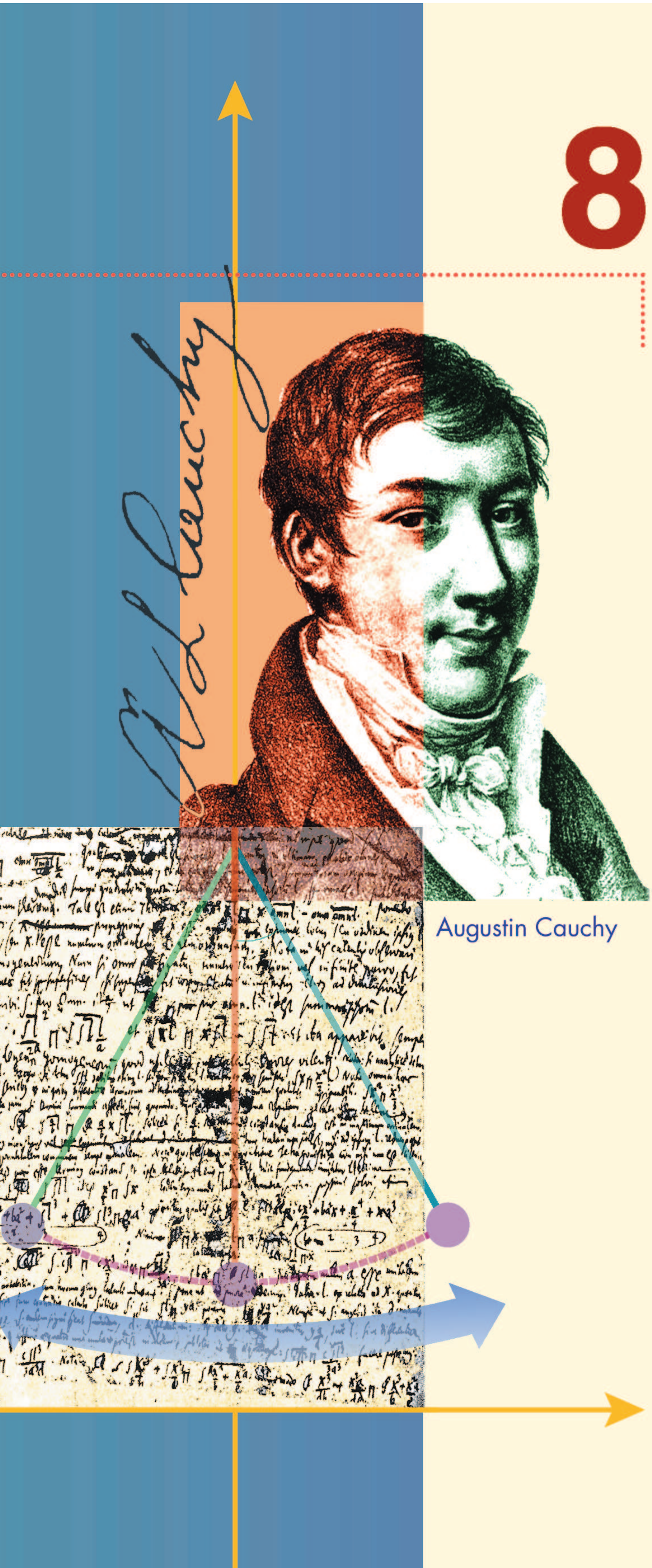

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8

PRINCIPLES OF INTEGRAL EVALUATION

In earlier chapters we obtained many basic integration formulas from the corresponding differentiation formulas. For example, knowing that the derivative of $\sin x$ is $\cos x$ enabled us to deduce that the integral of $\cos x$ is $\sin x$. Subsequently, we expanded our integration repertoire by introducing the method of u-substitution. That method enabled us to integrate many functions by transforming the integrand of an unfamiliar integral into a familiar form. However, u-substitution alone is not adequate to handle the wide variety of integrals that arise in applications, so additional integration techniques are still needed. In this chapter we will discuss some of those techniques, and we will provide a more systematic procedure for attacking unfamiliar integrals. We will talk more about numerical approximations of definite integrals, and we will explore the idea of integrating over infinite intervals.

8.1 AN OVERVIEW OF INTEGRATION METHODS

In this section we will give a brief overview of methods for evaluating integrals, and we will review the integration formulas that were discussed in earlier sections.

METHODS FOR APPROACHING INTEGRATION PROBLEMS

There are three basic approaches for evaluating unfamiliar integrals:

- **Technology**—CAS programs such as *Mathematica*, *Maple*, and *Derive* are capable of evaluating extremely complicated integrals, and for both the computer and handheld calculator such programs are increasingly available.
- **Tables**—Prior to the development of CAS programs, scientists relied heavily on tables to evaluate difficult integrals arising in applications. Such tables were compiled over many years, incorporating the skills and experience of many people. One such table appears in the endpapers of this text, but more comprehensive tables appear in various reference books such as the *CRC Standard Mathematical Tables and Formulae*, CRC Press, Inc., 1996.
- **Transformation Methods**—Transformation methods are methods for converting unfamiliar integrals into familiar integrals. These include u-substitution, algebraic manipulation of the integrand, and other methods that we will discuss in this chapter.

None of the three methods is perfect; for example, CAS programs often encounter integrals that they cannot evaluate and they sometimes produce answers that are excessively complicated, tables are not exhaustive and hence may not include a particular integral of interest, and transformation methods rely on human ingenuity that may prove to be inadequate in difficult problems.

In this chapter we will focus on transformation methods and tables, so it will *not be necessary* to have a CAS such as *Mathematica*, *Maple*, or *Derive*. However, if you have a CAS, then you can use it to confirm the results in the examples, and there are exercises that are designed to be solved with a CAS. If you have a CAS, keep in mind that many of the algorithms that it uses are based on the methods we will discuss here, so an understanding of these methods will help you to use your technology in a more informed way.

A REVIEW OF FAMILIAR INTEGRATION FORMULAS

The following is a list of basic integrals that we have encountered thus far:

CONSTANTS, POWERS, EXPONENTIALS

1. $\int du = u + C$

2. $\int a\,du = a\int du = au + C$

3. $\int u^r\,du = \frac{u^{r+1}}{r+1} + C,\ r \neq -1$

4. $\int \frac{du}{u} = \ln|u| + C$

5. $\int e^u\,du = e^u + C$

6. $\int b^u\,du = \frac{b^u}{\ln b} + C,\ b > 0, b \neq 1$

TRIGONOMETRIC FUNCTIONS

7. $\int \sin u\,du = -\cos u + C$

8. $\int \cos u\,du = \sin u + C$

9. $\int \sec^2 u\,du = \tan u + C$

10. $\int \csc^2 u\,du = -\cot u + C$

11. $\int \sec u \tan u\,du = \sec u + C$

12. $\int \csc u \cot u\,du = -\csc u + C$

13. $\int \tan u\,du = -\ln|\cos u| + C$

14. $\int \cot u\,du = \ln|\sin u| + C$

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HYPERBOLIC FUNCTIONS

15. $\int \sinh u\, du = \cosh u + C$

16. $\int \cosh u\, du = \sinh u + C$

17. $\int \operatorname{sech}^2 u\, du = \tanh u + C$

18. $\int \operatorname{csch}^2 u\, du = -\coth u + C$

19. $\int \operatorname{sech} u \tanh u\, du = -\operatorname{sech} u + C$

20. $\int \operatorname{csch} u \coth u\, du = -\operatorname{csch} u + C$

ALGEBRAIC FUNCTIONS $(a > 0)$

21. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \frac{u}{a} + C \qquad (|u| < a)$

22. $\int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1} \frac{u}{a} + C$

23. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \sec^{-1} \left|\frac{u}{a}\right| + C \qquad (0 < a < |u|)$

24. $\int \frac{du}{\sqrt{a^2 + u^2}} = \ln(u + \sqrt{u^2 + a^2}) + C$

25. $\int \frac{du}{\sqrt{u^2 - a^2}} = \ln\left|u + \sqrt{u^2 - a^2}\right| + C \qquad (0 < a < |u|)$

26. $\int \frac{du}{a^2 - u^2} = \frac{1}{2a} \ln\left|\frac{a + u}{a - u}\right| + C$

27. $\int \frac{du}{u\sqrt{a^2 - u^2}} = -\frac{1}{a} \ln\left|\frac{a + \sqrt{a^2 - u^2}}{u}\right| + C \qquad (0 < |u| < a)$

28. $\int \frac{du}{u\sqrt{a^2 + u^2}} = -\frac{1}{a} \ln\left|\frac{a + \sqrt{a^2 + u^2}}{u}\right| + C$

REMARK. Formula 23 is equivalent to Formula 23 of Section 7.6 (verify). Formula 25 is a generalization of a result in Theorem 7.8.6. Readers who did not cover Section 7.8 can ignore Formulas 24–28 for now, since we will develop other methods for obtaining them in this chapter.

