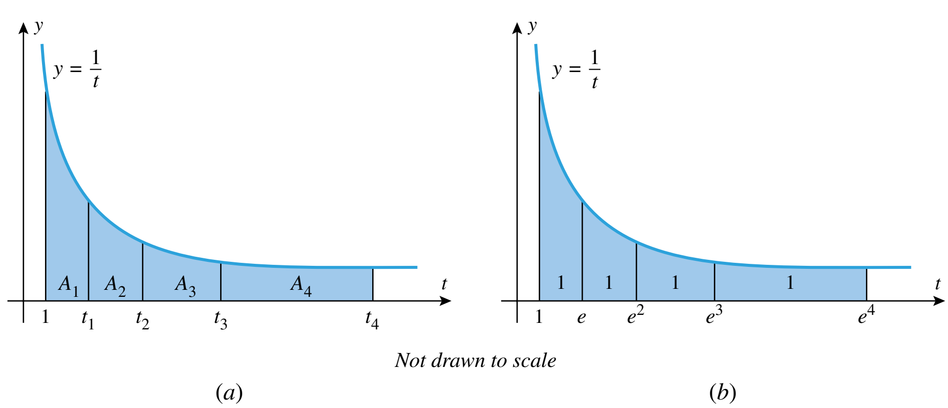

Figure 7.5.3

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Isaac Newton, the Belgian Jesuit priest, Gregory of St. Vincent (1584–1667), and Gregory's student, Alfons A. de Sarasa (1618–1667), it was shown that by taking the points to be

$$t_1 = e,\quad t_2 = e^2,\quad t_3 = e^3, \ldots,\quad t_n = e^n, \ldots$$

each of the areas would be 1 (Figure 7.5.3*b*). Thus, in modern integral notation

$$\int_1^{e^n} \frac{1}{t}\, dt = n$$

which can be expressed as

$$\int_1^{e^n} \frac{1}{t}\, dt = \ln(e^n)$$

By comparing the upper limit of the integral and the expression inside the logarithm, it is a natural leap to the more general result

$$\int_1^{x} \frac{1}{t}\, dt = \ln x$$

which today we take as the formal definition of the natural logarithm.

EXERCISE SET 7.5 Graphing Utility C CAS

1. Sketch the curve $y = 1/t$, and shade a region under the curve whose area is
(a) $\ln 2$ (b) $-\ln 0.5$ (c) 2.

2. Sketch the curve $y = 1/t$, and shade two different regions under the curve whose areas are $\ln 1.5$.

3. Given that $\ln a = 2$ and $\ln c = 5$, find
(a) $\displaystyle\int_1^{ac} \frac{1}{t}\, dt$ (b) $\displaystyle\int_1^{1/c} \frac{1}{t}\, dt$
(c) $\displaystyle\int_1^{a/c} \frac{1}{t}\, dt$ (d) $\displaystyle\int_1^{a^3} \frac{1}{t}\, dt$.

4. Given that $\ln a = 9$, find
(a) $\displaystyle\int_1^{\sqrt{a}} \frac{1}{t}\, dt$ (b) $\displaystyle\int_1^{2a} \frac{1}{t}\, dt$
(c) $\displaystyle\int_1^{2/a} \frac{1}{t}\, dt$ (d) $\displaystyle\int_2^{a} \frac{1}{t}\, dt$.

5. Approximate $\ln 5$ using the midpoint rule with $n = 10$, and estimate the magnitude of the error by comparing your answer to that produced directly by a calculating utility.

6. Approximate $\ln 3$ using the midpoint rule with $n = 20$, and estimate the magnitude of the error by comparing your answer to that produced directly by a calculating utility.

7. Simplify the expression and state the values of x for which your simplification is valid.
(a) $e^{-\ln x}$ (b) $e^{\ln x^2}$
(c) $\ln\left(e^{-x^2}\right)$ (d) $\ln(1/e^x)$
(e) $\exp(3\ln x)$ (f) $\ln(xe^x)$
(g) $\ln\left(e^{x-\sqrt[3]{x}}\right)$ (h) $e^{x-\ln x}$

8. (a) Let $f(x) = e^{-2x}$. Find the simplest exact value of the function $f(\ln 3)$.
(b) Let $f(x) = e^x + 3e^{-x}$. Find the simplest exact value of the function $f(\ln 2)$.

In Exercises 9 and 10, express the given quantity as a power of e.

9. (a) 3^π (b) $2^{\sqrt{2}}$

10. (a) π^{-x} (b) x^{2x}, $x > 0$

In Exercises 11 and 12, find the limits by making appropriate substitutions in the limits given in Theorem 7.5.6.

11. (a) $\displaystyle\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^{2x}$ (b) $\displaystyle\lim_{x\to 0}\,(1+2x)^{1/x}$

12. (a) $\displaystyle\lim_{x\to+\infty}\left(1+\frac{1}{3x}\right)^{x}$ (b) $\displaystyle\lim_{x\to 0}\,(1+x)^{1/(3x)}$

In Exercises 13 and 14, find $g'(x)$ using Part 2 of the Fundamental Theorem of Calculus, and check your answer by evaluating the integral and then differentiating.

13. $g(x) = \displaystyle\int_1^x (t^2 - t)\, dt$ **14.** $g(x) = \displaystyle\int_\pi^x (1 - \cos t)\, dt$

In Exercises 15 and 16, find the derivative using Formula (17), and check your answer by evaluating the integral and then differentiating.

15. (a) $\dfrac{d}{dx}\displaystyle\int_1^{x^3} \frac{1}{t}\,dt$ (b) $\dfrac{d}{dx}\displaystyle\int_1^{\ln x} e^t\,dt$

16. (a) $\dfrac{d}{dx}\displaystyle\int_{-1}^{x^2} \sqrt{t+1}\,dt$ (b) $\dfrac{d}{dx}\displaystyle\int_{\pi}^{1/x} \sin t\,dt$

17. Let $F(x) = \displaystyle\int_0^x \frac{\cos t}{t^2+3}\,dt$. Find

(a) $F(0)$ (b) $F'(0)$ (c) $F''(0)$.

18. Let $F(x) = \displaystyle\int_2^x \sqrt{3t^2+1}\,dt$. Find

(a) $F(2)$ (b) $F'(2)$ (c) $F''(2)$.

C **19.** (a) Use Formula (17) to find

$$\frac{d}{dx}\int_1^{x^2} t\sqrt{1+t}\,dt$$

(b) Use a CAS to evaluate the integral and differentiate the resulting function.

(c) Use the simplification command of the CAS, if necessary, to confirm that answers in parts (a) and (b) are the same.

20. Show that

(a) $\dfrac{d}{dx}\left[\displaystyle\int_x^a f(t)\,dt\right] = -f(x)$

(b) $\dfrac{d}{dx}\left[\displaystyle\int_{g(x)}^a f(t)\,dt\right] = -f(g(x))g'(x)$.

In Exercises 21 and 22, use the results in Exercise 20 to find the derivative.

21. (a) $\dfrac{d}{dx}\displaystyle\int_x^1 \sin(t^2)\,dt$ (b) $\dfrac{d}{dx}\displaystyle\int_{\tan x}^3 \frac{t^2}{1+t^2}\,dt$

22. (a) $\dfrac{d}{dx}\displaystyle\int_x^0 (t^2+1)^{40}\,dt$ (b) $\dfrac{d}{dx}\displaystyle\int_{1/x}^{\pi} \cos^3 t\,dt$

23. Find

$$\frac{d}{dx}\left[\int_{3x}^{x^2} \frac{t-1}{t^2+1}\,dt\right]$$

by writing

$$\int_{3x}^{x^2} \frac{t-1}{t^2+1}\,dt = \int_{3x}^{0} \frac{t-1}{t^2+1}\,dt + \int_0^{x^2} \frac{t-1}{t^2+1}\,dt$$

24. Use Exercise 20(b) and the idea in Exercise 23 to show that

$$\frac{d}{dx}\int_{h(x)}^{g(x)} f(t)\,dt = f(g(x))g'(x) - f(h(x))h'(x)$$

25. Use the result obtained in Exercise 24 to perform the following differentiations:

(a) $\dfrac{d}{dx}\displaystyle\int_{x^2}^{x^3} \sin^2 t\,dt$ (b) $\dfrac{d}{dx}\displaystyle\int_{-x}^{x} \frac{1}{1+t}\,dt$.

26. Prove that the function

$$F(x) = \int_x^{3x} \frac{1}{t}\,dt$$

is constant on the interval $(0, +\infty)$ by using Exercise 24 to find $F'(x)$. What is that constant?

27. Let $F(x) = \int_0^x f(t)\,dt$, where f is the function whose graph is shown in the accompanying figure.

(a) Find $F(0)$, $F(3)$, $F(5)$, $F(7)$, and $F(10)$.

(b) On what subintervals of the interval $[0, 10]$ is F increasing? Decreasing?

(c) Where does F have its maximum value? Its minimum value?

(d) Sketch the graph of F.

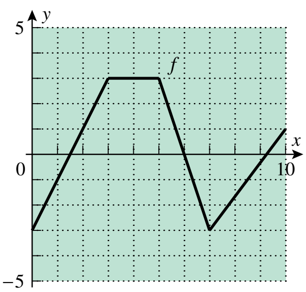


Figure Ex-27

28. Use the appropriate values found in part (a) of Exercise 27 to find the average value of f over the interval $[0, 10]$.

In Exercises 29 and 30, express $F(x)$ in a piecewise form that does not involve an integral.

29. $F(x) = \displaystyle\int_{-1}^{x} |t|\,dt$

30. $F(x) = \displaystyle\int_0^x f(t)\,dt$, where $f(x) = \begin{cases} x, & 0 \le x \le 2 \\ 2, & x > 2 \end{cases}$

In Exercises 31–34, use Formula (10) to solve the initial-value problem.

31. $\dfrac{dy}{dx} = \sqrt[3]{x};\ y(1) = 2$ **32.** $\dfrac{dy}{dx} = \dfrac{x+1}{\sqrt{x}};\ y(1) = 0$

33. $\dfrac{dy}{dx} = \sec^2 x - \sin x;\ y(\pi/4) = 1$

34. $\dfrac{dy}{dx} = xe^{x^2};\ y(0) = 0$

35. Suppose that at time $t = 0$ there are P_0 individuals who have disease X, and suppose that a certain model for the spread of the disease predicts that the disease will spread at the rate of $r(t)$ individuals per day. Write a formula for the number of individuals who will have disease X after x days.

36. Suppose that $v(t)$ is the velocity function of a particle moving along an s-axis. Write a formula for the coordinate of the particle at time T if the particle is at s_1 at time $t = 1$.

37. The accompanying figure shows the graphs of $y = f(x)$ and $y = \int_0^x f(t)\,dt$. Determine which graph is which, and explain your reasoning.

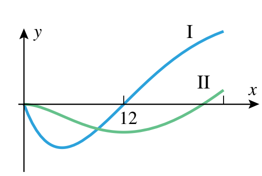


Figure Ex-37

38. (a) Make a conjecture about the value of the limit

$$\lim_{k\to 0} \int_1^b t^{k-1}\,dt \quad (b>0)$$

(b) Check your conjecture by evaluating the integral and finding the limit. [*Hint:* Interpret the limit as the definition of the derivative of an exponential function.]

39. Let $F(x) = \int_0^x f(t)\,dt$, where f is the function graphed in the accompanying figure.
(a) Where do the relative minima of F occur?
(b) Where do the relative maxima of F occur?
(c) Where does the absolute maximum of F on the interval [0, 5] occur?
(d) Where does the absolute minimum of F on the interval [0, 5] occur?
(e) Where is F concave up? Concave down?
(f) Sketch the graph of F.

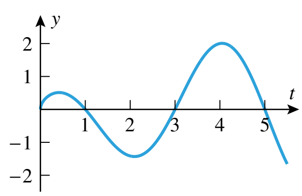


Figure Ex-39

C **40.** CAS programs have commands for working with most of the important nonelementary functions. Check your CAS documentation for information about the error function erf(x) [see Formula (11)], and then complete the following.
(a) Generate the graph of erf(x).
(b) Use the graph to make a conjecture about the existence and location of any relative maxima and minima of erf(x).
(c) Check your conjecture in part (b) using the derivative of erf(x).
(d) Use the graph to make a conjecture about the existence and location of any inflection points of erf(x).
(e) Check your conjecture in part (d) using the second derivative of erf(x).
(f) Use the graph to make a conjecture about the existence of horizontal asymptotes of erf(x).
(g) Check your conjecture in part (f) by using the CAS to find the limits of erf(x) as $x \to \pm\infty$.

41. The Fresnel sine and cosine functions $S(x)$ and $C(x)$ were defined in Formulas (12) and (13) and graphed in Figure 7.5.2. Their derivatives were given in Formulas (14) and (15).
(a) At what points does $C(x)$ have relative minima? Relative maxima?
(b) Where do the inflection points of $C(x)$ occur?
(c) Confirm that your answers in parts (a) and (b) are consistent with the graph of $C(x)$.

42. Find the limit

$$\lim_{h\to 0} \frac{1}{h} \int_x^{x+h} \ln t\,dt$$

43. Find a function f and a number a such that

$$2 + \int_a^x f(t)\,dt = e^{3x}$$

44. (a) Give a geometric argument to show that

$$\frac{1}{x+1} < \int_x^{x+1} \frac{1}{t}\,dt < \frac{1}{x}, \quad x>0$$

(b) Use the result in part (a) to prove that

$$\frac{1}{x+1} < \ln\left(1+\frac{1}{x}\right) < \frac{1}{x}, \quad x>0$$

(c) Use the result in part (b) to prove that

$$e^{\frac{x}{x+1}} < \left(1+\frac{1}{x}\right)^x < e, \quad x>0$$

and hence that

$$\lim_{x\to+\infty} \left(1+\frac{1}{x}\right)^x = e$$

(d) Use the inequality in part (c) to prove that

$$\left(1+\frac{1}{x}\right)^x < e < \left(1+\frac{1}{x}\right)^{x+1}, \quad x>0$$

45. Use a graphing utility to generate the graph of

$$y = \left(1+\frac{1}{x}\right)^{x+1} - \left(1+\frac{1}{x}\right)^x$$

in the window [0, 100] × [0, 0.2], and use that graph and part (d) of Exercise 44 to make a rough estimate of the error in the approximation

$$e \approx \left(1+\frac{1}{50}\right)^{50}$$

46. Prove: If f is continuous on an open interval I and a is any point in I, then

$$F(x) = \int_a^x f(t)\,dt$$

is continuous on I.

7.6 DERIVATIVES AND INTEGRALS INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

A common problem in trigonometry is to find an angle whose trigonometric functions are known. As you may recall, problems of this type involve the computation of "arc functions" such as $\arcsin x$, $\arccos x$, $\arctan x$, *and so forth. In this section we will consider this idea from the viewpoint of inverse functions, with the goal of developing derivative formulas for the inverse trigonometric functions. We will also derive some related integration formulas that involve inverse trigonometric functions.*

INVERSE TRIGONOMETRIC FUNCTIONS

None of the six basic trigonometric functions is one-to-one because they all repeat periodically and hence do not pass the horizontal line test. Thus, to define inverse trigonometric functions we must first restrict the domains of the trigonometric functions to make them one-to-one. The top part of Figure 7.6.1 shows how these restrictions are made for $\sin x$, $\cos x$, $\tan x$, and $\sec x$. (Inverses of $\cot x$ and $\csc x$ are of lesser importance and will be left for the exercises.) The inverses of these restricted functions are denoted by

$$\sin^{-1} x, \quad \cos^{-1} x, \quad \tan^{-1} x, \quad \sec^{-1} x$$

(or alternatively by $\arcsin x$, $\arccos x$, $\arctan x$, $\operatorname{arcsec} x$) and are defined as follows:

7.6.1 DEFINITION. The ***inverse sine function***, denoted by $\sin^{-1}$, is defined to be the inverse of the restricted sine function

$$\sin x, \quad -\pi/2 \le x \le \pi/2$$

7.6.2 DEFINITION. The ***inverse cosine function***, denoted by $\cos^{-1}$, is defined to be the inverse of the restricted cosine function

$$\cos x, \quad 0 \le x \le \pi$$

7.6.3 DEFINITION. The ***inverse tangent function***, denoted by $\tan^{-1}$, is defined to be the inverse of the restricted tangent function

$$\tan x, \quad -\pi/2 < x < \pi/2$$

7.6.4 DEFINITION.* The ***inverse secant function***, denoted by $\sec^{-1}$, is defined to be the inverse of the restricted secant function

$$\sec x, \quad 0 \le x \le \pi \text{ with } x \ne \pi/2$$

REMARK. The notations $\sin^{-1} x$, $\cos^{-1} x, \ldots$ are reserved exclusively for the inverse trigonometric functions and are not used for reciprocals of the trigonometric functions. For example, to denote the reciprocal $1/\sin x$ in exponent form, we would write $(\sin x)^{-1}$ and *never* $\sin^{-1} x$.

The graphs of the inverse trigonometric functions, which are shown in the bottom part of Figure 7.6.1, are obtained by reflecting the graphs in the top part of the figure about the line $y = x$. If you have trouble visualizing these relationships, then look at Figure 7.6.2

*There is no universal agreement on the definition of $\sec^{-1} x$, and some mathematicians prefer to restrict the domain of $\sec x$ so that $0 \le x < \pi/2$ or $\pi \le x < 3\pi/2$, which was the definition used in some earlier editions of this text. Each definition has advantages and disadvantages, but we have changed to the current definition to conform with the conventions used by the CAS programs *Mathematica*, *Maple*, and *Derive*.