EXERCISE SET 8.1

Review: Without looking at the text, complete the following integration formulas and then check your results by referring to the list of formulas at the beginning of this section.

Constants, Powers, Exponentials

$\int du =$ $\qquad \int a\, du =$

$\int u^r\, du =$ $\qquad \int \frac{du}{u} =$

$\int e^u\, du =$ $\qquad \int b^u\, du =$

Trigonometric Functions

$\int \sin u\, du =$ $\qquad \int \cos u\, du =$

$\int \sec^2 u\, du =$ $\qquad \int \csc^2 u\, du =$

$\int \sec u \tan u\, du =$ $\qquad \int \csc u \cot u\, du =$

$\int \tan u\, du =$ $\qquad \int \cot u\, du =$

Algebraic Functions

$\int \frac{du}{\sqrt{1-u^2}} =$ $\quad \int \frac{du}{1+u^2} =$

$\int \frac{du}{u\sqrt{u^2-1}} =$ $\quad \int \frac{du}{\sqrt{1+u^2}} =$

$\int \frac{du}{\sqrt{u^2-1}} =$ $\quad \int \frac{du}{1-u^2} =$

$\int \frac{du}{u\sqrt{1-u^2}} =$ $\quad \int \frac{du}{u\sqrt{1+u^2}} =$

Hyperbolic Functions

$\int \sinh u\, du =$ $\quad \int \cosh u\, du =$

$\int \operatorname{sech}^2 u\, du =$ $\quad \int \operatorname{csch}^2 u\, du =$

$\int \operatorname{sech} u \tanh u\, du =$

$\int \operatorname{csch} u \coth u\, du =$

In Exercises 1–30, evaluate the integrals by making appropriate u-substitutions and applying the formulas reviewed in this section.

1. $\int (3-2x)^3\, dx$ **2.** $\int \sqrt{4+9x}\, dx$

3. $\int x \sec^2(x^2)\, dx$ **4.** $\int 4x \tan(x^2)\, dx$

5. $\int \frac{\sin 3x}{2+\cos 3x}\, dx$ **6.** $\int \frac{1}{4+9x^2}\, dx$

7. $\int e^x \sinh(e^x)\, dx$ **8.** $\int \frac{\sec(\ln x)\tan(\ln x)}{x}\, dx$

9. $\int e^{\cot x} \csc^2 x\, dx$ **10.** $\int \frac{x}{\sqrt{1-x^4}}\, dx$

11. $\int \cos^5 7x \sin 7x\, dx$ **12.** $\int \frac{\cos x}{\sin x\sqrt{\sin^2 x+1}}\, dx$

13. $\int \frac{e^x}{\sqrt{4+e^{2x}}}\, dx$ **14.** $\int \frac{e^{\tan^{-1} x}}{1+x^2}\, dx$

15. $\int \frac{e^{\sqrt{x-2}}}{\sqrt{x-2}}\, dx$

16. $\int (3x+1)\cot(3x^2+2x)\, dx$

17. $\int \frac{\cosh\sqrt{x}}{\sqrt{x}}\, dx$ **18.** $\int \frac{dx}{x \ln x}$

19. $\int \frac{dx}{\sqrt{x}\, 3^{\sqrt{x}}}$

20. $\int \sec(\sin\theta)\tan(\sin\theta)\cos\theta\, d\theta$

21. $\int \frac{\operatorname{csch}^2(2/x)}{x^2}\, dx$ **22.** $\int \frac{dx}{\sqrt{x^2-3}}$

23. $\int \frac{e^{-x}}{4-e^{-2x}}\, dx$ **24.** $\int \frac{\cos(\ln x)}{x}\, dx$

25. $\int \frac{e^x}{\sqrt{1-e^{2x}}}\, dx$ **26.** $\int \frac{\sinh(x^{-1/2})}{x^{3/2}}\, dx$

27. $\int \frac{x}{\sec(x^2)}\, dx$ **28.** $\int \frac{e^x}{\sqrt{4-e^{2x}}}\, dx$

29. $\int x 4^{-x^2}\, dx$ **30.** $\int 2^{\pi x}\, dx$

8.2 INTEGRATION BY PARTS

In this section we will discuss an integration technique that is essentially an antiderivative formulation of the formula for differentiating a product of two functions.

THE PRODUCT RULE AND INTEGRATION BY PARTS

We saw in Section 5.3 that the u-substitution method of integration is based on the chain rule for differentiation. In this section we will examine a method of integration that is based on the *product rule* for differentiation. To motivate the general formula, we will consider the problem of evaluating $\int x \cos x\, dx$. Our approach to this problem will be by means of a two-step process. The first step is to choose a function whose derivative is the sum of *two* functions, *one* of which is $x \cos x$. For example, the function $x \sin x$ has this property, since by the product rule

$$\frac{d}{dx}(x \sin x) = x\cos x + \sin x$$

(Note that $x \sin x$ may be obtained from $x \cos x$ by integrating the $\cos x$ "part" of $x \cos x$ while leaving the x "part" alone.) The second step in evaluating $\int x \cos x\, dx$ is to subtract from our chosen function an antiderivative for the "extra" function that is produced by the product rule. What results will then be an antiderivative for $x \cos x$. For example, from the function $x \sin x$, we would need to subtract an antiderivative of $\sin x$. Since $-\cos x$ is an antiderivative of $\sin x$, we conclude that

$$x \sin x - (-\cos x) = x \sin x + \cos x$$

is an antiderivative of $x \cos x$. Indeed, this conclusion is easily verified since

$$\frac{d}{dx}(x \sin x + \cos x) = x \cos x + \sin x - \sin x = x \cos x$$

It follows that

$$\int x \cos x\, dx = x \sin x + \cos x + C$$

This two-step process is an illustration of a method of integration known as ***integration by parts***. More generally, suppose that we wish to evaluate an integral of the form $\int f(x)g(x)\, dx$. If $G(x)$ is an antiderivative of $g(x)$, then by the product rule for derivatives, the function $f(x)G(x)$ satisfies the equation

$$\frac{d}{dx}(f(x)G(x)) = f(x)g(x) + f'(x)G(x)$$

Consequently, if we subtract an antiderivative for $f'(x)G(x)$ from the function $f(x)G(x)$, the result will be an antiderivative for $f(x)g(x)$. We may express this conclusion symbolically by writing

$$\int f(x)g(x)\, dx = f(x)G(x) - \int f'(x)G(x)\, dx \qquad (1)$$

which is one version of the integration by parts formula. By using this formula we can sometimes reduce a difficult integration problem to an easier one.