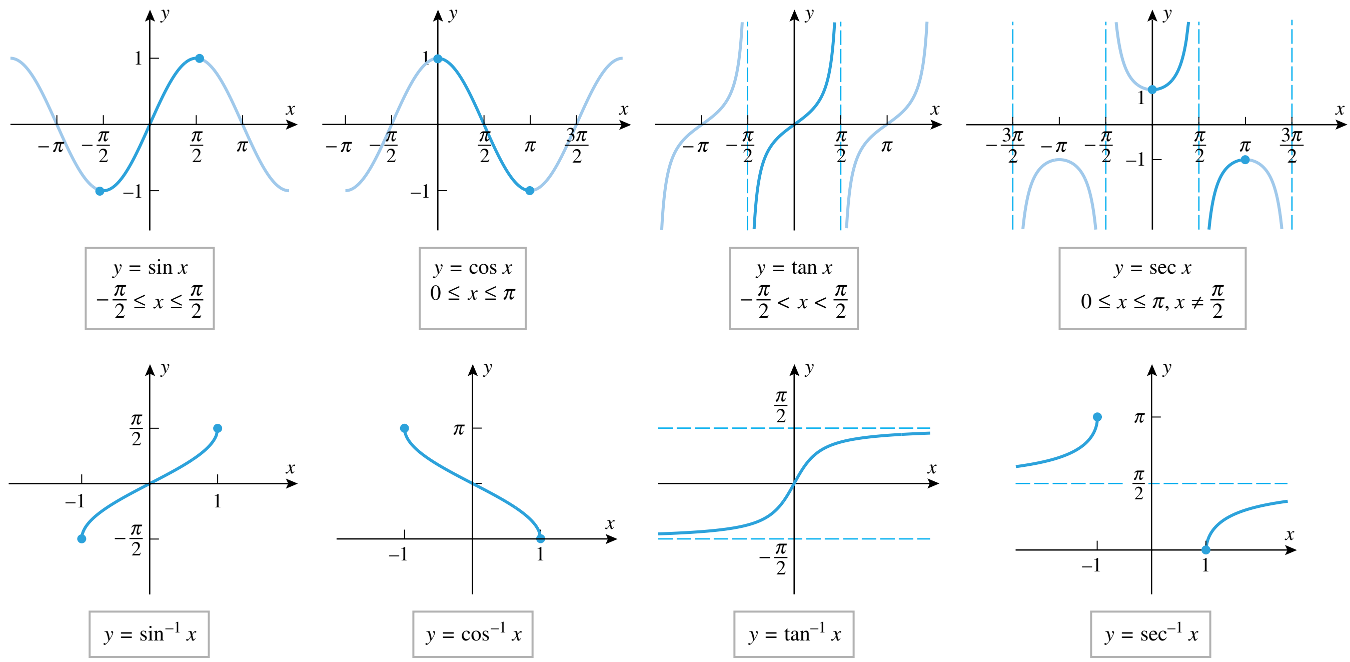


Figure 7.6.1

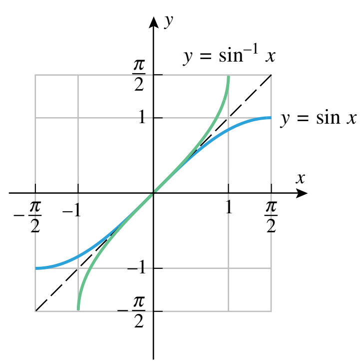


Figure 7.6.2

for a more detailed illustration for the inverse sine. It may also help to keep in mind that reflection about $y = x$ converts vertical lines to horizontal lines, and vice versa, and that x-intercepts reflect into y-intercepts, and vice versa.

Table 7.6.1 summarizes the basic properties of the inverse sine, cosine, tangent, and secant functions. You should confirm that the domains and ranges listed in this table are consistent with the graphs in the bottom part of Figure 7.6.1.

Table 7.6.1

FUNCTION	DOMAIN	RANGE	BASIC RELATIONSHIPS
$\sin^{-1}$	$[-1, 1]$	$[-\pi/2, \pi/2]$	$\sin^{-1}(\sin x) = x$ if $-\pi/2 \le x \le \pi/2$ $\sin(\sin^{-1} x) = x$ if $-1 \le x \le 1$
$\cos^{-1}$	$[-1, 1]$	$[0, \pi]$	$\cos^{-1}(\cos x) = x$ if $0 \le x \le \pi$ $\cos(\cos^{-1} x) = x$ if $-1 \le x \le 1$
$\tan^{-1}$	$(-\infty, +\infty)$	$(-\pi/2, \pi/2)$	$\tan^{-1}(\tan x) = x$ if $-\pi/2 < x < \pi/2$ $\tan(\tan^{-1} x) = x$ if $-\infty < x < +\infty$
$\sec^{-1}$	$(-\infty, -1] \cup [1, +\infty)$	$[0, \pi/2) \cup (\pi/2, \pi]$	$\sec^{-1}(\sec x) = x$ if $0 \le x \le \pi, x \ne \pi/2$ $\sec(\sec^{-1} x) = x$ if $\lvert x\rvert \ge 1$

EVALUATING INVERSE TRIGONOMETRIC FUNCTIONS

A common problem in trigonometry is to find an angle whose sine is known. For example, you might want to find an angle x in radian measure such that

$$\sin x = \tfrac{1}{2} \tag{1}$$

and, more generally, for a given value of y in the interval $-1 \le y \le 1$ you might want to solve the equation

$$\sin x = y \tag{2}$$

Because $\sin x$ repeats periodically, such equations have infinitely many solutions for x; however, if we solve this equation as

$$x = \sin^{-1} y$$

then we isolate the specific solution that lies in the interval $[-\pi/2, \pi/2]$, since this is the range of the inverse sine. For example, Figure 7.6.3 shows four solutions of Equation (1), namely, $-11\pi/6$, $-7\pi/6$, $\pi/6$, and $5\pi/6$. Of these, $\pi/6$ is the solution in the interval $[-\pi/2, \pi/2]$, so

$$\sin^{-1}\left(\tfrac{1}{2}\right) = \pi/6 \tag{3}$$

FOR THE READER. Refer to the documentation for your calculating utility to determine how to calculate inverse sines, inverse cosines, and inverse tangents; and then confirm Equation (3) numerically by showing that

$$\sin^{-1}(0.5) \approx 0.523598775598\ldots \approx \pi/6$$

In general, if we view $x = \sin^{-1} y$ as an angle in radian measure whose sine is y, then the restriction $-\pi/2 \le \theta \le \pi/2$ imposes the geometric requirement that the angle x terminate in either the first or fourth quadrant or on an axis adjacent to those quadrants.

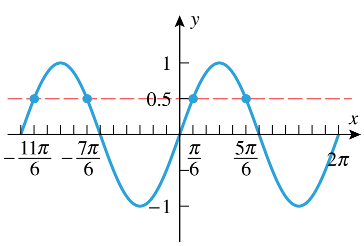


Figure 7.6.3

Example 1 Find exact values of

(a) $\sin^{-1}(1/\sqrt{2}\,)$ (b) $\sin^{-1}(-1)$

by inspection, and confirm your results numerically using a calculating utility.

Solution (a). Because $\sin^{-1}(1/\sqrt{2}\,) > 0$, we can view $x = \sin^{-1}(1/\sqrt{2}\,)$ as that angle in the first quadrant such that $\sin\theta = 1/\sqrt{2}$. Thus, $\sin^{-1}(1/\sqrt{2}\,) = \pi/4$. You can confirm this with your calculating utility by showing that $\sin^{-1}(1/\sqrt{2}\,) \approx 0.785 \approx \pi/4$.

Solution (b). Because $\sin^{-1}(-1) < 0$, we can view $x = \sin^{-1}(-1)$ as an angle in the fourth quadrant (or an adjacent axis) such that $\sin x = -1$. Thus, $\sin^{-1}(-1) = -\pi/2$. You can confirm this with your calculating utility by showing that $\sin^{-1}(-1) \approx -1.57 \approx -\pi/2$. ◀

FOR THE READER. If $x = \cos^{-1} y$ is viewed as an angle in radian measure whose cosine is y, in what possible quadrants can x lie? Answer the same question for $x = \tan^{-1} y$ and $x = \sec^{-1} y$.

FOR THE READER. Most calculators do not provide a direct method for calculating inverse secants. In such situations the identity

$$\sec^{-1} x = \cos^{-1}(1/x) \tag{4}$$

is useful (Exercise 16). Use this formula to show that

$$\sec^{-1}(2.25) \approx 1.11 \quad \text{and} \quad \sec^{-1}(-2.25) \approx 2.03$$

If you have a calculating utility (such as a CAS) that can find $\sec^{-1} x$ directly, use it to check these values.

IDENTITIES FOR INVERSE TRIGONOMETRIC FUNCTIONS

If we interpret $\sin^{-1} x$ as an angle in radian measure whose sine is x, and if that angle is *nonnegative*, then we can represent $\sin^{-1} x$ geometrically as an angle in a right triangle in which the hypotenuse has length 1 and the side opposite to the angle $\sin^{-1} x$ has length x (Figure 7.6.4*a*). By the Theorem of Pythagoras the side adjacent to the angle $\sin^{-1} x$ has length $\sqrt{1-x^2}$. Moreover, the third angle in Figure 7.6.4*a* is $\cos^{-1} x$, since the cosine of that angle is x (Figure 7.6.4*b*). This triangle motivates a number of useful identities involving inverse trigonometric functions that are valid for $-1 \le x \le 1$; for example,

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2} \tag{5}$$

$$\cos(\sin^{-1} x) = \sqrt{1-x^2} \tag{6}$$

$$\sin(\cos^{-1} x) = \sqrt{1-x^2} \tag{7}$$

$$\tan(\sin^{-1} x) = \frac{x}{\sqrt{1-x^2}} \tag{8}$$

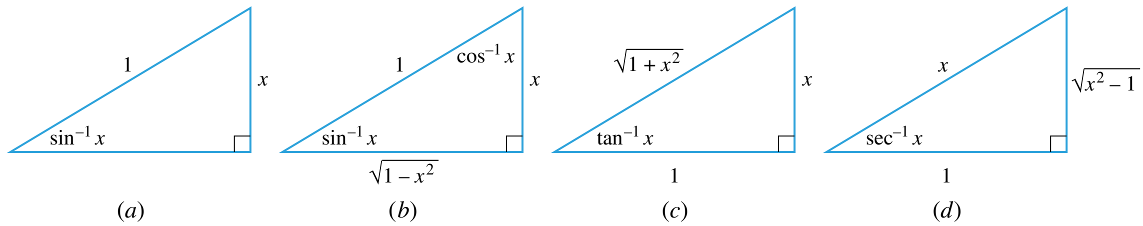


Figure 7.6.4

In a similar manner, $\tan^{-1} x$ and $\sec^{-1} x$ can be represented as angles in the right triangles shown in Figures 7.6.4*c* and 7.6.4*d* (verify). Those triangles reveal more useful identities; for example,

$$\sec(\tan^{-1} x) = \sqrt{1+x^2} \tag{9}$$

$$\sin(\sec^{-1} x) = \frac{\sqrt{x^2-1}}{x} \qquad (x \geq 1) \tag{10a}$$

REMARK. We leave it as an exercise to use (4) and (7) to obtain the following identity that is valid for $x \geq 1$ and $x \leq -1$ (Exercise 80):

$$\sin(\sec^{-1} x) = \frac{\sqrt{x^2-1}}{|x|} \qquad (|x| \geq 1) \tag{10b}$$

REMARK. There is nothing to be gained by memorizing these identities; what is important to understand is the *method* that was used to obtain them.

Referring to Figure 7.6.1, observe that the inverse sine and inverse tangent are odd functions; that is,

$$\sin^{-1}(-x) = -\sin^{-1}(x) \quad \text{and} \quad \tan^{-1}(-x) = -\tan^{-1}(x) \tag{11–12}$$

Example 2 Figure 7.6.5 shows a computer-generated graph of $y = \sin^{-1}(\sin x)$. One might think that this graph should be the line $y = x$, since $\sin^{-1}(\sin x) = x$. Why isn't it?

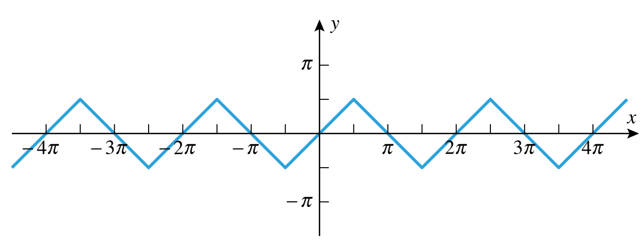


Figure 7.6.5

Solution. The relationship $\sin^{-1}(\sin x) = x$ is valid on the interval $-\pi/2 \leq x \leq \pi/2$, so we can say with certainty that the graphs of $y = \sin^{-1}(\sin x)$ and $y = x$ coincide on this interval (which is confirmed by Figure 7.6.5). However, outside of this interval the relationship $\sin^{-1}(\sin x) = x$ does not hold. For example, if x lies in the interval $\pi/2 \leq x \leq 3\pi/2$, then the quantity $x - \pi$ lies in the interval $-\pi/2 \leq x \leq \pi/2$, so

$$\sin^{-1}[\sin(x-\pi)] = x - \pi$$

Thus, by using the identity $\sin(x - \pi) = -\sin x$ and the fact that $\sin^{-1}$ is an odd function, we can express $\sin^{-1}(\sin x)$ as

$$\sin^{-1}(\sin x) = \sin^{-1}[-\sin(x-\pi)] = -\sin^{-1}[\sin(x-\pi)] = -(x-\pi)$$

This shows that on the interval $\pi/2 \leq x \leq 3\pi/2$ the graph of $y = \sin^{-1}(\sin x)$ coincides with the line $y = -(x - \pi)$, which has slope -1 and an x-intercept at $x = \pi$. This agrees with Figure 7.6.5. ◀

DERIVATIVES OF THE INVERSE TRIGONOMETRIC FUNCTIONS

Recall that if f is a one-to-one function whose derivative is known, then there are two basic ways to obtain a derivative formula for $f^{-1}(x)$—we can rewrite the equation $y = f^{-1}(x)$ as $x = f(y)$, and differentiate implicitly, or we can apply Formula (4) or (5) of Section 7.1. Here we will use implicit differentiation to obtain the derivative formula for $y = \sin^{-1} x$. Rewriting this equation as $x = \sin y$ and differentiating implicitly, we obtain

$$\frac{d}{dx}[x] = \frac{d}{dx}[\sin y]$$

$$1 = \cos y \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\cos(\sin^{-1} x)}$$

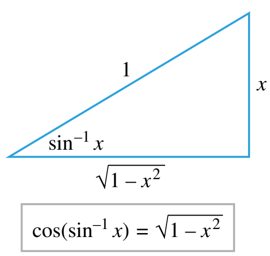


Figure 7.6.6

At this point we have succeeded in obtaining the derivative; however, this derivative formula can be simplified by applying Formula (6), which is derived from the triangle in Figure 7.6.6. This yields

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

Thus, we have shown that

$$\frac{d}{dx}[\sin^{-1} x] = \frac{1}{\sqrt{1-x^2}} \qquad (-1 < x < 1) \tag{13}$$

If u is a differentiable function of x, then (13) and the chain rule produce the following generalized derivative formula:

$$\frac{d}{dx}[\sin^{-1} u] = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad (-1 < u < 1) \tag{14}$$

The method used to obtain this formula can also be used to obtain generalized derivative formulas for the other inverse trigonometric functions. These formulas, which are valid for $-1 < u < 1$, are

$$\frac{d}{dx}[\sin^{-1} u] = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad (-1 < u < 1) \tag{15}$$

$$\frac{d}{dx}[\tan^{-1} u] = \frac{1}{1+u^2}\frac{du}{dx} \qquad (-\infty < u < +\infty) \tag{16}$$

$$\frac{d}{dx}[\sec^{-1} u] = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx} \qquad (1 < |u|) \tag{17}$$

DIFFERENTIABILITY OF THE INVERSE TRIGONOMETRIC FUNCTIONS

In the derivation of (13) we *assumed* that $\sin^{-1} x$ is differentiable. However, we can establish the differentiability with the help of Theorem 7.1.6. Since $f(x) = \sin x$ and $f'(x) = \cos x$, it follows from that theorem that the function $f^{-1}(x) = \sin^{-1} x$ will be differentiable at any value of x where $\cos(\sin^{-1} x) \neq 0$ or from (6) where $\sqrt{1-x^2} \neq 0$. Thus, $\sin^{-1} x$ is differentiable on the interval $(-1, 1)$. The differentiability of the remaining inverse trigonometric functions can be deduced similarly.

REMARK. Observe that $\sin^{-1} x$ is only differentiable on the interval $(-1, 1)$, even though its domain is $[-1, 1]$. However, it can be seen geometrically that $\sin^{-1}$ cannot be differentiable at $x = \pm 1$. Just observe that the graph of $y = \sin x$ has horizontal tangent lines at $(\pi/2, 1)$ and $(-\pi/2, -1)$ and that these become points of vertical tangency for $y = \sin^{-1} x$ when reflected around the line $y = x$.

Example 3 Find dy/dx if

(a) $y = \sin^{-1}(x^3)$ (b) $y = \sec^{-1}(e^x)$

Solution (a). From (14)

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(x^3)^2}}(3x^2) = \frac{3x^2}{\sqrt{1-x^6}}$$

Solution (b). From (17)

$$\frac{dy}{dx} = \frac{1}{e^x\sqrt{(e^x)^2-1}}(e^x) = \frac{1}{\sqrt{e^{2x}-1}} \quad ◀$$

INTEGRATION FORMULAS

Differentiation formulas (14)–(17) yield useful integration formulas. Those most commonly needed are

$$\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C \tag{18}$$

$$\int \frac{du}{1+u^2} = \tan^{-1} u + C \tag{19}$$

$$\int \frac{du}{|u|\sqrt{u^2-1}} = \sec^{-1} u + C \tag{20}$$

Example 4 Evaluate $\displaystyle\int \frac{dx}{1+3x^2}$.