In practice, it is usual to rewrite (1) by letting

$$u = f(x), \quad du = f'(x)\, dx$$
$$v = G(x), \quad dv = G'(x)\, dx = g(x)\, dx$$

This yields the following alternative form for (1):

$$\int u\, dv = uv - \int v\, du \qquad (2)$$

To illustrate the use of Formula (2) we will reevaluate $\int x \cos x\, dx$. The first step is to make a choice of u and dv. We will let $u = x$ and $dv = \cos x\, dx$ from which it follows that $du = dx$ and $v = \sin x$. Then, from Formula (2)

$$\int x \cos x\, dx = x \sin x - \int \sin x\, dx$$
$$= x \sin x - (-\cos x) + C = x \sin x + \cos x + C$$

REMARK. In the calculation of $v = \sin x$ from $dv = \cos x\, dx$, we omitted a constant of integration. Had we included a constant of integration and written $v = \sin x + C_1$, the constant C_1 would have eventually canceled out [Exercise 62(a)]. This is always the case in integration by parts [Exercise 62(b)], and it is common to omit consideration of a constant of integration when going from dv to v. However, in certain cases a clever choice of a constant of integration can simplify the computation of $\int v\, du$ [Exercises 63–65].

REMARK. To use integration by parts successfully, the choice of u and dv must be made so that the new integral is easier than the original. For example, if we decided above to let

$$u = \cos x, \quad dv = x\,dx, \quad du = -\sin x\,dx, \quad v = \frac{x^2}{2}$$

then we would have obtained

$$\int x\cos x\,dx = \frac{x^2}{2}\cos x - \int \frac{x^2}{2}(-\sin x)\,dx = \frac{x^2}{2}\cos x + \frac{1}{2}\int x^2 \sin x\,dx$$

For this choice of u and dv, the new integral is actually more complicated than the original. In general there are no hard and fast rules for choosing u and dv; it is mainly a matter of experience that comes from lots of practice.

For the case in which the integrand is the product of different "types" of functions, an interesting mnemonic device was suggested by Herbert Kasube in his article "A Technique for Integration by Parts" (*American Mathematical Monthly*, Vol. 90, 1983, pp. 210–211). In this article the author suggests the use of the acronym LIATE, which is short for **l**ogarithmic, **i**nverse trigonometric, **a**lgebraic, **t**rigonometric, and **e**xponential. According to the author, when the integrand of an integration by parts problem consists of the product of two different types of functions, we should let u designate the function that appears first in LIATE, and let dv denote the rest. For example, since the integrand of $\int x\cos x\,dx$ is the product of the algebraic function x with the trigonometric function $\cos x$, we should let $u = x$ and $dv = \cos x\,dx$, which agrees with our choice in the reevaluation of this integral. Although LIATE does not always produce the correct choice of u and dv, it does work much of the time.

Example 1 Evaluate $\int xe^x\,dx$.

Solution. In this case the integrand is the product of the algebraic function x with the exponential function e^x. According to LIATE we should let

$$u = x \quad \text{and} \quad dv = e^x\,dx$$

so that

$$du = dx \quad \text{and} \quad v = e^x$$

Thus, from (2)

$$\int xe^x\,dx = \int u\,dv = uv - \int v\,du = xe^x - \int e^x\,dx = xe^x - e^x + C$$ ◀

In some cases there is only one reasonable choice of u and dv.

Example 2 Evaluate $\int \ln x\,dx$.

Solution. One choice is to let $u = 1$ and $dv = \ln x\,dx$. But with this choice finding v is equivalent to evaluating $\int \ln x\,dx$ and we have gained nothing. Therefore, the only reasonable choice is to let $u = \ln x$ and $dv = dx$, so that $du = (1/x)\,dx$ and $v = x$. Thus, from (2)

$$\int \ln x\,dx = \int u\,dv = uv - \int v\,du = x\ln x - \int dx = x\ln x - x + C$$ ◀

REPEATED INTEGRATION BY PARTS

It is sometimes necessary to use integration by parts more than once in the same problem.

Example 3 Evaluate $\int x^2 e^{-x}\,dx$.

Solution. Let

$$u = x^2, \quad dv = e^{-x}\,dx, \quad du = 2x\,dx, \quad v = -e^{-x}$$

so that from (2)

$$\int x^2e^{-x}\,dx = \int u\,dv = uv - \int v\,du = x^2(-e^{-x}) - \int -e^{-x}(2x)\,dx$$
$$= -x^2e^{-x} + 2\int xe^{-x}\,dx$$

The last integral is similar to the original except that we have replaced x^2 by x. Another integration by parts applied to $\int xe^{-x}\,dx$ will complete the problem. We let

$$u = x, \quad dv = e^{-x}\,dx, \quad du = dx, \quad v = -e^{-x}$$

so that

$$\int xe^{-x}\,dx = x(-e^{-x}) - \int -e^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} + C$$

Since $-xe^{-x} - e^{-x}$ is an antiderivative for xe^{-x}, it follows that

$$\int x^2e^{-x}\,dx = -x^2e^{-x} + 2\int xe^{-x}\,dx = -x^2e^{-x} + 2(-xe^{-x} - e^{-x}) + C$$
$$= -(x^2 + 2x + 2)e^{-x} + C$$ ◀

Note that the integrand in Example 3 is of the form $p(x)q(x)$, where $p(x) = x^2$ is a *polynomial* and $q(x) = e^{-x}$ is a function that can be *repeatedly integrated*. For integrands of this form, repeated integration by parts can be done more efficiently by means of a procedure known as ***tabular integration by parts***. The procedure depends on the fact that repeated differentiation of a polynomial eventually results in 0. Since the method is easier to illustrate than to describe. we will show how tabular integration by parts may be used to evaluate the integral in Example 3. The first step is to create the following table:

REPEATED DIFFERENTIATION		REPEATED ANTIDIFFERENTIATION
x^2	+	e^{-x}
$2x$	−	$-e^{-x}$
2	+	e^{-x}
0		$-e^{-x}$

The entries in the left column of the table are obtained by starting with $p(x) = x^2$ and repeatedly *differentiating* until 0 results. The entries in the right column are obtained by starting with $q(x) = e^{-x}$ and repeatedly *integrating* until an entry is opposite the 0 in the left column. The diagonal segments shown in the table are alternately labeled with + and − signs. To evaluate $\int x^2e^{-x}\,dx$, we sum the products of the entries joined by a diagonal, incorporating the sign of the corresponding diagonal into each product. It follows that

$$\int x^2e^{-x}\,dx = -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C = -(x^2 + 2x + 2)e^{-x} + C$$

which agrees with our result in Example 3.

A second example should make the procedure clear.

Example 4 In Example 9 of Section 5.3 we evaluated $\int x^2\sqrt{x-1}\,dx$ using u-substitution. Evaluate this integral using tabular integration by parts.

Solution. The integrand is the product of a polynomial $p(x) = x^2$ and a function

$$q(x) = \sqrt{x-1} = (x-1)^{1/2}$$

that can be repeatedly integrated. First we form the table:

REPEATED DIFFERENTIATION		REPEATED ANTIDIFFERENTIATION
x^2	$+$	$(x-1)^{1/2}$
$2x$	$-$	$\frac{2}{3}(x-1)^{3/2}$
2	$+$	$\frac{4}{15}(x-1)^{5/2}$
0		$\frac{8}{105}(x-1)^{7/2}$

Then it follows that

$$\int x^2\sqrt{x-1}\,dx = \tfrac{2}{3}x^2(x-1)^{3/2} - \tfrac{8}{15}x(x-1)^{5/2} + \tfrac{16}{105}(x-1)^{7/2} + C$$

We leave it for the reader to show that this solution is equivalent to that of Example 9 in Section 5.3. ◀

The next illustration of repeated integration by parts deserves special attention.

Example 5 Evaluate $\int e^x \cos x\,dx$.