Solution. Substituting

$$u = \sqrt{3}x, \quad du = \sqrt{3}\,dx$$

yields

$$\int \frac{dx}{1+3x^2} = \frac{1}{\sqrt{3}}\int \frac{du}{1+u^2} = \frac{1}{\sqrt{3}}\tan^{-1} u + C = \frac{1}{\sqrt{3}}\tan^{-1}(\sqrt{3}x) + C \quad ◀$$

Example 5 Evaluate $\displaystyle\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$.

Solution. Substituting

$$u = e^x, \quad du = e^x\,dx$$

yields

$$\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx = \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C = \sin^{-1}(e^x) + C \quad ◀$$

Example 6 Evaluate $\displaystyle\int \frac{dx}{a^2+x^2}$, where $a \neq 0$ is a constant.

Solution. Some simple algebra and an appropriate u-substitution will allow us to use (19).

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\int \frac{\frac{dx}{a}}{1+\left(\frac{x}{a}\right)^2} = \frac{1}{a}\int \frac{du}{1+u^2} = \frac{1}{a}\tan^{-1} u + C = \frac{1}{a}\tan^{-1}\frac{x}{a} + C \quad ◀$$

$$\boxed{\begin{aligned} u &= \tfrac{x}{a} \\ du &= \tfrac{dx}{a} \end{aligned}}$$

The method of Example 6 leads to the following generalizations of (18), (19), and (20) for $a > 0$:

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a} + C \tag{21}$$

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C \tag{22}$$

$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a} + C \tag{23}$$

Example 7 Evaluate $\displaystyle\int \frac{dx}{\sqrt{2 - x^2}}$.

Solution. Applying (21) with $u = x$ and $a = \sqrt{2}$ yields

$$\int \frac{dx}{\sqrt{2 - x^2}} = \sin^{-1}\frac{x}{\sqrt{2}} + C$$ ◀

EXERCISE SET 7.6 Graphing Utility C CAS

1. Find the exact value of
(a) $\sin^{-1}(-1)$ (b) $\cos^{-1}(-1)$
(c) $\tan^{-1}(-1)$ (d) $\sec^{-1}(1)$.

2. Find the exact value of
(a) $\sin^{-1}\left(\frac{1}{2}\sqrt{3}\right)$ (b) $\cos^{-1}\left(\frac{1}{2}\right)$
(c) $\tan^{-1}(1)$ (d) $\sec^{-1}(-2)$.

3. Given that $\theta = \sin^{-1}\left(-\frac{1}{2}\sqrt{3}\right)$, find the exact values of $\cos\theta$, $\tan\theta$, $\cot\theta$, $\sec\theta$, and $\csc\theta$.

4. Given that $\theta = \cos^{-1}\left(\frac{1}{2}\right)$, find the exact values of $\sin\theta$, $\tan\theta$, $\cot\theta$, $\sec\theta$, and $\csc\theta$.

5. Given that $\theta = \tan^{-1}\left(\frac{4}{3}\right)$, find the exact values of $\sin\theta$, $\cos\theta$, $\cot\theta$, $\sec\theta$, and $\csc\theta$.

6. Given that $\theta = \sec^{-1} 2.6$, find the exact values of $\sin\theta$, $\cos\theta$, $\tan\theta$, $\cot\theta$, and $\csc\theta$.

7. Find the exact value of
(a) $\sin^{-1}(\sin \pi/7)$ (b) $\sin^{-1}(\sin \pi)$
(c) $\sin^{-1}(\sin 5\pi/7)$ (d) $\sin^{-1}(\sin 630)$.

8. Find the exact value of
(a) $\cos^{-1}(\cos \pi/7)$ (b) $\cos^{-1}(\cos \pi)$
(c) $\cos^{-1}(\cos 12\pi/7)$ (d) $\cos^{-1}(\cos 200)$.

9. For which values of x is it true that
(a) $\cos^{-1}(\cos x) = x$ (b) $\cos(\cos^{-1} x) = x$
(c) $\tan^{-1}(\tan x) = x$ (d) $\tan(\tan^{-1} x) = x$

In Exercises 10 and 11, find the exact value of the given quantity.

10. $\sec\left[\sin^{-1}\left(-\frac{3}{4}\right)\right]$

11. $\sin\left[2\cos^{-1}\left(\frac{3}{5}\right)\right]$

In Exercises 12 and 13, complete the identities using the triangle method (Figure 7.6.4).

12. (a) $\sin(\cos^{-1} x) = ?$ (b) $\tan(\cos^{-1} x) = ?$
(c) $\csc(\tan^{-1} x) = ?$ (d) $\sin(\tan^{-1} x) = ?$

13. (a) $\cos(\tan^{-1} x) = ?$ (b) $\tan(\cos^{-1} x) = ?$
(c) $\sin(\sec^{-1} x) = ?$ (d) $\cot(\sec^{-1} x) = ?$

14. (a) Use a calculating utility set to radian measure to make tables of values of $y = \sin^{-1} x$ and $y = \cos^{-1} x$ for $x = -1, -0.8, -0.6, \ldots, 0, 0.2, \ldots, 1$. Round your answers to two decimal places.
(b) Plot the points obtained in part (a), and use the points to sketch the graphs of $y = \sin^{-1} x$ and $y = \cos^{-1} x$. Confirm that your sketches agree with those in Figure 7.6.1.
(c) Use your graphing utility to graph $y = \sin^{-1} x$ and $y = \cos^{-1} x$; confirm that the graphs agree with those in Figure 7.6.1.

The function $\cot^{-1} x$ is defined to be the inverse of the restricted cotangent function

$\cot x, \quad 0 < x < \pi$

and the function $\csc^{-1} x$ is defined to be the inverse of the restricted cosecant function

$\csc x, \quad -\pi/2 < x < \pi/2, \quad x \neq 0$

Use these definitions in Exercises 15 and 16 and in all subsequent exercises that involve these functions.

15. (a) Sketch the graphs of $\cot^{-1} x$ and $\csc^{-1} x$.
(b) Find the domain and range of $\cot^{-1} x$ and $\csc^{-1} x$.

16. Show that

(a) $\cot^{-1} x = \begin{cases} \tan^{-1}(1/x), & \text{if } x > 0 \\ \pi + \tan^{-1}(1/x), & \text{if } x < 0 \end{cases}$

(b) $\sec^{-1} x = \cos^{-1} \dfrac{1}{x}$, if $|x| \geq 1$

(c) $\csc^{-1} x = \sin^{-1} \dfrac{1}{x}$, if $|x| \geq 1$.

17. Most scientific calculators have keys for the values of only $\sin^{-1} x$, $\cos^{-1} x$, and $\tan^{-1} x$. The formulas in Exercise 16 show how a calculator can be used to obtain values of $\cot^{-1} x$, $\sec^{-1} x$, and $\csc^{-1} x$ for positive values of x. Use these formulas and a calculator to find numerical values for each of the following inverse trigonometric functions. Express your answers in degrees, rounded to the nearest tenth of a degree.
(a) $\cot^{-1} 0.7$ (b) $\sec^{-1} 1.2$ (c) $\csc^{-1} 2.3$

18. (a) Use Theorem 7.1.6 to prove that

$$\frac{d}{dx}[\cot^{-1} x]\bigg|_{x=0} = -1$$

(b) Use part (a) above, part (a) of Exercise 16, and the chain rule to show that

$$\frac{d}{dx}[\cot^{-1} x] = -\frac{1}{\sqrt{1+x^2}}$$

for $-\infty < x < +\infty$.

(c) Conclude from (b) that

$$\frac{d}{dx}[\cot^{-1} u] = -\frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$$

for $-\infty < u < +\infty$.

19. (a) Use part (c) of Exercise 16, and the chain rule to show that

$$\frac{d}{dx}[\csc^{-1} x] = -\frac{1}{|x|\sqrt{x^2-1}}$$

for $1 < |x|$.

(b) Conclude from (a) that

$$\frac{d}{dx}[\csc^{-1} u] = -\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$$

for $1 < |u|$.

In Exercises 20–22, use a calculating utility to approximate the solution of each equation. Where radians are used, express your answer to four decimal places, and where degrees are used, express it to the nearest tenth of a degree. [*Note:* In each part, the solution is not in the range of the relevant inverse trigonometric function.]

20. (a) $\sin x = 0.37$, $\pi/2 < x < \pi$
(b) $\sin\theta = -0.61$, $180° < \theta < 270°$

21. (a) $\cos x = -0.85$, $\pi < x < 3\pi/2$
(b) $\cos\theta = 0.23$, $-90° < \theta < 0°$

22. (a) $\tan x = 3.16$, $-\pi < x < -\pi/2$
(b) $\tan\theta = -0.45$, $90° < \theta < 180°$

In Exercises 23–30, find dy/dx.

23. (a) $y = \sin^{-1}\left(\frac{1}{3}x\right)$ (b) $y = \cos^{-1}(2x+1)$

24. (a) $y = \tan^{-1}(x^2)$ (b) $y = \cot^{-1}(\sqrt{x})$

25. (a) $y = \sec^{-1}(x^7)$ (b) $y = \csc^{-1}(e^x)$

26. (a) $y = (\tan x)^{-1}$ (b) $y = \dfrac{1}{\tan^{-1} x}$

27. (a) $y = \sin^{-1}(1/x)$ (b) $y = \cos^{-1}(\cos x)$

28. (a) $y = \ln(\cos^{-1} x)$ (b) $y = \sqrt{\cot^{-1} x}$

29. (a) $y = e^x \sec^{-1} x$ (b) $y = x^2\left(\sin^{-1} x\right)^3$

30. (a) $y = \sin^{-1} x + \cos^{-1} x$ (b) $y = \sec^{-1} x + \csc^{-1} x$

In Exercises 31 and 32, find dy/dx by implicit differentiation.

31. $x^3 + x\tan^{-1} y = e^y$

32. $\sin^{-1}(xy) = \cos^{-1}(x-y)$

In Exercises 33–46, evaluate the integral.

33. $\displaystyle\int_0^{1/\sqrt{2}} \frac{dx}{\sqrt{1-x^2}}$

34. $\displaystyle\int \frac{dx}{\sqrt{1-4x^2}}$

35. $\displaystyle\int_{-1}^{1} \frac{dx}{1+x^2}$

36. $\displaystyle\int \frac{dx}{1+16x^2}$

37. $\displaystyle\int_{\sqrt{2}}^{2} \frac{dx}{x\sqrt{x^2-1}}$

38. $\displaystyle\int_{-\sqrt{2}}^{-2/\sqrt{3}} \frac{dx}{x\sqrt{x^2-1}}$

39. $\displaystyle\int \frac{\sec^2 x\,dx}{\sqrt{1-\tan^2 x}}$

40. $\displaystyle\int_{\ln 2}^{\ln(2/\sqrt{3})} \frac{e^{-x}\,dx}{\sqrt{1-e^{-2x}}}$

41. $\displaystyle\int \frac{e^x}{1+e^{2x}}\,dx$

42. $\displaystyle\int \frac{t}{t^4+1}\,dt$

43. $\displaystyle\int_1^3 \frac{dx}{\sqrt{x}\,(x+1)}$

44. $\displaystyle\int \frac{\sin\theta}{\cos^2\theta+1}\,d\theta$

45. $\displaystyle\int \frac{dx}{x\sqrt{1-(\ln x)^2}}$

46. $\displaystyle\int \frac{dx}{x\sqrt{9x^2-1}}$

47. Derive integration Formula (21).

48. Derive integration Formula (23).

In Exercises 49–54, use Formulas (21), (22), and (23) to evaluate the integrals.

49. (a) $\displaystyle\int \frac{dx}{\sqrt{9-x^2}}$ (b) $\displaystyle\int \frac{dx}{5+x^2}$ (c) $\displaystyle\int \frac{dx}{x\sqrt{x^2-\pi}}$

50. (a) $\displaystyle\int \frac{e^x}{4+e^{2x}}\,dx$ (b) $\displaystyle\int \frac{dx}{\sqrt{9-4x^2}}$ (c) $\displaystyle\int \frac{dy}{y\sqrt{5y^2-3}}$

51. $\displaystyle\int_0^1 \frac{x}{\sqrt{4-3x^4}}\,dx$ **52.** $\displaystyle\int_1^2 \frac{1}{\sqrt{x}\sqrt{4-x}}\,dx$

53. $\displaystyle\int_0^{2/\sqrt{3}} \frac{1}{4+9x^2}\,dx$ **54.** $\displaystyle\int_1^{\sqrt{2}} \frac{x}{3+x^4}\,dx$

55. In each part, sketch the graph and check your work with a graphing utility.
(a) $y = \sin^{-1} 2x$ (b) $y = \tan^{-1} \frac{1}{2}x$

56. (a) Use a calculating utility to evaluate $\sin^{-1}(\sin^{-1} 0.25)$ and $\sin^{-1}(\sin^{-1} 0.9)$, and explain what you think is happening in the second calculation.
(b) For what values of x in the interval $-1 \le x \le 1$ will your calculating utility produce a real value for the function $\sin^{-1}(\sin^{-1} x)$?

57. An Earth-observing satellite has horizon sensors that can measure the angle θ shown in the accompanying figure. Let R be the radius of the Earth (assumed spherical) and h the distance between the satellite and the Earth's surface.
(a) Show that $\sin\theta = \dfrac{R}{R+h}$.
(b) Find θ, to the nearest degree, for a satellite that is 10,000 km from the Earth's surface (use $R = 6378$ km).

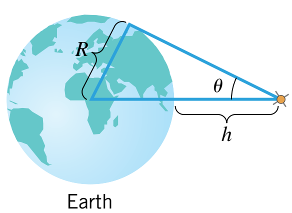


Figure Ex-57

58. The number of hours of daylight on a given day at a given point on the Earth's surface depends on the latitude λ of the point, the angle γ through which the Earth has moved in its orbital plane during the time period from the vernal equinox (March 21), and the angle of inclination ϕ of the Earth's axis of rotation measured from ecliptic north ($\phi \approx 23.45^\circ$). The number of hours of daylight h can be approximated by the formula

$$h = \begin{cases} 24, & D \ge 1 \\ 12 + \frac{2}{15}\sin^{-1} D, & |D| < 1 \\ 0, & D \le -1 \end{cases}$$

where

$$D = \frac{\sin\phi \sin\gamma \tan\lambda}{\sqrt{1 - \sin^2\phi \sin^2\gamma}}$$

and $\sin^{-1} D$ is in degree measure. Given that Fairbanks, Alaska, is located at a latitude of $\lambda = 65^\circ$ N and also that $\gamma = 90^\circ$ on June 20 and $\gamma = 270^\circ$ on December 20, approximate
(a) the maximum number of daylight hours at Fairbanks to one decimal place
(b) the minimum number of daylight hours at Fairbanks to one decimal place.

[*Note:* This problem was adapted from *TEAM, A Path to Applied Mathematics*, The Mathematical Association of America, Washington, D.C., 1985.]

59. A soccer player kicks a ball with an initial speed of 14 m/s at an angle θ with the horizontal (see the accompanying figure). The ball lands 18 m down the field. If air resistance is neglected, then the ball will have a parabolic trajectory and the horizontal range R will be given by

$$R = \frac{v^2}{g}\sin 2\theta$$

where v is the initial speed of the ball and g is the acceleration due to gravity. Using $g = 9.8\ \text{m}/s^2$, approximate two values of θ, to the nearest degree, at which the ball could have been kicked. Which angle results in the shorter time of flight? Why?

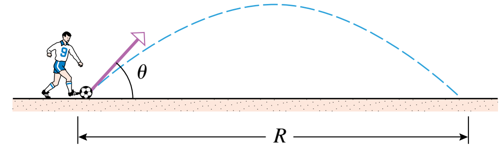


Figure Ex-59

60. The ***law of cosines*** states that

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

where a, b, and c are the lengths of the sides of a triangle and θ is the angle formed by sides a and b. Find θ, to the nearest degree, for the triangle with $a = 2$, $b = 3$, and $c = 4$.

61. An airplane is flying at a constant height of 3000 ft above water at a speed of 400 ft/s. The pilot is to release a survival package so that it lands in the water at a sighted point P. If air resistance is neglected, then the package will follow a parabolic trajectory whose equation relative to the coordinate system in the accompanying figure is

$$y = 3000 - \frac{g}{2v^2}x^2$$

where g is the acceleration due to gravity and v is the speed of the airplane. Using $g = 32\ \text{ft/s}^2$, find the "line of sight" angle θ, to the nearest degree, that will result in the package hitting the target point.

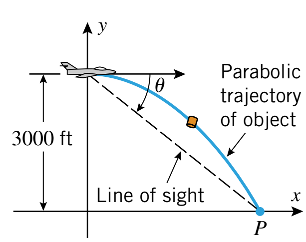


Figure Ex-61

62. (a) A camera is positioned x feet from the base of a missile launching pad (see the accompanying figure). If a missile of length a feet is launched vertically, show that when the base of the missile is b feet above the camera lens, the angle θ subtended at the lens by the missile is

$$\theta = \cot^{-1}\frac{x}{a+b} - \cot^{-1}\frac{x}{b}$$

(b) How far from the launching pad should the camera be positioned to maximize the angle θ subtended at the lens by the missile?

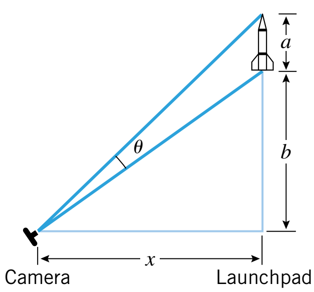


Figure Ex-62

63. A student wants to find the area enclosed by the graphs of $y = 1/\sqrt{1-x^2}$, $y = 0$, $x = 0$, and $x = 0.8$.
(a) Show that the exact area is $\sin^{-1} 0.8$.
(b) The student uses a calculator to approximate the result in part (a) to two decimal places and obtains an incorrect answer of 53.13. What was the student's error? Find the correct approximation.