Solution. Let

$$u = e^x,\quad dv = \cos x\,dx,\quad du = e^x\,dx,\quad v = \sin x$$

Thus,

$$\int e^x \cos x\,dx = \int u\,dv = uv - \int v\,du = e^x \sin x - \int e^x \sin x\,dx \tag{3}$$

Since the integral $\int e^x \sin x\,dx$ is similar in form to the original integral $\int e^x \cos x\,dx$, it seems that nothing has been accomplished. However, let us integrate this new integral by parts. We let

$$u = e^x,\quad dv = \sin x\,dx,\quad du = e^x\,dx,\quad v = -\cos x$$

Thus,

$$\int e^x \sin x\,dx = \int u\,dv = uv - \int v\,du = -e^x \cos x + \int e^x \cos x\,dx$$

Together with Equation (3) this yields

$$\int e^x \cos x\,dx = e^x \sin x + e^x \cos x - \int e^x \cos x\,dx \tag{4}$$

It appears that we are going in circles since our original integral has now reappeared on the right side of this equation. However, at this point it is helpful to remind ourselves of the *meaning* of Equation (4). Equation (4) is a symbolic way of stating that if $F(x)$ is any antiderivative of $e^x \cos x$, then the function $e^x \sin x + e^x \cos x - F(x)$ is also an antiderivative of $e^x \cos x$. In other words,

$$\begin{aligned} e^x \cos x &= \frac{d}{dx}[e^x \sin x + e^x \cos x - F(x)] = \frac{d}{dx}[e^x \sin x + e^x \cos x] - F'(x) \\ &= \frac{d}{dx}[e^x \sin x + e^x \cos x] - e^x \cos x \end{aligned}$$

Equivalently,

$$2e^x \cos x = \frac{d}{dx}[e^x \sin x + e^x \cos x]$$

or

$$e^x \cos x = \frac{1}{2}\frac{d}{dx}[e^x \sin x + e^x \cos x] = \frac{d}{dx}\left[\frac{1}{2}(e^x \sin x + e^x \cos x)\right]$$

[Note that this last equation may also be verified by direct computation of the derivative of $\frac{1}{2}(e^x \sin x + e^x \cos x)$.] It follows that

$$\int e^x \cos x\, dx = \tfrac{1}{2}(e^x \sin x + e^x \cos x) + C \tag{5}$$

We can also obtain Equation (5) directly from Equation (4) by an informal argument. The idea is to "solve" Equation (4) for $\int e^x \cos x\, dx$, adding the (necessary) constant of integration only at the very end. That is, from Equation (4) we obtain

$$2\int e^x \cos x\, dx = e^x \sin x + e^x \cos x$$

or

$$\int e^x \cos x\, dx = \tfrac{1}{2}(e^x \sin x + e^x \cos x)$$

Since the left side of this equation is an indefinite integral, we need a constant of integration C on the right side. Adding this C to the right side results in Equation (5). Although informal arguments such as these can save time, they must be used with care (Exercise 66).

INTEGRATION BY PARTS FOR DEFINITE INTEGRALS

For definite integrals the formula corresponding to (2) is

$$\int_a^b u\, dv = uv\Big]_a^b - \int_a^b v\, du \tag{6}$$

REMARK. It is important to keep in mind that the variables u and v in this formula are functions of x and that the limits of integration in (6) are limits on the variable x. Sometimes it is helpful to emphasize this by writing (6) as

$$\int_{x=a}^{x=b} u\, dv = uv\Big]_{x=a}^{x=b} - \int_{x=a}^{x=b} v\, du \tag{7}$$

The next example illustrates how integration by parts can be used to integrate the inverse trigonometric functions.

Example 6 Evaluate $\displaystyle\int_0^1 \tan^{-1} x\, dx$.

Solution. Let

$$u = \tan^{-1} x, \quad dv = dx, \quad du = \frac{1}{1+x^2}\, dx, \quad v = x$$

Thus,

$$\int_0^1 \tan^{-1} x\, dx = \int_0^1 u\, dv = uv\Big]_0^1 - \int_0^1 v\, du$$

$$= x \tan^{-1} x\Big]_0^1 - \int_0^1 \frac{x}{1+x^2}\, dx$$

The limits of integration refer to x; that is, $x = 0$ and $x = 1$.

But

$$\int_0^1 \frac{x}{1+x^2}\, dx = \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}\, dx = \frac{1}{2}\ln(1+x^2)\Big]_0^1 = \frac{1}{2}\ln 2$$

so

$$\int_0^1 \tan^{-1} x\, dx = x\tan^{-1} x\Big]_0^1 - \frac{1}{2}\ln 2 = \left(\frac{\pi}{4} - 0\right) - \frac{1}{2}\ln 2 = \frac{\pi}{4} - \ln\sqrt{2}$$ ◀

REDUCTION FORMULAS

Integration by parts can be used to derive ***reduction formulas*** for integrals. These are formulas that express an integral involving a power of a function in terms of an integral that involves a *lower* power of that function. For example, if n is a positive integer and $n \geq 2$, then integration by parts can be used to obtain the reduction formulas

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2}x\,dx \tag{8}$$

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx \tag{9}$$

To illustrate how such formulas can be obtained, let us derive (9). We begin by writing $\cos^n x$ as $\cos^{n-1} x \cdot \cos x$ and letting

$$u = \cos^{n-1} x \qquad dv = \cos x\,dx$$

$$du = (n-1)\cos^{n-2} x(-\sin x)\,dx \qquad v = \sin x$$

$$= -(n-1)\cos^{n-2} x \sin x\,dx$$

so that

$$\int \cos^n x\,dx = \int \cos^{n-1}x\cos x\,dx = \int u\,dv = uv - \int v\,du$$

$$= \cos^{n-1}x\sin x + (n-1)\int \sin^2 x\cos^{n-2}x\,dx$$

$$= \cos^{n-1}x\sin x + (n-1)\int (1-\cos^2 x)\cos^{n-2}x\,dx$$

$$= \cos^{n-1}x\sin x + (n-1)\int \cos^{n-2}x\,dx - (n-1)\int \cos^n x\,dx$$

We now appeal to an informal argument and "solve" for $\int \cos^n x\,dx$. (See our comments following Example 5.) Transposing the last term on the right to the left side yields

$$n\int \cos^n x\,dx = \cos^{n-1}x\sin x + (n-1)\int \cos^{n-2}x\,dx$$

from which (9) follows.

Reduction formulas (8) and (9) reduce the exponent of sine (or cosine) by 2. Thus, if the formulas are applied repeatedly, the exponent can eventually be reduced to 0 if n is even or 1 if n is odd, at which point the integration can be completed. We will discuss this method in more detail in the next section, but for now, here is an example that illustrates how reduction formulas work.

Example 7 Evaluate $\int \cos^4 x\,dx$.

Solution. From (9) with $n = 4$

$$\int \cos^4 x\,dx = \frac{1}{4}\cos^3 x\sin x + \frac{3}{4}\int \cos^2 x\,dx$$

Now apply (9) with $n = 2$.

$$= \frac{1}{4}\cos^3 x\sin x + \frac{3}{4}\left(\frac{1}{2}\cos x\sin x + \frac{1}{2}\int dx\right)$$

$$= \frac{1}{4}\cos^3 x\sin x + \frac{3}{8}\cos x\sin x + \frac{3}{8}x + C$$

◀

EXERCISE SET 8.2

In Exercises 1–40, evaluate the integral.

1. $\int xe^{-x}\,dx$

2. $\int xe^{3x}\,dx$

3. $\int x^2e^x\,dx$

4. $\int x^2e^{-2x}\,dx$

5. $\int x\sin 2x\,dx$

6. $\int x\cos 3x\,dx$

7. $\int x^2\cos x\,dx$

8. $\int x^2\sin x\,dx$

9. $\int \sqrt{x}\ln x\,dx$

10. $\int x\ln x\,dx$

11. $\int (\ln x)^2\,dx$

12. $\int \frac{\ln x}{\sqrt{x}}\,dx$

13. $\int \ln(2x+3)\,dx$

14. $\int \ln(x^2+4)\,dx$

15. $\int \sin^{-1}x\,dx$

16. $\int \cos^{-1}(2x)\,dx$

17. $\int \tan^{-1}(2x)\,dx$

18. $\int x\tan^{-1}x\,dx$

19. $\int e^x\sin x\,dx$

20. $\int e^{2x}\cos 3x\,dx$

21. $\int e^{ax}\sin bx\,dx$

22. $\int e^{-3\theta}\sin 5\theta\,d\theta$

23. $\int \sin(\ln x)\,dx$

24. $\int \cos(\ln x)\,dx$

25. $\int x\sec^2 x\,dx$

26. $\int x\tan^2 x\,dx$

27. $\int x^3e^{x^2}\,dx$

28. $\int \frac{xe^x}{(x+1)^2}\,dx$

29. $\int_0^1 xe^{-5x}\,dx$

30. $\int_0^2 xe^{2x}\,dx$

31. $\int_1^e x^2\ln x\,dx$

32. $\int_{\sqrt{e}}^e \frac{\ln x}{x^2}\,dx$

33. $\int_{-2}^2 \ln(x+3)\,dx$

34. $\int_0^{1/2} \sin^{-1}x\,dx$

35. $\int_2^4 \sec^{-1}\sqrt{\theta}\,d\theta$

36. $\int_1^2 x\sec^{-1}x\,dx$

37. $\int_0^{\pi/2} x\sin 4x\,dx$

38. $\int_0^{\pi} (x+x\cos x)\,dx$

39. $\int_1^3 \sqrt{x}\tan^{-1}\sqrt{x}\,dx$

40. $\int_0^2 \ln(x^2+1)\,dx$

41. In each part, evaluate the integral by making a u-substitution and then integrating by parts.
(a) $\int e^{\sqrt{x}}\,dx$ (b) $\int \cos\sqrt{x}\,dx$

42. Prove that tabular integration by parts gives the correct answer for
$$\int p(x)q(x)\,dx$$
where $p(x)$ is any *quadratic polynomial* and $q(x)$ is any function that can be repeatedly integrated.

In Exercises 43–46, evaluate the integral using tabular integration by parts.

43. $\int (3x^2-x+2)e^{-x}\,dx$

44. $\int (x^2+x+1)\sin x\,dx$

45. $\int 8x^4\cos 2x\,dx$

46. $\int x^3\sqrt{2x+1}\,dx$

47. (a) Find the area of the region enclosed by $y=\ln x$, the line $x=e$, and the x-axis.
(b) Find the volume of the solid generated when the region in part (a) is revolved about the x-axis.

48. Find the area of the region between $y=x\sin x$ and $y=x$ for $0\le x\le \pi/2$.

49. Find the volume of the solid generated when the region between $y=\sin x$ and $y=0$ for $0\le x\le\pi$ is revolved about the y-axis.

50. Find the volume of the solid generated when the region enclosed between $y=\cos x$ and $y=0$ for $0\le x\le \pi/2$ is revolved about the y-axis.

51. A particle moving along the x-axis has velocity function $v(t)=t^2e^{-t}$. How far does the particle travel from time $t=0$ to $t=5$?

52. The study of sawtooth waves in electrical engineering leads to integrals of the form
$$\int_{-\pi/\omega}^{\pi/\omega} t\sin(k\omega t)\,dt$$
where k is an integer and ω is a nonzero constant. Evaluate the integral.

53. Use reduction formula (8) to evaluate
(a) $\int \sin^3 x\,dx$ (b) $\int_0^{\pi/4} \sin^4 x\,dx$.

54. Use reduction formula (9) to evaluate
(a) $\int \cos^5 x\,dx$ (b) $\int_0^{\pi/2} \cos^6 x\,dx$.

55. Derive reduction formula (8).

56. In each part, use integration by parts or other methods to derive the reduction formula.
(a) $\int \sec^n x\,dx = \frac{\sec^{n-2}x\tan x}{n-1}+\frac{n-2}{n-1}\int \sec^{n-2}x\,dx$
(b) $\int \tan^n x\,dx = \frac{\tan^{n-1}x}{n-1}-\int \tan^{n-2}x\,dx$
(c) $\int x^ne^x\,dx = x^ne^x - n\int x^{n-1}e^x\,dx$

In Exercises 57 and 58, use the reduction formulas in Exercise 56 to evaluate the integrals.

57. (a) $\int \tan^4 x\,dx$ (b) $\int \sec^4 x\,dx$ (c) $\int x^3 e^x\,dx$

58. (a) $\int x^2 e^{3x}\,dx$ (b) $\int_0^1 xe^{-\sqrt{x}}\,dx$

[*Hint:* First make a substitution.]

59. Let f be a function whose second derivative is continuous on $[-1, 1]$. Show that

$$\int_{-1}^{1} xf''(x)\,dx = f'(1) + f'(-1) - f(1) + f(-1)$$

60. Recall from Theorem 7.1.4 and the discussion preceding it that if $f'(x) > 0$, then the function f is increasing and has an inverse function. The purpose of this problem is to show that if this condition is satisfied and if f' is continuous, then a definite integral of f^{-1} can be expressed in terms of a definite integral of f.

(a) Use integration by parts to show that

$$\int_a^b f(x)\,dx = bf(b) - af(a) - \int_a^b xf'(x)\,dx$$

(b) Use the result in part (a) to show that if $y = f(x)$, then

$$\int_a^b f(x)\,dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} f^{-1}(y)\,dy$$

(c) Show that if we let $\alpha = f(a)$ and $\beta = f(b)$, then the result in part (b) can be written as

$$\int_\alpha^\beta f^{-1}(x)\,dx = \beta f^{-1}(\beta) - \alpha f^{-1}(\alpha) - \int_{f^{-1}(\alpha)}^{f^{-1}(\beta)} f(x)\,dx$$

61. In each part, use the result in Exercise 60 to obtain the equation, and then confirm that the equation is correct by performing the integrations.

(a) $\int_0^{1/2} \sin^{-1} x\,dx = \frac{1}{2}\sin^{-1}\left(\frac{1}{2}\right) - \int_0^{\pi/6} \sin x\,dx$

(b) $\int_e^{e^2} \ln x\,dx = (2e^2 - e) - \int_1^2 e^x\,dx$

62. (a) In the integral $\int x\cos x\,dx$, let

$$u = x, \quad dv = \cos x\,dx,$$
$$du = dx, \quad v = \sin x + C_1$$

Show that the constant C_1 cancels out, thus giving the same solution obtained by omitting C_1.

(b) Show that in general

$$uv - \int v\,du = u(v + C_1) - \int (v + C_1)\,du$$

thereby justifying the omission of the constant of integration when calculating v in integration by parts.

63. Evaluate $\int \ln(x+1)\,dx$ using integration by parts. Simplify the computation of $\int v\,du$ by introducing a constant of integration $C_1 = 1$ when going from dv to v.

64. Evaluate $\int \ln(2x+3)\,dx$ using integration by parts. Simplify the computation of $\int v\,du$ by introducing a constant of integration $C_1 = \frac{3}{2}$ when going from dv to v. Compare your solution with your answer to Exercise 13.

65. Evaluate $\int x\tan^{-1} x\,dx$ using integration by parts. Simplify the computation of $\int v\,du$ by introducing a constant of integration $C_1 = \frac{1}{2}$ when going from dv to v.

66. What equation results if integration by parts is applied to the integral

$$\int \frac{1}{x\ln x}\,dx$$

with the choices

$$u = \frac{1}{\ln x} \quad \text{and} \quad dv = \frac{1}{x}\,dx?$$

In what sense is this equation true? In what sense is it false?

8.3 TRIGONOMETRIC INTEGRALS

In the last section we derived reduction formulas for integrating positive integer powers of sine, cosine, tangent, and secant. In this section we will show how to work with those reduction formulas, and we will discuss methods for integrating other kinds of integrals that involve trigonometric functions.

INTEGRATING POWERS OF SINE AND COSINE

We begin by recalling two reduction formulas from the preceding section.