64. Find the area of the region enclosed by the graphs of $y = 1/\sqrt{1-9x^2}$, $y = 0$, $x = 0$, and $x = 1/6$.

65. Estimate the value of k $(0 < k < 1)$ so that the region enclosed by $y = 1/\sqrt{1-x^2}$, $y = x$, $x = 0$, and $x = k$ has an area of 1 square unit.

66. Find the area of the region enclosed by the graphs of $y = \sin^{-1} x$, $x = 0$, and $y = \pi/2$.

67. Estimate the area of the region in the first quadrant enclosed by $y = \sin 2x$ and $y = \sin^{-1} x$.

68. Suppose that a particle moves along a line so that its velocity v at time t is given by

$$v(t) = \frac{3}{t^2+1} - 0.5t, \quad t \geq 0$$

where t is in seconds and v is in centimeters per second (cm/sec). Estimate the times at which the particle is 2 cm from its starting position.

69. Find the volume of the solid generated when the region bounded by $x = 2$, $x = -2$, $y = 0$, and $y = 1/\sqrt{4+x^2}$ is revolved about the x-axis.

70. (a) Find the volume V of the solid generated when the region bounded by $y = 1/(1+x^4)$, $y = 0$, $x = 1$, and $x = b$ $(b > 1)$ is revolved about the y-axis.
(b) Find $\lim_{b\to+\infty} V$.

71. Estimate the value of k $(k > 0)$ so that the region enclosed by $y = 1/(1+kx^2)$, $y = 0$, $x = 0$, and $x = 2$ has an area of 0.6 square unit.

C **72.** Consider the region enclosed by $y = \sin^{-1} x$, $y = 0$, and $x = 1$. Find the volume of the solid generated by revolving the region about the x-axis using
(a) disks (b) cylindrical shells.

73. Given points $A(2, 1)$ and $B(5, 4)$, find the point P in the interval $[2, 5]$ on the x-axis that maximizes angle APB.

74. The lower edge of a painting, 10 ft in height, is 2 ft above an observer's eye level. Assuming that the best view is obtained when the angle subtended at the observer's eye by the painting is maximum, how far from the wall should the observer stand?

75. Use Theorem 4.8.2 (the Mean-Value Theorem) to prove that

$$\frac{x}{1+x^2} < \tan^{-1} x < x \quad (x > 0)$$

76. Find $\lim_{n\to+\infty} \sum_{k=1}^{n} \frac{n}{n^2+k^2}$. [*Hint:* Interpret this as the limit of a Riemann sum in which the interval [0, 1] is divided into n subintervals of equal width.]

77. Prove:
(a) $\sin^{-1}(-x) = -\sin^{-1} x$
(b) $\tan^{-1}(-x) = -\tan^{-1} x$.

78. Prove:
(a) $\cos^{-1}(-x) = \pi - \cos^{-1} x$
(b) $\sec^{-1}(-x) = \pi - \sec^{-1} x$

79. Prove:
(a) $\sin^{-1} x = \tan^{-1}\dfrac{x}{\sqrt{1-x^2}} \quad (|x| < 1)$
(b) $\cos^{-1} x = \dfrac{\pi}{2} - \tan^{-1}\dfrac{x}{\sqrt{1-x^2}} \quad (|x| < 1)$

80. Prove:

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

provided $-\pi/2 < \tan^{-1} x + \tan^{-1} y < \pi/2$. [*Hint:* Use an identity for $\tan(\alpha+\beta)$.]

81. Use the result in Exercise 80 to show that
(a) $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \pi/4$
(b) $2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} = \pi/4$.

82. Use identities (4) and (7) to obtain identity (10b).

7.7 L'HÔPITAL'S RULE; INDETERMINATE FORMS

In this section we will discuss a general method for using derivatives to find limits. This method will enable us to establish limits with certainty that earlier in the text we were only able to conjecture using numerical or graphical evidence. The method that we will discuss in this section is an extremely powerful tool that is used internally by many computer programs to calculate limits of various types.

INDETERMINATE FORMS OF TYPE 0/0

In earlier sections we discussed limits that can be determined by inspection or by some appropriate algebraic manipulation. Two special exceptions to this were the limits in Theorem 2.6.3,

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \quad \text{and} \quad \lim_{x\to 0} \frac{1-\cos x}{x} = 0 \tag{1–2}$$

Equation (1) was shown using the Squeezing Theorem (2.6.2) and some careful manipulation of inequalities, and Equation (2) then followed using the identity $\sin^2 x + \cos^2 x = 1$. These in turn were used in Section 3.4 to derive the derivatives of the sine and cosine functions. Equations (1) and (2) are really special cases of these derivatives, as can be seen by

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\sin x - \sin 0}{x-0} = \frac{d}{dx}(\sin x)\bigg|_{x=0} = \cos 0 = 1$$

and

$$\lim_{x\to 0} \frac{1-\cos x}{x} = \lim_{x\to 0} -\frac{\cos x - 1}{x} = -\left(\lim_{x\to 0} \frac{\cos x - \cos 0}{x-0}\right)$$
$$= -\left(\frac{d}{dx}(\cos x)\bigg|_{x=0}\right) = \sin 0 = 0$$

What makes the limits in (1) and (2) bothersome is the fact that the numerator and denominator both approach 0 as $x \to 0$. Such limits are called ***indeterminate forms of type* 0/0**. As illustrated above, the definition of a derivative provides an important class of examples of indeterminate forms of type 0/0. Our goal here is to develop a general method, based on the derivative, for evaluating indeterminate forms.

L'HÔPITAL'S RULE

Consider the limit

$$\lim_{x\to 0} \frac{e^{2x}-1}{\sin x} \tag{3}$$

Unlike (1) and (2), the limit in (3) is not easily seen as the evaluation of the derivative of a function at $x = 0$. However, (3) can be expressed as the ratio of two derivatives.

$$\lim_{x\to 0} \frac{e^{2x}-1}{\sin x} = \lim_{x\to 0} \frac{(e^{2x}-e^{2\cdot 0})/(x-0)}{(\sin x - \sin 0)/(x-0)} = \frac{\dfrac{d}{dx}(e^{2x})\bigg|_{x=0}}{\dfrac{d}{dx}(\sin x)\bigg|_{x=0}} = \frac{2e^{(2\cdot 0)}}{\cos 0} = 2 \tag{4}$$

The method of (4) can be stated more generally. Suppose that f and g are differentiable functions at $x = a$ and that

$$\lim_{x\to a} \frac{f(x)}{g(x)} \tag{5}$$

is an indeterminate form of type 0/0, that is

$$\lim_{x\to a} f(x) = 0 \quad \text{and} \quad \lim_{x\to a} g(x) = 0 \tag{6}$$

In particular, the differentiability of f and g at $x = a$ implies that f and g are continuous at $x = a$, and hence from (6)

$$f(a) = \lim_{x\to a} f(x) = 0 \quad \text{and} \quad g(a) = \lim_{x\to a} g(x) = 0$$

Further, since f and g are differentiable at $x = a$,

$$\lim_{x \to a} \frac{f'(x)}{x - a} = \lim_{x \to a} \frac{f'(x) - f(a)}{x - a} = f'(a)$$

and

$$\lim_{x \to a} \frac{g(x)}{x - a} = \lim_{x \to a} \frac{g(x) - g(a)}{x - a} = g'(a)$$

If $g'(a) \neq 0$ then the indeterminate form in (5) can be evaluated as the ratio of derivative values.

$$\lim_{x \to a} \frac{f'(x)}{g(x)} = \lim_{x \to a} \frac{f(x)/(x-a)}{g(x)/(x-a)} = \frac{\displaystyle\lim_{x \to a} \frac{f(x) - f(a)}{x - a}}{\displaystyle\lim_{x \to a} \frac{g(x) - g(a)}{x - a}} = \frac{f'(a)}{g'(a)} \tag{7}$$

If $f'(x)$ and $g'(x)$ are continuous at $x = a$, the result in (7) is a special case of L'Hôpital's* ***rule***, which converts an indeterminate form of type $0/0$ into a new limit involving derivatives. Moreover, L'Hôpital's rule is also true for limits at $-\infty$ and at $+\infty$. We state this result in Theorem 7.7.1, but omit the proof.

7.7.1 THEOREM **(*L'Hôpital's Rule for Form* 0/0).** *Suppose that f and g are differentiable functions on an open interval containing $x = a$, except possibly at $x = a$, and that*

$$\lim_{x \to a} f(x) = 0 \quad \textit{and} \quad \lim_{x \to a} g(x) = 0$$

If $\lim_{x \to a} [f'(x)/g'(x)]$ has a finite limit or if this limit is $+\infty$ or $-\infty$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Moreover, this statement is also true in the case of a limit as $x \to a^-$, $x \to a^+$, $x \to -\infty$, or as $x \to +\infty$.

REMARK. Note that in L'Hôpital's rule the numerator and denominator are differentiated separately, which is not the same as differentiating $f(x)/g(x)$.

In the following examples we will apply L'Hôpital's rule using the following three-step process:

Step 1. Check that the limit of $f(x)/g(x)$ is an indeterminate form. If it is not, then L'Hôpital's rule cannot be used.

Step 2. Differentiate f and g separately.

Step 3. Find the limit of $f'(x)/g'(x)$. If this limit is finite, $+\infty$, or $-\infty$, then it is equal to the limit of $f(x)/g(x)$.

*GUILLAUME FRANCOIS ANTOINE DE L'HÔPITAL (1661–1704). French mathematician. L'Hôpital, born to parents of the French high nobility, held the title of Marquis de Sainte-Mesme Comte d'Autrement. He showed mathematical talent quite early and at age 15 solved a difficult problem about cycloids posed by Pascal. As a young man he served briefly as a cavalry officer, but resigned because of nearsightedness. In his own time he gained fame as the author of the first textbook ever published on differential calculus, *L'Analyse des Infiniment Petits pour l'Intelligence des Lignes Courbes* (1696). L'Hôpital's rule appeared for the first time in that book. Actually, L'Hôpital's rule and most of the material in the calculus text were due to John Bernoulli, who was L'Hôpital's teacher. L'Hôpital dropped his plans for a book on integral calculus when Leibniz informed him that he intended to write such a text. L'Hôpital was apparently generous and personable, and his many contacts with major mathematicians provided the vehicle for disseminating major discoveries in calculus throughout Europe.

Example 1 In each part confirm that the limit is an indeterminate form of type 0/0, and evaluate it using L'Hôpital's rule.

(a) $\lim_{x\to 2}\dfrac{x^2-4}{x-2}$ (b) $\lim_{x\to 0}\dfrac{\sin 2x}{x}$ (c) $\lim_{x\to \pi/2}\dfrac{1-\sin x}{\cos x}$ (d) $\lim_{x\to 0}\dfrac{e^x-1}{x^3}$

(e) $\lim_{x\to 0^-}\dfrac{\tan x}{x^2}$ (f) $\lim_{x\to 0}\dfrac{1-\cos x}{x^2}$ (g) $\lim_{x\to +\infty}\dfrac{x^{-4/3}}{\sin(1/x)}$

Solution (a). The numerator and denominator have a limit of 0, so the limit is a 0/0 indeterminate form. Applying L'Hôpital's rule yields

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{\dfrac{d}{dx}[x^2-4]}{\dfrac{d}{dx}[x-2]}=\lim_{x\to 2}\frac{2x}{1}=4$$

This limit can also be recognized as the derivative of $y=x^2$ at $x=2$,

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=\frac{d}{dx}(x^2)\bigg|_{x=2}=2\cdot 2=4$$

Finally, observe that this limit could have been obtained by factoring

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{(x-2)(x+2)}{x-2}=\lim_{x\to 2}(x+2)=4$$

Solution (b). The numerator and denominator have a limit of 0, so the limit is a 0/0 indetermiate form. Applying L'Hôpital's rule yields

$$\lim_{x\to 0}\frac{\sin 2x}{x}=\lim_{x\to 0}\frac{\dfrac{d}{dx}[\sin 2x]}{\dfrac{d}{dx}[x]}=\lim_{x\to 0}\frac{2\cos 2x}{1}=2$$

Observe that this result agrees with that obtained by substitution in Example 2(b) of Section 2.6.

Solution (c). The numerator and denominator have a limit of 0, so the limit is a 0/0 indetermiate form. Applying L'Hôpital's rule yields

$$\lim_{x\to \pi/2}\frac{1-\sin x}{\cos x}=\lim_{x\to \pi/2}\frac{\dfrac{d}{dx}[1-\sin x]}{\dfrac{d}{dx}[\cos x]}=\lim_{x\to \pi/2}\frac{-\cos x}{-\sin x}=\frac{0}{-1}=0$$

Solution (d). The numerator and denominator have a limit of 0, so the limit is a 0/0 indetermiate form. Applying L'Hôpital's rule yields

$$\lim_{x\to 0}\frac{e^x-1}{x^3}=\lim_{x\to 0}\frac{\dfrac{d}{dx}[e^x-1]}{\dfrac{d}{dx}[x^3]}=\lim_{x\to 0}\frac{e^x}{3x^2}=+\infty$$

Solution (e). The numerator and denominator have a limit of 0, so the limit is a 0/0 indetermiate form. Applying L'Hôpital's rule yields

$$\lim_{x\to 0^-}\frac{\tan x}{x^2}=\lim_{x\to 0^-}\frac{\sec^2 x}{2x}=-\infty$$

Solution (f). The numerator and denominator have a limit of 0, so the limit is a $0/0$ indetermiate form. Applying L'Hôpital's rule yields

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{\sin x}{2x}$$

Since the new limit is another indeterminate form of type $0/0$, we apply L'Hôpital's rule again:

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{\sin x}{2x}=\lim_{x\to 0}\frac{\cos x}{2}=\frac{1}{2}$$

Solution (g). The numerator and denominator have a limit of 0, so the limit is a $0/0$ indetermiate form. Applying L'Hôpital's rule yields

$$\lim_{x\to+\infty}\frac{x^{-4/3}}{\sin(1/x)}=\lim_{x\to+\infty}\frac{-\frac{4}{3}x^{-7/3}}{(-1/x^2)\cos(1/x)}=\lim_{x\to+\infty}\frac{\frac{4}{3}x^{-1/3}}{\cos(1/x)}=\frac{0}{1}=0 \qquad \blacktriangleleft$$

WARNING. Applying L'Hôpital's rule to limits that are not indeterminate forms can lead to incorrect results. For example, in the limit

$$\lim_{x\to 0}\frac{x+6}{x+2}=\frac{6}{2}=3$$

the numerator approaches 6 and the denominator approaches 2, so the limit is not an indeterminate form of type $0/0$. However, if we ignore this and blindly apply L'Hôpital's rule, we reach the following *erroneous* conclusion:

$$\lim_{x\to 0}\frac{x+6}{x+2}=\lim_{x\to 0}\frac{\frac{d}{dx}[x+6]}{\frac{d}{dx}[x+2]}=\lim_{x\to 0}\frac{1}{1}=1$$

INDETERMINATE FORMS OF TYPE ∞/∞

When we want to indicate that the limit (or the one-sided limits) of a function are $+\infty$ or $-\infty$ without being specific about the sign, we will say that the limit is ∞. For example,

$$\lim_{x\to a^+} f(x)=\infty \quad\text{means}\quad \lim_{x\to a^+} f(x)=+\infty \quad\text{or}\quad \lim_{x\to a^+} f(x)=-\infty$$

$$\lim_{x\to +\infty} f(x)=\infty \quad\text{means}\quad \lim_{x\to +\infty} f(x)=+\infty \quad\text{or}\quad \lim_{x\to +\infty} f(x)=-\infty$$

$$\lim_{x\to a} f(x)=\infty \quad\text{means}\quad \lim_{x\to a^+} f(x)=\pm\infty \quad\text{and}\quad \lim_{x\to a^-} f(x)=\pm\infty$$

The limit of a ratio, $f(x)/g(x)$, in which the numerator has limit ∞ and the denominator has limit ∞ is called an ***indeterminate form of type*** $\boldsymbol{\infty/\infty}$. The following version of L'Hôpital's rule, which we state without proof, can often be used to evaluate limits of this type.

7.7.2 THEOREM (***L'Hôpital's Rule for Form*** ∞/∞). *Suppose that f and g are differentiable functions on an open interval containing $x=a$, except possibly at $x=a$, and that*

$$\lim_{x\to a} f(x)=\infty \quad\textit{and}\quad \lim_{x\to a} g(x)=\infty$$

If $\lim_{x\to a}[f'(x)/g'(x)]$ has a finite limit orf if this limit is $+\infty$ or $-\infty$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Moreover, this statement is also true in the case of a limit as $x\to a^-$, $x\to a^+$, $x\to-\infty$, or as $x\to+\infty$.

Example 2 In each part confirm that the limit is an indeterminate form of type ∞/∞ and apply L'Hôpital's rule.