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx \tag{1}$$

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx \tag{2}$$

In the case where $n = 2$, these formulas yield

$$\int \sin^2 x\,dx = -\frac{1}{2}\sin x\cos x + \frac{1}{2}\int dx = \frac{1}{2}x - \frac{1}{2}\sin x\cos x + C \tag{3}$$

$$\int \cos^2 x\,dx = \frac{1}{2}\cos x\sin x + \frac{1}{2}\int dx = \frac{1}{2}x + \frac{1}{2}\sin x\cos x + C \tag{4}$$

Alternative forms of these integration formulas can be derived from the trigonometric identities

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x) \quad \text{and} \quad \cos^2 x = \tfrac{1}{2}(1 + \cos 2x) \tag{5–6}$$

which follow from the double-angle formulas

$$\cos 2x = 1 - 2\sin^2 x \quad \text{and} \quad \cos 2x = 2\cos^2 x - 1$$

These identities yield

$$\int \sin^2 x\,dx = \frac{1}{2}\int (1 - \cos 2x)\,dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C \tag{7}$$

$$\int \cos^2 x\,dx = \frac{1}{2}\int (1 + \cos 2x)\,dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C \tag{8}$$

Observe that the antiderivatives in Formulas (3) and (4) involve both sines and cosines, whereas those in (7) and (8) involve sines alone. However, the apparent discrepancy is easy to resolve by using the identity

$$\sin 2x = 2\sin x\cos x$$

to rewrite (7) and (8) in forms (3) and (4), or conversely.

In the case where $n = 3$, the reduction formulas for integrating $\sin^3 x$ and $\cos^3 x$ yield

$$\int \sin^3 x\,dx = -\frac{1}{3}\sin^2 x\cos x + \frac{2}{3}\int \sin x\,dx = -\frac{1}{3}\sin^2 x\cos x - \frac{2}{3}\cos x + C \tag{9}$$

$$\int \cos^3 x\,dx = \frac{1}{3}\cos^2 x\sin x + \frac{2}{3}\int \cos x\,dx = \frac{1}{3}\cos^2 x\sin x + \frac{2}{3}\sin x + C \tag{10}$$

If desired, Formula (9) can be expressed in terms of cosines alone by using the identity $\sin^2 x = 1 - \cos^2 x$, and Formula (10) can be expressed in terms of sines alone by using the identity $\cos^2 x = 1 - \sin^2 x$. We leave it for you to do this and confirm that

$$\int \sin^3 x\,dx = \tfrac{1}{3}\cos^3 x - \cos x + C \tag{11}$$

$$\int \cos^3 x\,dx = \sin x - \tfrac{1}{3}\sin^3 x + C \tag{12}$$

FOR THE READER. When asked to integrate $\sin^3 x$ and $\cos^3 x$, the *Maple* CAS produces forms (11) and (12). However, the *Mathematica* CAS produces

$$\int \sin^3 x\,dx = -\tfrac{3}{4}\cos x + \tfrac{1}{12}\cos 3x + C$$

$$\int \cos^3 x\,dx = \tfrac{3}{4}\sin x + \tfrac{1}{12}\sin 3x + C$$

See if you can reconcile *Mathematica*'s results with (11) and (12).

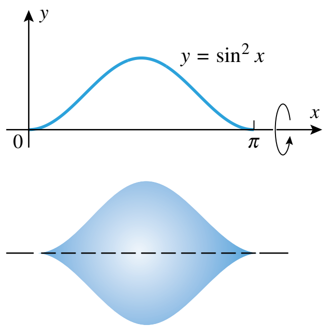


Figure 8.3.1

We leave it as an exercise to obtain the following formulas by first applying the reduction formulas, and then using appropriate trigonometric identities.

$$\int \sin^4 x\, dx = \tfrac{3}{8}x - \tfrac{1}{4}\sin 2x + \tfrac{1}{32}\sin 4x + C \qquad (13)$$

$$\int \cos^4 x\, dx = \tfrac{3}{8}x + \tfrac{1}{4}\sin 2x + \tfrac{1}{32}\sin 4x + C \qquad (14)$$

Example 1 Find the volume V of the solid that is obtained when the region under the curve $y = \sin^2 x$ over the interval $[0, \pi]$ is revolved about the x-axis (Figure 8.3.1).

Solution. Using the method of disks, Formula (5) of Section 6.2 yields

$$V = \int_0^\pi \pi \sin^4 x\, dx = \pi\left[\tfrac{3}{8}x - \tfrac{1}{4}\sin 2x + \tfrac{1}{32}\sin 4x\right]_0^\pi = \tfrac{3}{8}\pi^2$$ ◀

INTEGRATING PRODUCTS OF SINES AND COSINES

If m and n are positive integers, then the integral

$$\int \sin^m x \cos^n x\, dx$$

can be evaluated by one of the three procedures stated in Table 8.3.1, depending on whether m and n are odd or even.

Table 8.3.1

$\int \sin^m x \cos^n x\, dx$	PROCEDURE	RELEVANT IDENTITIES
n odd	• Split off a factor of $\cos x$. • Apply the relevant identity. • Make the substitution $u = \sin x$.	$\cos^2 x = 1 - \sin^2 x$
m odd	• Split off a factor of $\sin x$. • Apply the relevant identity. • Make the substitution $u = \cos x$.	$\sin^2 x = 1 - \cos^2 x$
$\begin{cases} m \text{ even} \\ n \text{ even} \end{cases}$	• Use the relevant identities to reduce the powers on $\sin x$ and $\cos x$.	$\begin{cases} \sin^2 x = \frac{1}{2}(1 - \cos 2x) \\ \cos^2 x = \frac{1}{2}(1 + \cos 2x) \end{cases}$

Example 2 Evaluate

(a) $\int \sin^4 x \cos^5 x\, dx$ (b) $\int \sin^4 x \cos^4 x\, dx$

Solution (a). Since $n = 5$ is odd, we will follow the first procedure in Table 8.3.1:

$$\begin{aligned}
\int \sin^4 x \cos^5 x\, dx &= \int \sin^4 x \cos^4 x \cos x\, dx \\
&= \int \sin^4 x (1 - \sin^2 x)^2 \cos x\, dx \\
&= \int u^4 (1 - u^2)^2\, du \\
&= \int (u^4 - 2u^6 + u^8)\, du \\
&= \tfrac{1}{5}u^5 - \tfrac{2}{7}u^7 + \tfrac{1}{9}u^9 + C \\
&= \tfrac{1}{5}\sin^5 x - \tfrac{2}{7}\sin^7 x + \tfrac{1}{9}\sin^9 x + C
\end{aligned}$$

Solution (b). Since $m = n = 4$, both exponents are even, so we will follow the third procedure in Table 8.3.1:

$$\int \sin^4 x \cos^4 x\, dx = \int (\sin^2 x)^2 (\cos^2 x)^2\, dx$$

$$= \int \left(\frac{1}{2}[1 - \cos 2x]\right)^2 \left(\frac{1}{2}[1 + \cos 2x]\right)^2 dx$$

$$= \frac{1}{16} \int (1 - \cos^2 2x)^2\, dx$$

$$= \frac{1}{16} \int \sin^4 2x\, dx$$

Note that this can be obtained more directly from the original integral using the identity $\sin x \cos x = \frac{1}{2}\sin 2x$.

$$= \frac{1}{32} \int \sin^4 u\, du$$

$u = 2x$
$du = 2\,dx$ or $dx = \frac{1}{2}\,du$

$$= \frac{1}{32}\left(\frac{3}{8}u - \frac{1}{4}\sin 2u + \frac{1}{32}\sin 4u\right) + C$$

Formula (13)

$$= \frac{3}{128}x - \frac{1}{128}\sin 4x + \frac{1}{1024}\sin 8x + C$$ ◀

Integrals of the form

$$\int \sin mx \cos nx\, dx, \quad \int \sin mx \sin nx\, dx, \quad \int \cos mx \cos nx\, dx \tag{15}$$

can be found by using the trigonometric identities

$$\sin\alpha \cos\beta = \tfrac{1}{2}[\sin(\alpha - \beta) + \sin(\alpha + \beta)] \tag{16}$$

$$\sin\alpha \sin\beta = \tfrac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)] \tag{17}$$

$$\cos\alpha \cos\beta = \tfrac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)] \tag{18}$$

to express the integrand as a sum or difference of sines and cosines.

Example 3 Evaluate $\int \sin 7x \cos 3x\, dx$.

Solution. Using (16) yields

$$\int \sin 7x \cos 3x\, dx = \frac{1}{2}\int (\sin 4x + \sin 10x)\, dx = -\frac{1}{8}\cos 4x - \frac{1}{20}\cos 10x + C$$ ◀

INTEGRATING POWERS OF TANGENT AND SECANT

The procedures for integrating powers of tangent and secant closely parallel those for sine and cosine. The idea is to use the following reduction formulas (which were derived in Exercise 56 of Section 8.2) to reduce the exponent in the integrand until the resulting integral can be evaluated:

$$\int \tan^n x\, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\, dx \tag{19}$$

$$\int \sec^n x\, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\, dx \tag{20}$$

In the case where n is odd, the exponent can be reduced to 1, leaving us with the problem of integrating $\tan x$ or $\sec x$. These integrals are given by

$$\int \tan x\,dx = \ln|\sec x| + C \tag{21}$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C \tag{22}$$

Formula (21) can be obtained by writing

$$\begin{aligned}\int \tan x\,dx &= \int \frac{\sin x}{\cos x}\,dx\\ &= -\ln|\cos x| + C && \boxed{\begin{aligned}u &= \cos x\\ du &= -\sin x\,dx\end{aligned}}\\ &= \ln|\sec x| + C && \boxed{\ln|\cos x| = -\ln\frac{1}{|\cos x|}}\end{aligned}$$

Formula (22) requires a trick. We write

$$\begin{aligned}\int \sec x\,dx &= \int \sec x\left(\frac{\sec x + \tan x}{\sec x + \tan x}\right)dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx\\ &= \ln|\sec x + \tan x| + C && \boxed{\begin{aligned}u &= \sec x + \tan x\\ du &= (\sec^2 x + \sec x \tan x)\,dx\end{aligned}}\end{aligned}$$

The following basic integrals occur frequently and are worth noting:

$$\int \tan^2 x\,dx = \tan x - x + C \tag{23}$$

$$\int \sec^2 x\,dx = \tan x + C \tag{24}$$

Formula (24) is already known to us, since the derivative of $\tan x$ is $\sec^2 x$. Formula (23) can be obtained by applying reduction formula (19) with $n = 2$ (verify) or, alternatively, by using the identity

$$1 + \tan^2 x = \sec^2 x$$

to write

$$\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx = \tan x - x + C$$

The formulas

$$\int \tan^3 x\,dx = \tfrac{1}{2}\tan^2 x - \ln|\sec x| + C \tag{25}$$

$$\int \sec^3 x\,dx = \tfrac{1}{2}\sec x \tan x + \tfrac{1}{2}\ln|\sec x + \tan x| + C \tag{26}$$

can be deduced from (21), (22), and reduction formulas (19) and (20) as follows:

$$\int \tan^3 x\,dx = \frac{1}{2}\tan^2 x - \int \tan x\,dx = \frac{1}{2}\tan^2 x - \ln|\sec x| + C$$

$$\int \sec^3 x\,dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\int \sec x\,dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$$

INTEGRATING PRODUCTS OF TANGENTS AND SECANTS

If m and n are positive integers, then the integral

$$\int \tan^m x \sec^n x\,dx$$

can be evaluated by one of the three procedures stated in Table 8.3.2, depending on whether m and n are odd or even.

Table 8.3.2

$\int \tan^m x \sec^n x \, dx$	PROCEDURE	RELEVANT IDENTITIES
n even	• Split off a factor of $\sec^2 x$. • Apply the relevant identity. • Make the substitution $u = \tan x$.	$\sec^2 x = \tan^2 x + 1$
m odd	• Split off a factor of $\sec x \tan x$. • Apply the relevant identity. • Make the substitution $u = \sec x$.	$\tan^2 x = \sec^2 x - 1$
m even, n odd	• Use the relevant identities to reduce the integrand to powers of $\sec x$ alone. • Then use the reduction formula for powers of $\sec x$.	$\tan^2 x = \sec^2 x - 1$

Example 4 Evaluate

(a) $\int \tan^2 x \sec^4 x \, dx$ (b) $\int \tan^3 x \sec^3 x \, dx$ (c) $\int \tan^2 x \sec x \, dx$

Solution (a). Since $n = 4$ is even, we will follow the first procedure in Table 8.3.2:

$$\begin{aligned}\int \tan^2 x \sec^4 x \, dx &= \int \tan^2 x \sec^2 x \sec^2 x \, dx \\ &= \int \tan^2 x (\tan^2 x + 1) \sec^2 x \, dx \\ &= \int u^2 (u^2 + 1) \, du \\ &= \tfrac{1}{5}u^5 + \tfrac{1}{3}u^3 + C = \tfrac{1}{5}\tan^5 x + \tfrac{1}{3}\tan^3 x + C\end{aligned}$$

Solution (b). Since $m = 3$ is odd, we will follow the second procedure in Table 8.3.2:

$$\begin{aligned}\int \tan^3 x \sec^3 x \, dx &= \int \tan^2 x \sec^2 x (\sec x \tan x) \, dx \\ &= \int (\sec^2 x - 1) \sec^2 x (\sec x \tan x) \, dx \\ &= \int (u^2 - 1) u^2 \, du \\ &= \tfrac{1}{5}u^5 - \tfrac{1}{3}u^3 + C = \tfrac{1}{5}\sec^5 x - \tfrac{1}{3}\sec^3 x + C\end{aligned}$$

Solution (c). Since $m = 2$ is even and $n = 1$ is odd, we will follow the third procedure in Table 8.3.2:

$$\begin{aligned}\int \tan^2 x \sec x \, dx &= \int (\sec^2 x - 1) \sec x \, dx \\ &= \int \sec^3 x \, dx - \int \sec x \, dx \\ &= \tfrac{1}{2}\sec x \tan x + \tfrac{1}{2}\ln|\sec x + \tan x| - \ln|\sec x + \tan x| + C \\ &= \tfrac{1}{2}\sec x \tan x - \tfrac{1}{2}\ln|\sec x + \tan x| + C\end{aligned}$$

See (26) and (22).

◀

AN ALTERNATIVE METHOD FOR INTEGRATING POWERS OF SINE, COSINE, TANGENT, AND SECANT

The methods in Tables 8.3.1 and 8.3.2 can sometimes be applied if $m = 0$ or $n = 0$ to integrate positive integer powers of sine, cosine, tangent, and secant without reduction formulas. For example, instead of using the reduction formula to integrate $\sin^3 x$, we can apply the second procedure in Table 8.3.1.

$$\begin{aligned}\int \sin^3 x\,dx &= \int (\sin^2 x)\sin x\,dx \\ &= \int (1-\cos^2 x)\sin x\,dx \qquad \boxed{\begin{aligned}u &= \cos x\\ du &= -\sin x\,dx\end{aligned}} \\ &= -\int (1-u^2)\,du \\ &= \tfrac{1}{3}u^3 - u + C = \tfrac{1}{3}\cos^3 x - \cos x + C\end{aligned}$$

which agrees with (11).