(a) $\displaystyle\lim_{x\to+\infty}\frac{x}{e^x}$ (b) $\displaystyle\lim_{x\to 0^+}\frac{\ln x}{\csc x}$

Solution (a). The numerator and denominator both have a limit of $+\infty$, so we have an indeterminate form of type ∞/∞. Applying L'Hôpital's rule yields

$$\lim_{x\to+\infty}\frac{x}{e^x} = \lim_{x\to+\infty}\frac{1}{e^x} = 0$$

Solution (b). The numerator has a limit of $-\infty$ and the denominator has a limit of $+\infty$, so we have an indeterminate form of type ∞/∞. Applying L'Hôpital's rule yields

$$\lim_{x\to 0^+}\frac{\ln x}{\csc x} = \lim_{x\to 0^+}\frac{1/x}{-\csc x \cot x} \tag{8}$$

This last limit is again an indeterminate form of type ∞/∞. Moreover, any additional applications of L'Hôpital's rule will yield powers of $1/x$ in the numerator and expressions involving $\csc x$ and $\cot x$ in the denominator; thus, repeated application of L'Hôpital's rule simply produces new indeterminate forms. We must try something else. The last limit in (8) can be rewritten as

$$\lim_{x\to 0^+}\left(-\frac{\sin x}{x}\tan x\right) = -\lim_{x\to 0^+}\frac{\sin x}{x}\cdot\lim_{x\to 0^+}\tan x = -(1)(0) = 0$$

Thus,

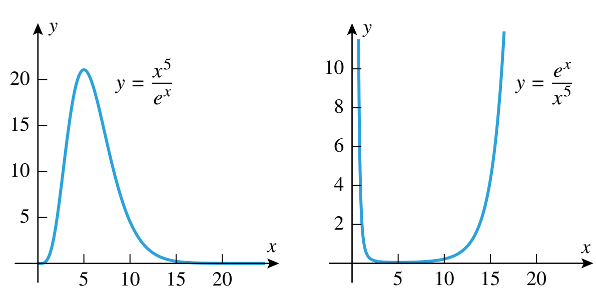
$$\lim_{x\to 0^+}\frac{\ln x}{\csc x} = 0$$

◀

ANALYZING THE GROWTH OF EXPONENTIAL FUNCTIONS USING L'HÔPITAL'S RULE

If n is any positive integer, then $x^n \to +\infty$ as $x \to +\infty$. Such integer powers of x are sometimes used as "measuring sticks" to describe how rapidly other functions grow. For example, we know that $e^x \to +\infty$ as $x \to +\infty$ and that the growth of e^x is very rapid (Table 7.2.3); however, the growth of x^n is also rapid when n is a high power, so it is reasonable to ask whether high powers of x grow more or less rapidly than e^x. One way to investigate this is to examine the behavior of the ratio x^n/e^x as $x\to+\infty$. For example, Figure 7.7.1*a* shows the graph of $y = x^5/e^x$. This graph suggests that $x^5/e^x \to 0$ as $x\to+\infty$, and this implies that the growth of the function e^x is sufficiently rapid that its values eventually overtake those of x^5 and force the ratio toward zero. Stated informally, "e^x eventually grows more rapidly than x^5." The same conclusion could have been reached by putting e^x on top and examining the behavior of e^x/x^5 as $x\to+\infty$ (Figure 7.7.1*b*). In this case the values of e^x eventually overtake those of x^5 and force the ratio toward $+\infty$. More generally, we can use L'Hôpital's rule to show that e^x *eventually grows more rapidly than any positive integer power of* x, that is,

$$\boxed{\lim_{x\to+\infty}\frac{x^n}{e^x} = 0} \quad\text{and}\quad \boxed{\lim_{x\to+\infty}\frac{e^x}{x^n} = +\infty} \tag{9–10}$$



(*a*) (*b*)

Figure 7.7.1

Both limits are indeterminate forms of type ∞/∞ that can be evaluated using L'Hôpital's rule. For example, to establish (9), we will need to apply L'Hôpital's rule n times. For this purpose, observe that successive differentiations of x^n reduce the exponent by 1 each time, thus producing a constant for the nth derivative. For example, the successive derivatives of x^3 are $3x^2$, $6x$, and 6. In general, the nth derivative of x^n is the constant $n(n-1)(n-2)\cdots 1 = n!$ (verify).* Thus, applying L'Hôpital's rule n times to (9) yields

$$\lim_{x\to+\infty}\frac{x^n}{e^x} = \lim_{x\to+\infty}\frac{n!}{e^x} = 0$$

Limit (10) can be established similarly.

INDETERMINATE FORMS OF TYPE $0\cdot\infty$

Thus far we have discussed indeterminate forms of type $0/0$ and ∞/∞. However, these are not the only possibilities; in general, the limit of an expression that has one of the forms

$$\frac{f(x)}{g(x)},\quad f(x)\cdot g(x),\quad f(x)^{g(x)},\quad f(x)-g(x),\quad f(x)+g(x)$$

is called an *indeterminate form* if the limits of $f(x)$ and $g(x)$ individually exert conflicting influences on the limit of the entire expression. For example, the limit

$$\lim_{x\to 0^+} x\ln x$$

is an ***indeterminate form of type* $\mathbf{0\cdot\infty}$** because the limit of the first factor is 0, the limit of the second factor is $-\infty$, and these two limits exert conflicting influences on the product. On the other hand, the limit

$$\lim_{x\to+\infty}[\sqrt{x}(1-x^2)]$$

is not an indeterminate form because the first factor has a limit of $+\infty$, the second factor has a limit of $-\infty$, and these influences work together to produce a limit of $-\infty$ for the product.

WARNING. It is tempting to argue that an indeterminate form of type $0\cdot\infty$ has value 0 since "zero times anything is zero." However, this is fallacious since $0\cdot\infty$ is not a product of numbers, but rather a statement about limits. For example, the following limits are of the form $0\cdot\infty$:

$$\lim_{x\to 0^+} x\cdot\frac{1}{x} = 1,\qquad \lim_{x\to 0^+} x^2\cdot\frac{1}{x} = 0,\qquad \lim_{x\to 0^+}\sqrt{x}\cdot\frac{1}{x} = +\infty$$

Indeterminate forms of type $0\cdot\infty$ can sometimes be evaluated by rewriting the product as a ratio, and then applying L'Hôpital's rule for indeterminate forms of type $0/0$ or ∞/∞.

Example 3 Evaluate

(a) $\lim\limits_{x\to 0^+} x\ln x$ (b) $\lim\limits_{x\to\pi/4}(1-\tan x)\sec 2x$

Solution (a). The factor x has a limit of 0 and the factor $\ln x$ has a limit of $-\infty$, so the stated problem is an indeterminate form of type $0\cdot\infty$. There are two possible approaches: we can rewrite the limit as

$$\lim_{x\to 0^+}\frac{\ln x}{1/x}\quad\text{or}\quad\lim_{x\to 0^+}\frac{x}{1/\ln x}$$

the first being an indeterminate form of type ∞/∞ and the second an indeterminate form of type $0/0$. However, the first form is the preferred initial choice because the derivative of $1/x$ is less complicated than the derivative of $1/\ln x$. That choice yields

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{\ln x}{1/x} = \lim_{x\to 0^+}\frac{1/x}{-1/x^2} = \lim_{x\to 0^+}(-x) = 0$$

*Recall that for $n\ge 1$ the expression $n!$, read ***n-factorial***, denotes the product of the first n positive integers.

Solution (b). The stated problem is an indeterminate form of type $0 \cdot \infty$. We will convert it to an indeterminate form of type $0/0$:

$$\lim_{x\to\pi/4}(1-\tan x)\sec 2x = \lim_{x\to\pi/4}\frac{1-\tan x}{1/\sec 2x} = \lim_{x\to\pi/4}\frac{1-\tan x}{\cos 2x}$$
$$= \lim_{x\to\pi/4}\frac{-\sec^2 x}{-2\sin 2x} = \frac{-2}{-2} = 1 \qquad \blacktriangleleft$$

INDETERMINATE FORMS OF TYPE $\infty - \infty$

A limit problem that leads to one of the expressions

$$(+\infty)-(+\infty),\quad (-\infty)-(-\infty),$$
$$(+\infty)+(-\infty),\quad (-\infty)+(+\infty)$$

is called an ***indeterminate form of type*** $\boldsymbol{\infty - \infty}$. Such limits are indeterminate because the two terms exert conflicting influences on the expression: one pushes it in the positive direction and the other pushes it in the negative direction. However, limit problems that lead to one of the expressions

$$(+\infty)+(+\infty),\qquad (+\infty)-(-\infty),$$
$$(-\infty)+(-\infty),\qquad (-\infty)-(+\infty)$$

are not indeterminate, since the two terms work together (those on the top produce a limit of $+\infty$ and those on the bottom produce a limit of $-\infty$).

Indeterminate forms of type $\infty - \infty$ can sometimes be evaluated by combining the terms and manipulating the result to produce an indeterminate form of type $0/0$ or ∞/∞.

Example 4 Evaluate $\displaystyle\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\sin x}\right)$.

Solution. Both terms have a limit of $+\infty$, so the stated problem is an indeterminate form of type $\infty - \infty$. Combining the two terms yields

$$\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\sin x}\right) = \lim_{x\to 0^+}\left(\frac{\sin x - x}{x\sin x}\right)$$

which is an indeterminate form of type $0/0$. Applying L'Hôpital's rule twice yields

$$\lim_{x\to 0^+}\left(\frac{\sin x - x}{x\sin x}\right) = \lim_{x\to 0^+}\frac{\cos x - 1}{\sin x + x\cos x}$$
$$= \lim_{x\to 0^+}\frac{-\sin x}{\cos x + \cos x - x\sin x} = \frac{0}{2} = 0 \qquad \blacktriangleleft$$

INDETERMINATE FORMS OF TYPE 0^0, ∞^0, 1^∞

Limits of the form

$$\lim f(x)^{g(x)}$$

give rise to ***indeterminate forms of the types*** $\boldsymbol{0^0}$, $\boldsymbol{\infty^0}$, ***and*** $\boldsymbol{1^\infty}$. (The interpretations of these symbols should be clear.) For example, the limit

$$\lim_{x\to 0^+}(1+x)^{1/x}$$

whose value we know to be e [see Formula (5) of Section 7.2] is an indeterminate form of type 1^∞. It is indeterminate because the expressions $1+x$ and $1/x$ exert two conflicting influences: the first approaches 1, which drives the expression toward 1, and the second approaches $+\infty$, which drives the expression toward $+\infty$.

Indeterminate forms of types 0^0, ∞^0, and 1^∞ can sometimes be evaluated by first introducing a dependent variable

$$y = f(x)^{g(x)}$$

and then calculating the limit of $\ln y$ by expressing it as

$$\lim \ln y = \lim\,[\ln(f(x)^{g(x)})] = \lim\,[g(x)\ln f(x)]$$

Once the limit of $\ln y$ is known, the limit of $y = f(x)^{g(x)}$ itself can generally be obtained by a method that we will illustrate in the next example.

Example 5 Show that $\lim_{x\to 0}(1+x)^{1/x} = e$.

Solution. As discussed above, we begin by introducing a dependent variable

$$y = (1+x)^{1/x}$$

and taking the natural logarithm of both sides:

$$\ln y = \ln(1+x)^{1/x} = \frac{1}{x}\ln(1+x) = \frac{\ln(1+x)}{x}$$

Thus,

$$\lim_{x\to 0}\ln y = \lim_{x\to 0}\frac{\ln(1+x)}{x}$$

which is an indeterminate form of type $0/0$, so by L'Hôpital's rule

$$\lim_{x\to 0}\ln y = \lim_{x\to 0}\frac{\ln(1+x)}{x} = \lim_{x\to 0}\frac{1/(1+x)}{1} = 1$$

Since we have shown that $\ln y \to 1$ as $x\to 0$, the continuity of the exponential function implies that $e^{\ln y}\to e^1$ as $x\to 0$, and this implies that $y\to e$ as $x\to 0$. Thus,

$$\lim_{x\to 0}(1+x)^{1/x} = e$$ ◀

Exercise Set 7.7

Graphing Utility C CAS

In Exercises 1 and 2, evaluate the given limit without using L'Hôpital's rule, and then check that your answer is correct using L'Hôpital's rule.

1. (a) $\lim_{x\to 2}\dfrac{x^2-4}{x^2+2x-8}$ (b) $\lim_{x\to+\infty}\dfrac{2x-5}{3x+7}$

2. (a) $\lim_{x\to 0}\dfrac{\sin x}{\tan x}$ (b) $\lim_{x\to 1}\dfrac{x^2-1}{x^3-1}$

In Exercises 3–36, find the limit.

3. $\lim_{x\to 1}\dfrac{\ln x}{x-1}$

4. $\lim_{x\to 0}\dfrac{\sin 2x}{\sin 5x}$

5. $\lim_{x\to 0}\dfrac{e^x-1}{\sin x}$

6. $\lim_{x\to 3}\dfrac{x-3}{3x^2-13x+12}$

7. $\lim_{\theta\to 0}\dfrac{\tan\theta}{\theta}$

8. $\lim_{t\to 0}\dfrac{te^t}{1-e^t}$

9. $\lim_{x\to\pi^+}\dfrac{\sin x}{x-\pi}$

10. $\lim_{x\to 0^+}\dfrac{\sin x}{x^2}$

11. $\lim_{x\to+\infty}\dfrac{\ln x}{x}$

12. $\lim_{x\to+\infty}\dfrac{e^{3x}}{x^2}$

13. $\lim_{x\to 0^+}\dfrac{\cot x}{\ln x}$

14. $\lim_{x\to 0^+}\dfrac{1-\ln x}{e^{1/x}}$

15. $\lim_{x\to+\infty}\dfrac{x^{100}}{e^x}$

16. $\lim_{x\to 0^+}\dfrac{\ln(\sin x)}{\ln(\tan x)}$

17. $\lim_{x\to 0}\dfrac{\sin^{-1}2x}{x}$

18. $\lim_{x\to 0}\dfrac{x-\tan^{-1}x}{x^3}$

19. $\lim_{x\to+\infty}xe^{-x}$

20. $\lim_{x\to\pi^-}(x-\pi)\tan\frac{1}{2}x$

21. $\lim_{x\to+\infty}x\sin\dfrac{\pi}{x}$

22. $\lim_{x\to 0^+}\tan x\ln x$

23. $\lim_{x\to\pi/2^-} \sec 3x \cos 5x$

24. $\lim_{x\to\pi} (x-\pi)\cot x$

25. $\lim_{x\to+\infty} (1-3/x)^x$

26. $\lim_{x\to 0} (1+2x)^{-3/x}$

27. $\lim_{x\to 0} (e^x + x)^{1/x}$

28. $\lim_{x\to+\infty} (1+a/x)^{bx}$

29. $\lim_{x\to 1} (2-x)^{\tan[(\pi/2)x]}$

30. $\lim_{x\to+\infty} [\cos(2/x)]^{x^2}$

31. $\lim_{x\to 0} (\csc x - 1/x)$

32. $\lim_{x\to 0} \left(\dfrac{1}{x^2} - \dfrac{\cos 3x}{x^2}\right)$

33. $\lim_{x\to+\infty} (\sqrt{x^2+x} - x)$

34. $\lim_{x\to 0} \left(\dfrac{1}{x} - \dfrac{1}{e^x-1}\right)$

35. $\lim_{x\to+\infty} [x - \ln(x^2+1)]$

36. $\lim_{x\to+\infty} [\ln x - \ln(1+x)]$

37. Use a CAS to check the answers you obtained in Exercises 31–36.

38. Show that for any positive integer n

(a) $\lim_{x\to+\infty} \dfrac{\ln x}{x^n} = 0$ (b) $\lim_{x\to+\infty} \dfrac{x^n}{\ln x} = +\infty$

39. (a) Find the error in the following calculation:

$$\lim_{x\to 1} \frac{x^3 - x^2 + x - 1}{x^3 - x^2} = \lim_{x\to 1} \frac{3x^2 - 2x + 1}{3x^2 - 2x} = \lim_{x\to 1} \frac{6x-2}{6x-2} = 1$$

(b) Find the correct answer.

40. Find $\lim_{x\to 1} \dfrac{x^4 - 4x^3 + 6x^2 - 4x + 1}{x^4 - 3x^3 + 3x^2 - x}$.

In Exercises 41–44, make a conjecture about the limit by graphing the function involved with a graphing utility; then check your conjecture using L'Hôpital's rule.

41. $\lim_{x\to+\infty} \dfrac{\ln(\ln x)}{\sqrt{x}}$

42. $\lim_{x\to 0^+} x^x$

43. $\lim_{x\to 0^+} (\sin x)^{3/\ln x}$

44. $\lim_{x\to(\pi/2)^-} \dfrac{4\tan x}{1+\sec x}$

In Exercises 45–48, make a conjecture about the equations of horizontal asymptotes, if any, by graphing the equation with a graphing utility; then check your answer using L'Hôpital's rule.

45. $y = \ln x - e^x$

46. $y = x - \ln(1+2e^x)$

47. $y = (\ln x)^{1/x}$

48. $y = \left(\dfrac{x+1}{x+2}\right)^x$

49. Limits of the type

$$0/\infty,\ \ \infty/0,\ \ 0^\infty,\ \ \infty\cdot\infty,\ \ +\infty+(+\infty),$$
$$+\infty-(-\infty),\ \ -\infty+(-\infty),\ \ -\infty-(+\infty)$$

are *not* indeterminate forms. Find the following limits by inspection.

(a) $\lim_{x\to 0^+} \dfrac{x}{\ln x}$ (b) $\lim_{x\to+\infty} \dfrac{x^3}{e^{-x}}$

(c) $\lim_{x\to(\pi/2)^-} (\cos x)^{\tan x}$ (d) $\lim_{x\to 0^+} (\ln x)\cot x$

(e) $\lim_{x\to 0^+} \left(\dfrac{1}{x} - \ln x\right)$ (f) $\lim_{x\to-\infty} (x + x^3)$

50. There is a myth that circulates among beginning calculus students which states that all indeterminate forms of types 0^0, ∞^0, and 1^∞ have value 1 because "anything to the zero power is 1" and "1 to any power is 1." The fallacy is that 0^0, ∞^0, and 1^∞ are not powers of numbers, but rather descriptions of limits. The following examples, which were suggested by Prof. Jack Staib of Drexel University, show that such indeterminate forms can have any positive real value:

(a) $\lim_{x\to 0^+} \left[x^{(\ln a)/(1+\ln x)}\right]$ (form 0^0)

(b) $\lim_{x\to+\infty} \left[x^{(\ln a)/(1+\ln x)}\right]$ (form ∞^0)

(c) $\lim_{x\to 0} \left[(x+1)^{(\ln a)/x}\right]$ (form 1^∞).

Verify these results.

In Exercises 51–54, verify that L'Hôpital's rule is of no help in finding the limit, then find the limit, if it exists, by some other method.

51. $\lim_{x\to+\infty} \dfrac{x + \sin 2x}{x}$

52. $\lim_{x\to+\infty} \dfrac{2x - \sin x}{3x + \sin x}$

53. $\lim_{x\to+\infty} \dfrac{x(2+\sin 2x)}{x+1}$

54. $\lim_{x\to+\infty} \dfrac{x(2+\sin x)}{x^2+1}$

55. The accompanying schematic diagram represents an electrical circuit consisting of an electromotive force that produces a voltage V, a resistor with resistance R, and an inductor with inductance L. It is shown in electrical circuit theory that if the voltage is first applied at time $t = 0$, then the current I flowing through the circuit at time t is given by

$$I = \frac{V}{R}(1 - e^{-Rt/L})$$

What is the effect on the current at a fixed time t if the resistance approaches 0 (i.e., $R \to 0^+$)?

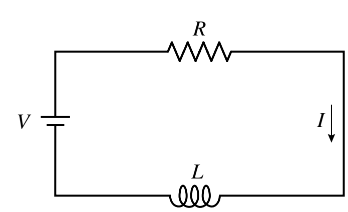


Figure Ex-55

56. (a) Show that $\lim_{x\to\pi/2} (\pi/2 - x)\tan x = 1$.

(b) Show that

$$\lim_{x\to\pi/2}\left(\frac{1}{\pi/2-x}-\tan x\right)=0$$

(c) It follows from part (b) that the approximation

$$\tan x \approx \frac{1}{\pi/2-x}$$

should be good for values of x near $\pi/2$. Use a calculator to find $\tan x$ and $1/(\pi/2-x)$ for $x=1.57$; compare the results.

C **57.** (a) Use a CAS to show that if k is a positive constant, then

$$\lim_{x\to+\infty} x(k^{1/x}-1)=\ln k$$

(b) Confirm this result using L'Hôpital's rule. [*Hint:* Express the limit in terms of $t=1/x$.]

(c) If n is a positive integer, then it follows from part (a) with $x=n$ that the approximation

$$n(\sqrt[n]{k}-1)\approx \ln k$$

should be good when n is large. Use this result and the square root key on a calculator to approximate the values of $\ln 0.3$ and $\ln 2$ with $n=1024$, then compare the values obtained with values of the logarithms generated directly from the calculator. [*Hint:* The nth roots for which n is a power of 2 can be obtained as successive square roots.]

58. Let $f(x)=x^2\sin(1/x)$.

(a) Are the limits $\lim_{x\to0^+} f(x)$ and $\lim_{x\to0^-} f(x)$ indeterminate forms?

(b) Use a graphing utility to generate the graph of f, and use the graph to make conjectures about the limits in part (a).

(c) Use the Squeezing Theorem (2.6.2) to confirm that your conjectures in part (b) are correct.

59. Find all values of k and l such that

$$\lim_{x\to0}\frac{k+\cos lx}{x^2}=-4$$

60. (a) Explain why L'Hôpital's rule does not apply to the problem

$$\lim_{x\to0}\frac{x^2\sin(1/x)}{\sin x}$$

(b) Find the limit.

61. Find $\lim_{x\to0^+}\dfrac{x\sin(1/x)}{\sin x}$ if it exists.

7.8 HYPERBOLIC FUNCTIONS AND HANGING CABLES

In this section we will study certain combinations of e^x and e^{-x}, called "hyperbolic functions." These functions, which arise in various engineering applications, have many properties in common with the trigonometric functions. This similarity is somewhat surprising, since there is little on the surface to suggest that there should be any relationship between exponential and trigonometric functions. This is because the relationship occurs within the context of complex numbers, a topic which we will leave for more advanced courses.

DEFINITIONS OF HYPERBOLIC FUNCTIONS

To introduce the hyperbolic functions, observe that the function e^x can be expressed in the following way as the sum of an even function and an odd function:

$$e^x=\underbrace{\frac{e^x+e^{-x}}{2}}_{\text{Even}}+\underbrace{\frac{e^x-e^{-x}}{2}}_{\text{Odd}}$$

These functions are sufficiently important that there are names and notation associated with them: the odd function is called the *hyperbolic sine* of x and the even function is called the *hyperbolic cosine* of x. They are denoted by

$$\sinh x=\frac{e^x-e^{-x}}{2} \quad\text{and}\quad \cosh x=\frac{e^x+e^{-x}}{2}$$

where sinh is pronounced "cinch" and cosh rhymes with "gosh." From these two building blocks we can create four more functions to produce the following set of six ***hyperbolic functions***.

7.8.1 DEFINITION.

Hyperbolic sine	$\sinh x = \dfrac{e^x - e^{-x}}{2}$
Hyperbolic cosine	$\cosh x = \dfrac{e^x + e^{-x}}{2}$
Hyperbolic tangent	$\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$
Hyperbolic cotangent	$\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$
Hyperbolic secant	$\operatorname{sech} x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$
Hyperbolic cosecant	$\operatorname{csch} x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}}$

REMARK. The terms "tanh," "sech," and "csch" are pronounced "tanch," "seech," and "coseech," respectively.

Example 1

$$\sinh 0 = \frac{e^0 - e^{-0}}{2} = \frac{1-1}{2} = 0$$

$$\cosh 0 = \frac{e^0 + e^{-0}}{2} = \frac{1+1}{2} = 1$$

$$\sinh 2 = \frac{e^2 - e^{-2}}{2} \approx 3.6269$$ ◀

GRAPHS OF THE HYPERBOLIC FUNCTIONS

The graphs of the hyperbolic functions, which are shown in Figure 7.8.1, can be generated with a graphing utility, but it is worthwhile to observe that the general shape of the graph of $y = \cosh x$ can be obtained by sketching the graphs of $y = \frac{1}{2}e^x$ and $y = \frac{1}{2}e^{-x}$ separately and adding the corresponding y-coordinates [see part (*a*) of the figure]. Similarly, the general shape of the graph of $y = \sinh x$ can be obtained by sketching the graphs of $y = \frac{1}{2}e^x$ and $y = -\frac{1}{2}e^{-x}$ separately and adding corresponding y-coordinates [see part (*b*) of the figure].

Observe that $\sinh x$ has a domain of $(-\infty, +\infty)$ and a range of $(-\infty, +\infty)$, whereas $\cosh x$ has a domain of $(-\infty, +\infty)$ and a range of $[1, +\infty)$. Observe also that $y = \frac{1}{2}e^x$ and $y = \frac{1}{2}e^{-x}$ are ***curvilinear asymptotes*** for $y = \cosh x$ in the sense that the graph of $y = \cosh x$ gets closer and closer to the graph of $y = \frac{1}{2}e^x$ as $x \to +\infty$ and gets closer and closer to the graph of $y = \frac{1}{2}e^{-x}$ as $x \to -\infty$. (See Exercise 2.3.) Similarly, $y = \frac{1}{2}e^x$ is a curvilinear asymptote for $y = \sinh x$ as $x \to +\infty$ and $y = -\frac{1}{2}e^{-x}$ is a curvilinear asymptote as $x \to -\infty$. Other properties of the hyperbolic functions are explored in the exercises.

HANGING CABLES AND OTHER APPLICATIONS

Hyperbolic functions arise in vibratory motions inside elastic solids and more generally in many problems where mechanical energy is gradually absorbed by a surrounding medium. They also occur when a homogeneous, flexible cable is suspended between two points, as with a telephone line hanging between two poles. Such a cable forms a curve, called a ***catenary*** (from the Latin *catena*, meaning "chain"). If, as in Figure 7.8.2, a coordinate system is introduced so that the low point of the cable lies at the point $(0, a)$ on the y-axis,

The design of the Gateway Arch near St. Louis is based on an inverted hyperbolic cosine curve.

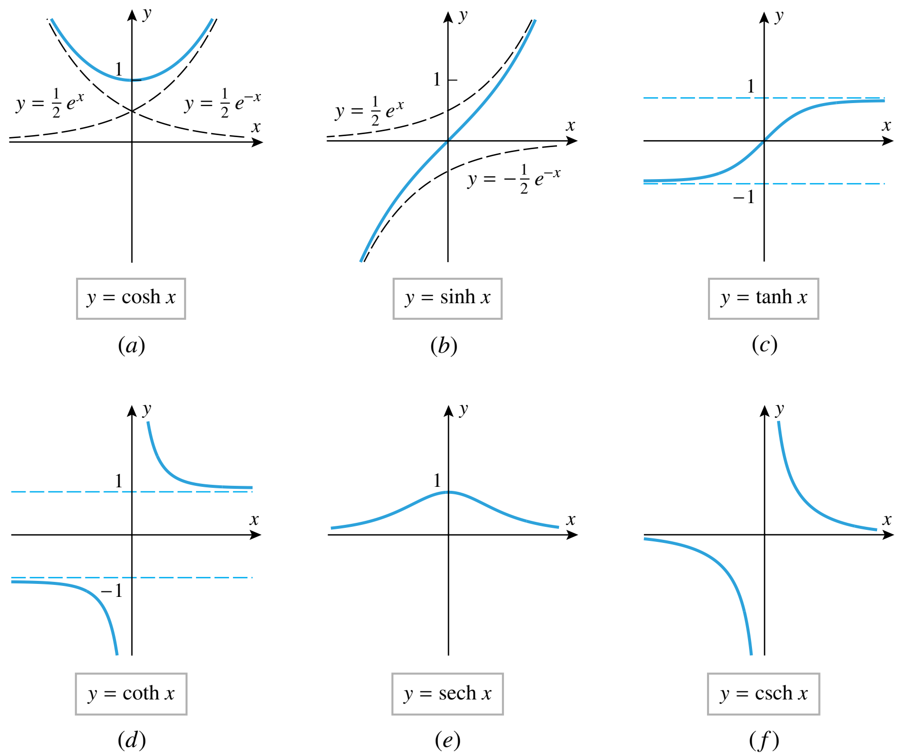


Figure 7.8.1

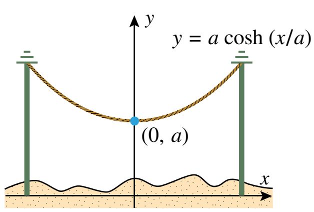


Figure 7.8.2

then it can be shown using principles of physics that the cable has the equation

$$y = a\cosh\left(\frac{x}{a}\right)$$

HYPERBOLIC IDENTITIES

The hyperbolic functions satisfy various identities that are similar to identities for trigonometric functions. The most fundamental of these is

$$\cosh^2 x - \sinh^2 x = 1 \tag{1}$$

which can be proved by writing

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\ &= \tfrac{1}{4}(e^{2x} + 2e^0 + e^{-2x}) - \tfrac{1}{4}(e^{2x} - 2e^0 + e^{-2x}) \\ &= 1\end{aligned}$$

Other hyperbolic identities can be derived in a similar manner or, alternatively, by performing algebraic operations on known identities. For example, if we divide (1) by $\cosh^2 x$, we obtain

$$1 - \tanh^2 x = \operatorname{sech}^2 x$$

and if we divide (1) by $\sinh^2 x$, we obtain

$$\coth^2 x - 1 = \operatorname{csch}^2 x$$

The following theorem summarizes some of the more useful hyperbolic identities. The proofs of those not already obtained are left as exercises.

7.8.2 THEOREM.

$$\cosh x + \sinh x = e^x \qquad \sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$
$$\cosh x - \sinh x = e^{-x} \qquad \cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$
$$\cosh^2 x - \sinh^2 x = 1 \qquad \sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$$
$$1 - \tanh^2 x = \operatorname{sech}^2 x \qquad \cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y$$
$$\coth^2 x - 1 = \operatorname{csch}^2 x \qquad \sinh 2x = 2\sinh x \cosh x$$
$$\cosh(-x) = \cosh x \qquad \cosh 2x = \cosh^2 x + \sinh^2 x$$
$$\sinh(-x) = -\sinh x \qquad \cosh 2x = 2\sinh^2 x + 1$$
$$\cosh 2x = 2\cosh^2 x - 1$$

WHY THEY ARE CALLED HYPERBOLIC FUNCTIONS

Recall that the parametric equations

$$x = \cos t, \quad y = \sin t \qquad (0 \le t \le 2\pi)$$

represent the unit circle $x^2 + y^2 = 1$ (Figure 7.8.3*a*), as may be seen by writing

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$$

If $0 \le t \le 2\pi$, then the parameter t can be interpreted as the angle in radians from the positive x-axis to the point $(\cos t, \sin t)$ or, alternatively, as twice the shaded area of the sector in Figure 7.8.3*a* (verify). Analogously, the parametric equations

$$x = \cosh t, \quad y = \sinh t \qquad (-\infty < t < +\infty)$$

represent a portion of the curve $x^2 - y^2 = 1$, as may be seen by writing

$$x^2 - y^2 = \cosh^2 t - \sinh^2 t = 1$$

and observing that $x = \cosh t > 0$. This curve, which is shown in Figure 7.8.3*b*, is the right half of a larger curve called the ***unit hyperbola***; this is the reason why the functions in this section are called *hyperbolic* functions. It can be shown that if $t \ge 0$, then the parameter t can be interpreted as twice the shaded area in Figure 7.8.3*b*. (We omit the details.)

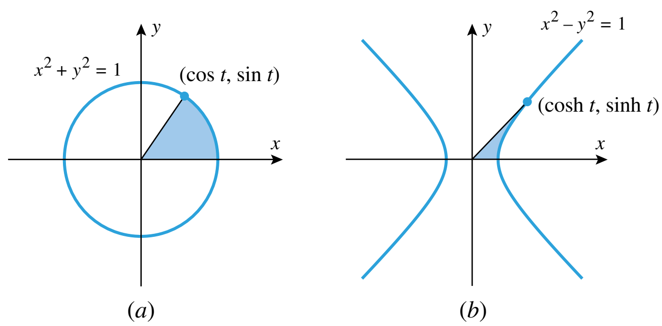


Figure 7.8.3

DERIVATIVE AND INTEGRAL FORMULAS

Derivative formulas for $\sinh x$ and $\cosh x$ can be obtained by expressing these functions in terms of e^x and e^{-x}:

$$\frac{d}{dx}[\sinh x] = \frac{d}{dx}\left[\frac{e^x - e^{-x}}{2}\right] = \frac{e^x + e^{-x}}{2} = \cosh x$$

$$\frac{d}{dx}[\cosh x] = \frac{d}{dx}\left[\frac{e^x + e^{-x}}{2}\right] = \frac{e^x - e^{-x}}{2} = \sinh x$$

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Derivatives of the remaining hyperbolic functions can be obtained by expressing them in terms of sinh and cosh and applying appropriate identities. For example,

$$\frac{d}{dx}[\tanh x] = \frac{d}{dx}\left[\frac{\sinh x}{\cosh x}\right] = \frac{\cosh x \frac{d}{dx}[\sinh x] - \sinh x \frac{d}{dx}[\cosh x]}{\cosh^2 x}$$

$$= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x$$

The following theorem provides a complete list of the generalized derivative formulas and corresponding integration formulas for the hyperbolic functions.

7.8.3 THEOREM.

$$\frac{d}{dx}[\sinh u] = \cosh u \frac{du}{dx} \qquad \int \cosh u\, du = \sinh u + C$$

$$\frac{d}{dx}[\cosh u] = \sinh u \frac{du}{dx} \qquad \int \sinh u\, du = \cosh u + C$$

$$\frac{d}{dx}[\tanh u] = \operatorname{sech}^2 u \frac{du}{dx} \qquad \int \operatorname{sech}^2 u\, du = \tanh u + C$$

$$\frac{d}{dx}[\coth u] = -\operatorname{csch}^2 u \frac{du}{dx} \qquad \int \operatorname{csch}^2 u\, du = -\coth u + C$$

$$\frac{d}{dx}[\operatorname{sech} u] = -\operatorname{sech} u \tanh u \frac{du}{dx} \qquad \int \operatorname{sech} u \tanh u\, du = -\operatorname{sech} u + C$$

$$\frac{d}{dx}[\operatorname{csch} u] = -\operatorname{csch} u \coth u \frac{du}{dx} \qquad \int \operatorname{csch} u \coth u\, du = -\operatorname{csch} u + C$$

Example 2

$$\frac{d}{dx}[\cosh(x^3)] = \sinh(x^3) \cdot \frac{d}{dx}[x^3] = 3x^2 \sinh(x^3)$$

$$\frac{d}{dx}[\ln(\tanh x)] = \frac{1}{\tanh x} \cdot \frac{d}{dx}[\tanh x] = \frac{\operatorname{sech}^2 x}{\tanh x}$$ ◀

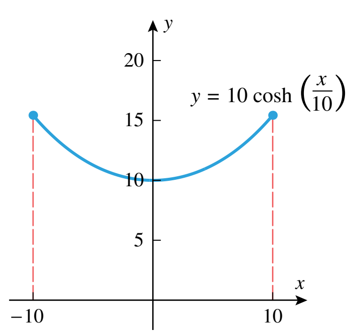


Figure 7.8.4

Example 3

$$\int \sinh^5 x \cosh x\, dx = \tfrac{1}{6}\sinh^6 x + C \qquad \boxed{\begin{array}{c} u = \sinh x \\ du = \cosh x\, dx \end{array}}$$

$$\int \tanh x\, dx = \int \frac{\sinh x}{\cosh x} dx$$

$$= \ln|\cosh x| + C \qquad \boxed{\begin{array}{c} u = \cosh x \\ du = \sinh x\, dx \end{array}}$$

$$= \ln(\cosh x) + C$$

We were justified in dropping the absolute value signs since $\cosh x > 0$ for all x. ◀

Example 4 Find the length of the catenary $y = 10\cosh(x/10)$ from $x = -10$ to $x = 10$ (Figure 7.8.4).

Solution. From Formula (4) of Section 6.4, the length L of the catenary is

$$\begin{aligned}
L &= \int_{-10}^{10} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \\
&= 2\int_{0}^{10} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \qquad \boxed{\text{By symmetry about the } y\text{-axis}} \\
&= 2\int_{0}^{10} \sqrt{1+\sinh^2\left(\frac{x}{10}\right)}\,dx \\
&= 2\int_{0}^{10} \cosh\left(\frac{x}{10}\right)dx \qquad \boxed{\text{By (1) and the fact that } \cosh x > 0} \\
&= 20\sinh\left(\frac{x}{10}\right)\Big]_0^{10} \\
&= 20[\sinh 1 - \sinh 0] = 20\sinh 1 = 20\left(\frac{e-e^{-1}}{2}\right) \approx 23.50
\end{aligned}$$

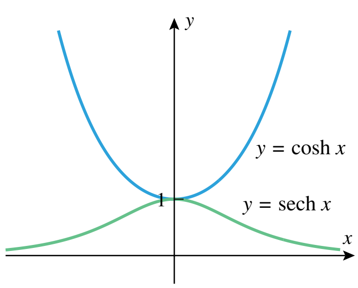
◀

REMARK. Computer algebra systems, such as *Mathematica*, *Maple*, and *Derive* have built-in capabilities for evaluating hyperbolic functions directly, but some calculators do not. However, if you need to evaluate a hyperbolic function on a calculator, you can do so by expressing it in terms of exponential functions, as in this example.

INVERSES OF HYPERBOLIC FUNCTIONS

Referring to Figure 7.8.1, it is evident that the graphs of $\sinh x$, $\tanh x$, $\coth x$, and $\operatorname{csch} x$ pass the horizontal line test, but the graphs of $\cosh x$ and $\operatorname{sech} x$ do not. In the latter case restricting x to be nonnegative makes the functions invertible (Figure 7.8.5). The graphs of the six inverse hyperbolic functions in Figure 7.8.6 were obtained by reflecting the graphs of the hyperbolic functions (with the appropriate restrictions) about the line $y = x$.

Table 7.8.1 summarizes the basic properties of the inverse hyperbolic functions. You should confirm that the domains and ranges listed in this table agree with the graphs in Figure 7.8.6.



With the restriction that $x \ge 0$, the curves $y = \cosh x$ and $y = \operatorname{sech} x$ pass the horizontal line test.

Figure 7.8.5

Table 7.8.1

FUNCTION	DOMAIN	RANGE	BASIC RELATIONSHIPS
$\sinh^{-1} x$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$\sinh^{-1}(\sinh x) = x$ if $-\infty < x < +\infty$ $\sinh(\sinh^{-1} x) = x$ if $-\infty < x < +\infty$
$\cosh^{-1} x$	$[1, +\infty)$	$[0, +\infty)$	$\cosh^{-1}(\cosh x) = x$ if $x \ge 0$ $\cosh(\cosh^{-1} x) = x$ if $x \ge 1$
$\tanh^{-1} x$	$(-1, 1)$	$(-\infty, +\infty)$	$\tanh^{-1}(\tanh x) = x$ if $-\infty < x < +\infty$ $\tanh(\tanh^{-1} x) = x$ if $-1 < x < 1$
$\coth^{-1} x$	$(-\infty, -1) \cup (1, +\infty)$	$(-\infty, 0) \cup (0, +\infty)$	$\coth^{-1}(\coth x) = x$ if $x < 0$ or $x > 0$ $\coth(\coth^{-1} x) = x$ if $x < -1$ or $x > 1$
$\operatorname{sech}^{-1} x$	$(0, 1]$	$[0, +\infty)$	$\operatorname{sech}^{-1}(\operatorname{sech} x) = x$ if $x \ge 0$ $\operatorname{sech}(\operatorname{sech}^{-1} x) = x$ if $0 < x \le 1$
$\operatorname{csch}^{-1} x$	$(-\infty, 0) \cup (0, +\infty)$	$(-\infty, 0) \cup (0, +\infty)$	$\operatorname{csch}^{-1}(\operatorname{csch} x) = x$ if $x < 0$ or $x > 0$ $\operatorname{csch}(\operatorname{csch}^{-1} x) = x$ if $x < 0$ or $x > 0$

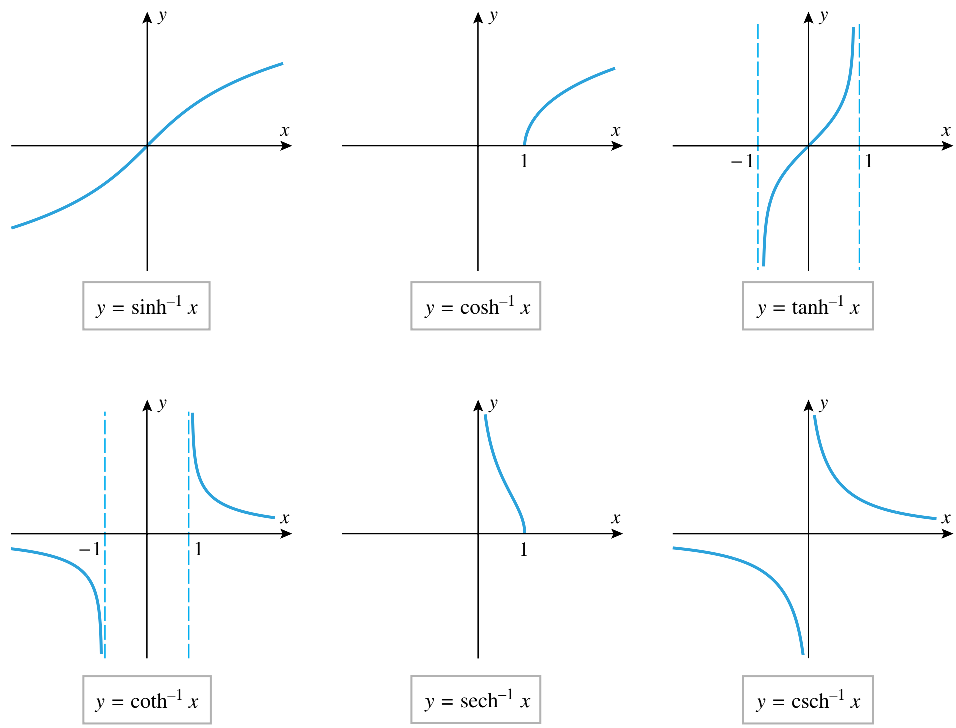


Figure 7.8.6

LOGARITHMIC FORMS OF INVERSE HYPERBOLIC FUNCTIONS

Because the hyperbolic functions are expressible in terms of e^x, it should not be surprising that the inverse hyperbolic functions are expressible in terms of natural logarithms; the next theorem shows that this is so.

7.8.4 THEOREM. *The following relationships hold for all x in the domains of the stated inverse hyperbolic functions:*

$$\sinh^{-1} x = \ln(x + \sqrt{x^2+1}\,) \qquad \cosh^{-1} x = \ln(x + \sqrt{x^2-1}\,)$$

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad \coth^{-1} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$$

$$\operatorname{sech}^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) \qquad \operatorname{csch}^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right)$$

We will show how to derive the first formula in this theorem, and leave the rest as exercises. The basic idea is to write the equation $x = \sinh y$ in terms of exponential functions and solve this equation for y as a function of x. This will produce the equation $y = \sinh^{-1} x$ with $\sinh^{-1} x$ expressed in terms of natural logarithms. Expressing $x = \sinh y$ in terms of exponentials yields

$$x = \sinh y = \frac{e^y - e^{-y}}{2}$$

which can be rewritten as

$$e^y - 2x - e^{-y} = 0$$

Multiplying this equation through by e^y we obtain

$$e^{2y} - 2xe^y - 1 = 0$$

and applying the quadratic formula yields

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}$$

Since $e^y > 0$, the solution involving the minus sign is extraneous and must be discarded. Thus,

$$e^y = x + \sqrt{x^2+1}$$

Taking natural logarithms yields

$$y = \ln(x + \sqrt{x^2+1}) \quad \text{or} \quad \sinh^{-1} x = \ln(x + \sqrt{x^2+1})$$

Example 5

$$\sinh^{-1} 1 = \ln(1 + \sqrt{1^2+1}) = \ln(1+\sqrt{2}) \approx 0.8814$$

$$\tanh^{-1}\left(\frac{1}{2}\right) = \frac{1}{2}\ln\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right) = \frac{1}{2}\ln 3 \approx 0.5493$$

◀

DERIVATIVES AND INTEGRALS INVOLVING INVERSE HYPERBOLIC FUNCTIONS

Theorem 7.1.6 can be used to establish the differentiability of the inverse hyperbolic functions (we omit the details), and formulas for the derivatives can be obtained from Theorem 7.8.4. For example,

$$\begin{aligned}\frac{d}{dx}[\sinh^{-1} x] = \frac{d}{dx}[\ln(x+\sqrt{x^2+1})] &= \frac{1}{x+\sqrt{x^2+1}}\left(1 + \frac{x}{\sqrt{x^2+1}}\right)\\ &= \frac{\sqrt{x^2+1}+x}{(x+\sqrt{x^2+1})(\sqrt{x^2+1})} = \frac{1}{\sqrt{x^2+1}}\end{aligned}$$

This computation leads to two integral formulas, a formula that involves $\sinh^{-1} x$ and an equivalent formula that involves logarithms:

$$\int \frac{dx}{\sqrt{x^2+1}} = \sinh^{-1} x + C = \ln(x+\sqrt{x^2+1}) + C$$

FOR THE READER. The derivative of $\sinh^{-1} x$ can also be obtained by letting $y = \sinh^{-1} x$ and differentiating the equation $x = \sinh y$ implicitly. Try it.

The following two theorems list the generalized derivative formulas and corresponding integration formulas for the inverse hyperbolic functions. Some of the proofs appear as exercises.

7.8.5 THEOREM.

$$\frac{d}{dx}(\sinh^{-1} u) = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx} \qquad \frac{d}{dx}(\coth^{-1} u) = \frac{1}{1-u^2}\frac{du}{dx}, \quad |u| > 1$$

$$\frac{d}{dx}(\cosh^{-1} u) = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}, \quad u > 1 \qquad \frac{d}{dx}(\operatorname{sech}^{-1} u) = -\frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}, \quad 0 < u < 1$$

$$\frac{d}{dx}(\tanh^{-1} u) = \frac{1}{1-u^2}\frac{du}{dx}, \quad |u| < 1 \qquad \frac{d}{dx}(\operatorname{csch}^{-1} u) = -\frac{1}{|u|\sqrt{1+u^2}}\frac{du}{dx}, \quad u \neq 0$$

7.8.6 THEOREM. *If $a > 0$, then*

$$\int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C \quad or \quad \ln(u+\sqrt{u^2+a^2}) + C$$

$$\int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C \quad or \quad \ln(u+\sqrt{u^2-a^2}) + C, \quad u > a$$

$$\int \frac{du}{a^2-u^2} = \begin{cases} \dfrac{1}{a}\tanh^{-1}\left(\dfrac{u}{a}\right) + C, & |u| < a \\ \dfrac{1}{a}\coth^{-1}\left(\dfrac{u}{a}\right) + C, & |u| > a \end{cases} \quad or \quad \frac{1}{2a}\ln\left|\frac{a+u}{a-u}\right| + C, \quad |u| \neq a$$

$$\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\left|\frac{u}{a}\right| + C \quad or \quad -\frac{1}{a}\ln\left(\frac{a+\sqrt{a^2-u^2}}{|u|}\right) + C, \quad 0 < |u| < a$$

$$\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\operatorname{csch}^{-1}\left|\frac{u}{a}\right| + C \quad or \quad -\frac{1}{a}\ln\left(\frac{a+\sqrt{a^2+u^2}}{|u|}\right) + C, \quad u \neq 0$$

Example 6 Evaluate $\displaystyle\int \frac{dx}{\sqrt{4x^2-9}}$, $x > \frac{3}{2}$.

Solution. Let $u = 2x$. Thus, $du = 2\,dx$ and

$$\int \frac{dx}{\sqrt{4x^2-9}} = \frac{1}{2}\int \frac{2\,dx}{\sqrt{4x^2-9}} = \frac{1}{2}\int \frac{du}{\sqrt{u^2-3^2}}$$
$$= \frac{1}{2}\cosh^{-1}\left(\frac{u}{3}\right) + C = \frac{1}{2}\cosh^{-1}\left(\frac{2x}{3}\right) + C$$

Alternatively, we can use the logarithmic equivalent of $\cosh^{-1}(2x/3)$,

$$\cosh^{-1}\left(\frac{2x}{3}\right) = \ln(2x+\sqrt{4x^2-9}) - \ln 3$$

(verify), and express the answer as

$$\int \frac{dx}{\sqrt{4x^2-9}} = \frac{1}{2}\ln(2x+\sqrt{4x^2-9}) + C$$ ◀

EXERCISE SET 7.8 Graphing Utility C CAS

In Exercises 1 and 2, approximate the expression to four decimal places.

1. (a) $\sinh 3$ (b) $\cosh(-2)$ (c) $\tanh(\ln 4)$
(d) $\sinh^{-1}(-2)$ (e) $\cosh^{-1} 3$ (f) $\tanh^{-1}\frac{3}{4}$

2. (a) $\operatorname{csch}(-1)$ (b) $\operatorname{sech}(\ln 2)$ (c) $\coth 1$
(d) $\operatorname{sech}^{-1}\frac{1}{2}$ (e) $\coth^{-1} 3$ (f) $\operatorname{csch}^{-1}(-\sqrt{3})$

3. In each part, find the exact numerical value of the expression.
(a) $\sinh(\ln 3)$ (b) $\cosh(-\ln 2)$
(c) $\tanh(2\ln 5)$ (d) $\sinh(-3\ln 2)$

4. In each part, rewrite the expression as a ratio of polynomials.
(a) $\cosh(\ln x)$ (b) $\sinh(\ln x)$
(c) $\tanh(2\ln x)$ (d) $\cosh(-\ln x)$

5. In each part, a value for one of the hyperbolic functions is given at an unspecified positive number x_0. Use appropriate identities to find the exact values of the remaining five hyperbolic functions at x_0.
(a) $\sinh x_0 = 2$ (b) $\cosh x_0 = \frac{5}{4}$ (c) $\tanh x_0 = \frac{4}{5}$

6. Obtain the derivative formulas for $\operatorname{csch} x$, $\operatorname{sech} x$, and $\coth x$ from the derivative formulas for $\sinh x$, $\cosh x$, and $\tanh x$.

7. Find the derivatives of $\sinh^{-1} x$, $\cosh^{-1} x$, and $\tanh^{-1} x$ by differentiating the equations $x = \sinh y$, $x = \cosh y$, and $x = \tanh y$ implicitly.

C **8.** Use a CAS to find the derivatives of $\sinh^{-1} x$, $\cosh^{-1} x$, $\tanh^{-1} x$, $\coth^{-1} x$, $\operatorname{sech}^{-1} x$, and $\operatorname{csch}^{-1} x$, and confirm that your answers are consistent with those in Theorem 7.8.5.

In Exercises 9–28, find dy/dx.

9. $y = \sinh(4x - 8)$ **10.** $y = \cosh(x^4)$

11. $y = \coth(\ln x)$ **12.** $y = \ln(\tanh 2x)$

13. $y = \operatorname{csch}(1/x)$ **14.** $y = \operatorname{sech}(e^{2x})$

15. $y = \sqrt{4x + \cosh^2(5x)}$ **16.** $y = \sinh^3(2x)$

17. $y = x^3 \tanh^2(\sqrt{x})$ **18.** $y = \sinh(\cos 3x)$

19. $y = \sinh^{-1}\left(\frac{1}{3}x\right)$ **20.** $y = \sinh^{-1}(1/x)$

21. $y = \ln(\cosh^{-1} x)$ **22.** $y = \cosh^{-1}(\sinh^{-1} x)$

23. $y = \dfrac{1}{\tanh^{-1} x}$ **24.** $y = (\coth^{-1} x)^2$

25. $y = \cosh^{-1}(\cosh x)$ **26.** $y = \sinh^{-1}(\tanh x)$

27. $y = e^x \operatorname{sech}^{-1}\sqrt{x}$ **28.** $y = (1 + x \operatorname{csch}^{-1} x)^{10}$

C **29.** Use a CAS to find the derivatives in Example 2. If the answers produced by the CAS do not match those in the text, then use appropriate identities to show that the answers are equivalent.

C **30.** For each of the derivatives you obtained in Exercises 9–28, use a CAS to check your answer. If the answer produced by the CAS does not match your own, show that the two answers are equivalent.

In Exercises 31–46, evaluate the integrals.

31. $\displaystyle\int \sinh^6 x \cosh x\, dx$ **32.** $\displaystyle\int \cosh(2x - 3)\, dx$

33. $\displaystyle\int \sqrt{\tanh x}\, \operatorname{sech}^2 x\, dx$ **34.** $\displaystyle\int \operatorname{csch}^2(3x)\, dx$

35. $\displaystyle\int \tanh x\, dx$ **36.** $\displaystyle\int \coth^2 x \operatorname{csch}^2 x\, dx$

37. $\displaystyle\int_{\ln 2}^{\ln 3} \tanh x \operatorname{sech}^3 x\, dx$ **38.** $\displaystyle\int_0^{\ln 3} \frac{e^x - e^{-x}}{e^x + e^{-x}}\, dx$

39. $\displaystyle\int \frac{dx}{\sqrt{1 + 9x^2}}$ **40.** $\displaystyle\int \frac{dx}{\sqrt{x^2 - 2}}$ $(x > \sqrt{2})$

41. $\displaystyle\int \frac{dx}{\sqrt{1 - e^{2x}}}$ $(x < 0)$ **42.** $\displaystyle\int \frac{\sin\theta\, d\theta}{\sqrt{1 + \cos^2\theta}}$

43. $\displaystyle\int \frac{dx}{x\sqrt{1 + 4x^2}}$ **44.** $\displaystyle\int \frac{dx}{\sqrt{9x^2 - 25}}$ $(x > 5/3)$

45. $\displaystyle\int_0^{1/2} \frac{dx}{1 - x^2}$ **46.** $\displaystyle\int_0^{\sqrt{3}} \frac{dt}{\sqrt{t^2 + 1}}$

C **47.** For each of the integrals you evaluated in Exercises 31–46, use a CAS to check your answer. If the answer produced by the CAS does not match your own, show that the two answers are equivalent.

48. Use a graphing utility to generate the graphs of $\sinh x$, $\cosh x$, and $\tanh x$ by expressing these functions in terms of e^x and e^{-x}. If your graphing utility can graph the hyperbolic functions directly, then generate the graphs that way as well.

49. Find the area enclosed by $y = \sinh 2x$, $y = 0$, and $x = \ln 3$.

50. Find the volume of the solid that is generated when the region enclosed by $y = \operatorname{sech} x$, $y = 0$, $x = 0$, and $x = \ln 2$ is revolved about the x-axis.

51. Find the volume of the solid that is generated when the region enclosed by $y = \cosh 2x$, $y = \sinh 2x$, $x = 0$, and $x = 5$ is revolved about the x-axis.

52. Approximate the positive value of the constant a such that the area enclosed by $y = \cosh ax$, $y = 0$, $x = 0$, and $x = 1$ is 2 square units. Express your answer to at least five decimal places.

53. Find the arc length of $y = \cosh x$ between $x = 0$ and $x = \ln 2$.

54. Find the arc length of the catenary $y = a\cosh(x/a)$ between $x = 0$ and $x = x_1$ $(x_1 > 0)$.

55. Prove that $\sinh x$ is an odd function of x and that $\cosh x$ is an even function of x, and check that this is consistent with the graphs in Figure 7.8.1.

In Exercises 56 and 57, prove the identities.

56. (a) $\cosh x + \sinh x = e^x$
(b) $\cosh x - \sinh x = e^{-x}$
(c) $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$
(d) $\sinh 2x = 2 \sinh x \cosh x$
(e) $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$
(f) $\cosh 2x = \cosh^2 x + \sinh^2 x$
(g) $\cosh 2x = 2\sinh^2 x + 1$
(h) $\cosh 2x = 2\cosh^2 x - 1$

57. (a) $1 - \tanh^2 x = \operatorname{sech}^2 x$
(b) $\tanh(x + y) = \dfrac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$
(c) $\tanh 2x = \dfrac{2\tanh x}{1 + \tanh^2 x}$

58. Prove:
(a) $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$, $x \geq 1$
(b) $\tanh^{-1} x = \dfrac{1}{2}\ln\left(\dfrac{1 + x}{1 - x}\right)$, $-1 < x < 1$.

59. Use Exercise 58 to obtain the derivative formulas for $\cosh^{-1} x$ and $\tanh^{-1} x$.

60. Prove:

$$\operatorname{sech}^{-1} x = \cosh^{-1}(1/x), \quad 0 < x \leq 1$$
$$\coth^{-1} x = \tanh^{-1}(1/x), \quad |x| > 1$$
$$\operatorname{csch}^{-1} x = \sinh^{-1}(1/x), \quad x \neq 0$$

61. Use Exercise 60 to express the integral

$$\int \frac{du}{1 - u^2}$$

entirely in terms of $\tanh^{-1}$.

62. Show that

(a) $\dfrac{d}{dx}[\operatorname{sech}^{-1}|x|] = -\dfrac{1}{x\sqrt{1 - x^2}}$

(b) $\dfrac{d}{dx}[\operatorname{csch}^{-1}|x|] = -\dfrac{1}{x\sqrt{1 + x^2}}$.

63. Find the limits, and confirm that they are consistent with the graphs in Figures 7.8.1 and 7.8.6.
(a) $\lim_{x\to+\infty} \sinh x$ (b) $\lim_{x\to-\infty} \sinh x$
(c) $\lim_{x\to+\infty} \tanh x$ (d) $\lim_{x\to-\infty} \tanh x$
(e) $\lim_{x\to+\infty} \sinh^{-1} x$ (f) $\lim_{x\to1^-} \tanh^{-1} x$

64. In each part, find the limit.
(a) $\lim_{x\to+\infty} (\cosh^{-1} x - \ln x)$ (b) $\lim_{x\to+\infty} \dfrac{\cosh x}{e^x}$

65. Use the first and second derivatives to show that the graph of $y = \tanh^{-1} x$ is always increasing and has an inflection point at the origin.

66. The integration formulas for $1/\sqrt{u^2 - a^2}$ in Theorem 7.8.6 are valid for $u > a$. Show that the following formula is valid for $u < -a$:

$$\int \frac{du}{\sqrt{u^2 - a^2}} = -\cosh^{-1}\left(-\frac{u}{a}\right) + C = \ln\left|u + \sqrt{u^2 - a^2}\right| + C$$

67. Show that $(\sinh x + \cosh x)^n = \sinh nx + \cosh nx$.

68. Show that

$$\int_{-a}^{a} e^{tx}\, dx = \frac{2\sinh at}{t}$$

69. A cable is suspended between two poles as shown in Figure 7.8.2. The equation of the curve formed by the cable is $y = a\cosh(x/a)$, where a is a positive constant. Suppose that the x-coordinates of the points of support are $x = -b$ and $x = b$, where $b > 0$.
(a) Show that the length L of the cable is given by

$$L = 2a \sinh \frac{b}{a}$$

(b) Show that the sag S (the vertical distance between the highest and lowest points on the cable) is given by

$$S = a \cosh \frac{b}{a} - a$$

Exercises 70 and 71 refer to the hanging cable described in Exercise 69.

70. Assuming that the cable is 120 ft long and the poles are 100 ft apart, approximate the sag in the cable by approximating a. Express your final answer to the nearest tenth of a foot. [*Hint:* First let $u = 50/a$.]

71. Assuming that the poles are 400 ft apart and the sag in the cable is 30 ft, approximate the length of the cable by approximating a. Express your final answer to the nearest tenth of a foot. [*Hint:* First let $u = 200/a$.]

72. The accompanying figure shows a person pulling a boat by holding a rope of length a attached to the bow and walking along the edge of a dock. If we assume that the rope is always tangent to the curve traced by the bow of the boat, then this curve, which is called a ***tractrix***, has the property that the segment of the tangent line between the curve and the y-axis has a constant length a. It can be proved that the equation of this tractrix is

$$y = a\,\text{sech}^{-1}\frac{x}{a} - \sqrt{a^2 - x^2}$$

(a) Show that to move the bow of the boat to a point (x, y), the person must walk a distance

$$D = a\,\text{sech}^{-1}\frac{x}{a}$$

from the origin.
(b) If the rope has a length of 15 m, how far must the person walk from the origin to bring the boat 10 m from the dock? Round your answer to two decimal places.
(c) Find the distance traveled by the bow along the tractrix as it moves from its initial position to the point where it is 5 m from the dock.

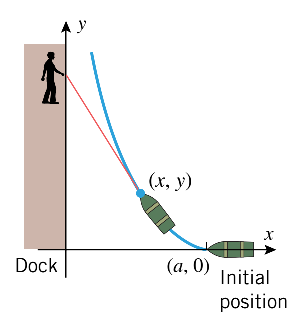


Figure Ex-72

SUPPLEMENTARY EXERCISES

Graphing Utility C CAS

1. (a) State conditions under which two functions, f and g, will be inverses, and give several examples of such functions.
(b) In words, what is the relationship between the graphs of $y = f(x)$ and $y = g(x)$ when f and g are inverse functions?
(c) What is the relationship between the domains and ranges of inverse functions f and g?
(d) What condition must be satisfied for a function f to have an inverse? Give some examples of functions that do not have inverses.
(e) If f and g are inverse functions and f is continuous, must g be continuous? Give a reasonable informal argument to support your answer.
(f) If f and g are inverse functions and f is differentiable, must g be differentiable? Give a reasonable informal argument to support your answer.

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2. (a) State the restrictions on the domains of $\sin x$, $\cos x$, $\tan x$, and $\sec x$ that are imposed to make those functions one-to-one in the definitions of $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, and $\sec^{-1} x$.
(b) Sketch the graphs of the restricted trigonometric functions in part (a) and their inverses.

3. In each part, find $f^{-1}(x)$ if the inverse exists.
(a) $f(x) = 8x^3 - 1$ (b) $f(x) = x^2 - 2x + 1$
(c) $f(x) = (e^x)^2 + 1$ (d) $f(x) = (x+2)/(x-1)$

4. Let $f(x) = (ax+b)/(cx+d)$. What conditions on a, b, c, d guarantee that f^{-1} exists? Find $f^{-1}(x)$.

5. Express the following function as a rational function of x:

$$3\ln\left(e^{2x}(e^x)^3\right) + 2\exp(\ln 1)$$

6. In each part, find the exact numerical value of the given expression.
(a) $\cos[\cos^{-1}(4/5) + \sin^{-1}(5/13)]$
(b) $\sin[\sin^{-1}(4/5) + \cos^{-1}(5/13)]$

7. In each part, prove the identity
(a) $\cosh 3x = 4\cosh^3 x - 3\cosh x$
(b) $\cosh \frac{1}{2}x = \sqrt{\frac{1}{2}(\cosh x + 1)}$
(c) $\sinh \frac{1}{2}x = \pm\sqrt{\frac{1}{2}(\cosh x - 1)}$

8. Suppose that $y = Ce^{kt}$, where C and k are constants, and let $Y = \ln y$. Show that the graph of Y versus t is a line, and state its slope and Y-intercept.

9. (a) Sketch the curves $y = \pm e^{-x/2}$ and $y = e^{-x/2}\sin 2x$ for $-\pi/2 \le x \le 3\pi/2$ in the same coordinate system, and check your work using a graphing utility.
(b) Find all x-intercepts of the curve $y = e^{-x/2}\sin 2x$ in the stated interval, and find the x-coordinates of all points where this curve intersects the curves $y = \pm e^{-x/2}$.

10. In each part, sketch the graph, and check your work with a graphing utility.
(a) $f(x) = 3\sin^{-1}(x/2)$
(b) $f(x) = \cos^{-1} x - \pi/2$
(c) $f(x) = 2\tan^{-1}(-3x)$
(d) $f(x) = \cos^{-1} x + \sin^{-1} x$

11. The design of the Gateway Arch in St. Louis, Missouri, by architect Eero Saarinan was implemented using equations provided by Dr. Hannskarl Badel. The equation used for the centerline of the arch was

$$y = 693.8597 - 68.7672\cosh(0.0100333x) \text{ ft}$$

for x between -299.2239 and 299.2239.
(a) Use a graphing utility to graph the centerline of the arch.
(b) Find the length of the centerline to four decimal places.
(c) For what values of x is the height of the arch 100 ft? Round your answers to four decimal places.
(d) Approximate, to the nearest degree, the acute angle that the tangent line to the centerline makes with the ground at the ends of the arch.

12. (a) Show that for $x > 0$ and $k \ne 0$ the equations

$$x^k = e^x \quad \text{and} \quad \frac{\ln x}{x} = \frac{1}{k}$$

have the same solutions.
(b) Use the graph of $y = (\ln x)/x$ to determine the values of k for which the equation $x^k = e^x$ has two distinct positive solutions.
(c) Estimate the positive solution(s) of $x^8 = e^x$.

13. (a) Show that the graphs of $y = \ln x$ and $y = x^{0.2}$ intersect.
(b) Approximate the solution(s) of the equation $\ln x = x^{0.2}$ to three decimal places.

14. Suppose that a hollow tube rotates with a constant angular velocity of ω rad/s about a horizontal axis at one end of the tube, as shown in the accompanying figure. Assume that an object is free to slide without friction in the tube while the tube is rotating. Let r be the distance from the object to the pivot point at time $t \ge 0$, and assume that the object is at rest and $r = 0$ when $t = 0$. It can be shown that if the tube is horizontal at time $t = 0$ and rotating as shown in the figure, then

$$r = \frac{g}{2\omega^2}[\sinh(\omega t) - \sin(\omega t)]$$

during the period that the object is in the tube. Assume that t is in seconds and r is in meters, and use $g = 9.8$ m/s^2 and $\omega = 2$ rad/s.
(a) Graph r versus t for $0 \le t \le 1$.
(b) Assuming that the tube has a length of 1 m, approximately how long does it take for the object to reach the end of the tube?
(c) Use the result of part (b) to approximate dr/dt at the instant that the object reaches the end of the tube.

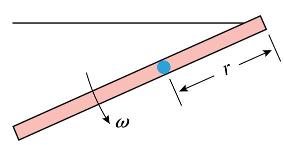

Figure Ex-14

15. In each part, use any appropriate method to find dy/dx.
(a) $y = e^{\ln(x^3+1)}$ (b) $y = \dfrac{a}{1+be^{-x}}$
(c) $y = \ln\left(\dfrac{\sqrt{x}\sqrt[3]{x+1}}{\sin x \sec x}\right)$ (d) $y = (1+x)^{1/x}$
(e) $y = x^{(e^x)}$ (f) $x^2 \sinh y = 1$

16. Show that the function $y = e^{ax}\sin bx$ satisfies

$$y'' - 2ay' + (a^2+b^2)y = 0$$

for any real constants a and b.

17. Show that the function $y = \tan^{-1} x$ satisfies

$$y'' = -2\sin y\cos^3 y$$

18. Show that for any constant a, the function $y = \sinh(ax)$ satisfies the equation $y'' = a^2 y$.

19. Find the value of b so that the line $y = x$ is tangent to the graph of $y = \log_b x$. Confirm your result by graphing both $y = x$ and $y = \log_b x$ in the same coordinate system.

20. In each part, find the value of k for which the graphs of $y = f(x)$ and $y = \ln x$ share a common tangent line at their point of intersection. Confirm your result by graphing $y = f(x)$ and $y = \ln x$ in the same coordinate system.
(a) $f(x) = \sqrt{x} + k$ (b) $f(x) = k\sqrt{x}$

In Exercises 21 and 22, find the absolute minimum m and the absolute maximum M of f on the given interval (if they exist), and state where the absolute extrema occur.

21. $f(x) = e^x/x^2$; $(0, +\infty)$ **22.** $f(x) = x^x$; $(0, +\infty)$

23. For $f(x) = 1/x$, find all values of x^* in the interval $[1, e]$ that satisfy Equation (7) in the Mean-Value Theorem for Integrals (5.6.2), and explain what these numbers represent.

C **24.** Suppose that the number of individuals at time t in a certain wildlife population is given by

$$N(t) = \frac{340}{1 + 9(0.77)^t}, \quad t \geq 0$$

where t is in years. At approximately what instant of time is the size of the population increasing most rapidly?

In Exercises 25–28, evaluate the integrals by hand, and check your answers with a CAS if you have one.

25. $\displaystyle\int_e^{e^2} \frac{dx}{x \ln x}$ **26.** $\displaystyle\int_0^1 \frac{dx}{\sqrt{e^x}}$

27. $\displaystyle\int_0^{\ln\sqrt{2}} \frac{1 + \cos(e^{-2x})}{e^{2x}}\,dx$

28. $\displaystyle\int \frac{e^{2x}}{e^x + 3}\,dx$
[*Hint:* Divide $e^x + 3$ into e^{2x}.]

29. Give a convincing geometric argument to show that

$$\int_1^e \ln x\,dx + \int_0^1 e^x\,dx = e$$

30. Find the limit by interpreting it as a limit of Riemann sums in which the interval $[0, 1]$ is divided into n subintervals of equal length.

$$\lim_{n\to+\infty} \frac{e^{1/n} + e^{2/n} + e^{3/n} + \cdots + e^{n/n}}{n}$$

31. (a) Divide the interval $[1, 2]$ into 5 subintervals of equal length, and use appropriate Riemann sums to show that

$$0.2\left[\tfrac{1}{1.2} + \tfrac{1}{1.4} + \tfrac{1}{1.6} + \tfrac{1}{1.8} + \tfrac{1}{2.0}\right] < \ln 2$$
$$< 0.2\left[\tfrac{1}{1.0} + \tfrac{1}{1.2} + \tfrac{1}{1.4} + \tfrac{1}{1.6} + \tfrac{1}{1.8}\right]$$

(b) Show that if the interval $[1, 2]$ is divided into n subintervals of equal length, then

$$\sum_{k=1}^{n} \frac{1}{n+k} < \ln 2 < \sum_{k=0}^{n-1} \frac{1}{n+k}$$

(c) Show that the difference between the two sums in part (b) is $1/(2n)$, and use this result to show that the sums in part (a) approximate $\ln 2$ with an error of at most 0.1.
(d) How large must n be to ensure that the sums in part (b) approximate $\ln 2$ to three decimal places?

32. Find the left endpoint, right endpoint, and midpoint approximations of the area under the curve $y = e^x$ over the interval $[0, 5]$ using $n = 5$ subintervals.

In Exercises 33 and 34, use a calculating utility to find the left endpoint, right endpoint, and midpoint approximations to the area under the curve $y = f(x)$ over the stated interval using $n = 10$ subintervals.

33. $y = \ln x$; $[1, 2]$ **34.** $y = e^x$; $[0, 1]$

35. Express the limit as a definite integral over $[0, 1]$, and then evaluate the limit by evaluating the integral.

$$\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} e^{x_k^*}\,\Delta x_k$$

36. Suppose that $\lim f(x) = \pm\infty$ and $\lim g(x) = \pm\infty$. In each of the four possible cases, state whether $\lim\,[f(x) - g(x)]$ is an indeterminate form, and give a reasonable informal argument to support your answer.

37. (a) Under what conditions will a limit of the form

$$\lim_{x\to a}\,[f(x)/g(x)]$$

be an indeterminate form?
(b) If $\lim_{x\to a} g(x) = 0$, must $\lim_{x\to a}\,[f(x)/g(x)]$ be an indeterminate form? Give some examples to support your
answer.

38. In each part, find the limit.
(a) $\displaystyle\lim_{x\to+\infty} (e^x - x^2)$ (b) $\displaystyle\lim_{x\to 1} \sqrt{\frac{\ln x}{x^4 - 1}}$
(c) $\displaystyle\lim_{x\to 0} \frac{a^x - 1}{x}, \quad a > 0$