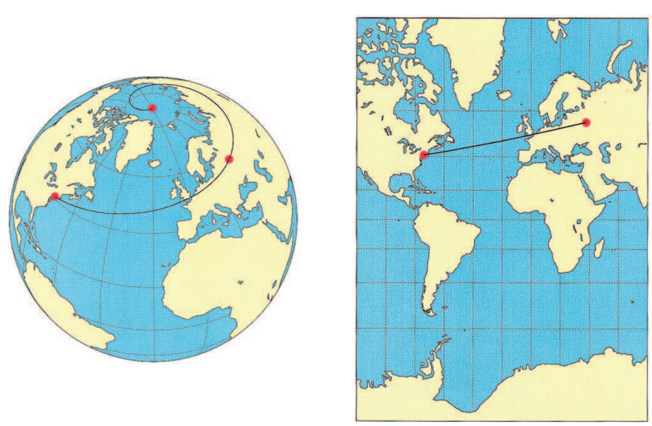
REMARK. With the aid of the identity $1 + \cot^2 x = \csc^2 x$ the techniques in Table 8.3.2 can be adapted to treat integrals of the form

$$\int \cot^m x \csc^n x\,dx$$

Also, there are reduction formulas for powers of cosecant and cotangent that are analogous to Formulas (19) and (20).

MERCATOR'S MAP OF THE WORLD

The integral of $\sec x$ plays an important role in the design of navigational maps for charting nautical and aeronautical courses. Sailors and pilots usually chart their courses along paths with constant compass headings; for example, the course might be 30° northeast or 135° southwest. Except for courses that are parallel to the equator or run due north or south, a course with constant compass heading spirals around the Earth toward one of the poles (as in Figure 8.3.2*a*). However, in 1569 the Flemish mathematician and geographer Gerhard Kramer (1512–1594) (better known by the Latin name Mercator) devised a world map, called the *Mercator projection*, in which spirals of constant compass headings appear as straight lines. This was extremely important because it enabled sailors to determine compass headings between two points by connecting them with a straight line on a map (Figure 8.3.2*b*).



A flight with constant compass heading from New York City to Moscow as it appears on a globe

(*a*)

A flight with constant compass heading from New York City to Moscow as it appears on a Mercator projection

(*b*)

Figure 8.3.2

If the Earth is assumed to be a sphere of radius 4000 mi, then the lines of latitude at 1° increments are equally spaced about 70 mi apart (why?). However, in the Mercator projection, the lines of latitude become wider apart toward the poles, so that two widely spaced latitude lines near the poles may be actually the same distance apart on the Earth as two closely spaced latitude lines near the equator. It can be proved that on a Mercator map in which the equatorial line has length L, the vertical distance D_β on the map between the equator (latitude 0°) and the line of latitude $\beta°$ is

$$D_\beta = \frac{L}{2\pi}\int_0^{\beta\pi/180} \sec x\, dx \qquad (27)$$

(see Exercises 59 and 60).

EXERCISE SET 8.3

In Exercises 1–52, evaluate the integral.

1. $\int \cos^5 x \sin x\, dx$
2. $\int \sin^4 3x \cos 3x\, dx$
3. $\int \sin ax \cos ax\, dx$
4. $\int \cos^2 3x\, dx$
5. $\int \sin^2 5\theta\, d\theta$
6. $\int \cos^3 at\, dt$
7. $\int \cos^5 \theta\, d\theta$
8. $\int \sin^3 x \cos^3 x\, dx$
9. $\int \sin^2 2t \cos^3 2t\, dt$
10. $\int \sin^3 2x \cos^2 2x\, dx$
11. $\int \sin^2 x \cos^2 x\, dx$
12. $\int \sin^2 x \cos^4 x\, dx$
13. $\int \sin x \cos 2x\, dx$
14. $\int \sin 3\theta \cos 2\theta\, d\theta$
15. $\int \sin x \cos(x/2)\, dx$
16. $\int \cos^{1/5} x \sin x\, dx$
17. $\int_0^{\pi/4} \cos^3 x\, dx$
18. $\int_0^{\pi/2} \sin^2 \frac{x}{2} \cos^2 \frac{x}{2}\, dx$
19. $\int_0^{\pi/3} \sin^4 3x \cos^3 3x\, dx$
20. $\int_{-\pi}^{\pi} \cos^2 5\theta\, d\theta$
21. $\int_0^{\pi/6} \sin 2x \cos 4x\, dx$
22. $\int_0^{2\pi} \sin^2 kx\, dx$
23. $\int \sec^2(3x+1)\, dx$
24. $\int \tan 5x\, dx$
25. $\int e^{-2x} \tan(e^{-2x})\, dx$
26. $\int \cot 3x\, dx$
27. $\int \sec 2x\, dx$
28. $\int \frac{\sec(\sqrt{x})}{\sqrt{x}}\, dx$
29. $\int \tan^2 x \sec^2 x\, dx$
30. $\int \tan^5 x \sec^4 x\, dx$
31. $\int \tan^3 4x \sec^4 4x\, dx$
32. $\int \tan^4 \theta \sec^4 \theta\, d\theta$
33. $\int \sec^5 x \tan^3 x\, dx$
34. $\int \tan^5 \theta \sec \theta\, d\theta$
35. $\int \tan^4 x \sec x\, dx$
36. $\int \tan^2 \frac{x}{2} \sec^3 \frac{x}{2}\, dx$
37. $\int \tan 2t \sec^3 2t\, dt$
38. $\int \tan x \sec^5 x\, dx$
39. $\int \sec^4 x\, dx$
40. $\int \sec^5 x\, dx$
41. $\int \tan^4 x\, dx$
42. $\int \tan^3 4x\, dx$
43. $\int \sqrt{\tan x} \sec^4 x\, dx$
44. $\int \tan x \sec^{3/2} x\, dx$
45. $\int_0^{\pi/6} \tan^2 2x\, dx$
46. $\int_0^{\pi/6} \sec^3 \theta \tan \theta\, d\theta$
47. $\int_0^{\pi/2} \tan^5 \frac{x}{2}\, dx$
48. $\int_0^{1/4} \sec \pi x \tan \pi x\, dx$
49. $\int \cot^3 x \csc^3 x\, dx$
50. $\int \cot^2 3t \sec 3t\, dt$
51. $\int \cot^3 x\, dx$
52. $\int \csc^4 x\, dx$
53. Let m, n be distinct nonnegative integers. Use Formulas (16)–(18) to prove:
   (a) $\int_0^{2\pi} \sin mx \cos nx\, dx = 0$
   (b) $\int_0^{2\pi} \cos mx \cos nx\, dx = 0$
   (c) $\int_0^{2\pi} \sin mx \sin nx\, dx = 0$.
54. Evaluate the integrals in Exercise 53 when m and n denote the *same* nonnegative integer.
55. Find the arc length of the curve $y = \ln(\cos x)$ over the interval $[0, \pi/4]$.

56. Find the volume of the solid generated when the region enclosed by $y = \tan x$, $y = 1$, and $x = 0$ is revolved about the x-axis.

57. Find the volume of the solid that results when the region enclosed by $y = \cos x$, $y = \sin x$, $x = 0$, and $x = \pi/4$ is revolved about the x-axis.

58. The region bounded below by the x-axis and above by the portion of $y = \sin x$ from $x = 0$ to $x = \pi$ is revolved about the x-axis. Find the volume of the resulting solid.

59. Use Formula (27) to show that if the length of the equatorial line on a Mercator projection is L, then the vertical distance D between the latitude lines at α° and β° on the same side of the equator (where $\alpha < \beta$) is

$$D = \frac{L}{2\pi}\ln\left|\frac{\sec\beta^\circ + \tan\beta^\circ}{\sec\alpha^\circ + \tan\alpha^\circ}\right|$$

60. Suppose that the equator has a length of 100 cm on a Mercator projection. In each part, use the result in Exercise 59 to answer the question.
(a) What is the vertical distance on the map between the equator and the line at 25° north latitude?
(b) What is the vertical distance on the map between New Orleans, Louisiana, at 30° north latitude and Winnepeg, Canada, at 50° north latitude?

61. (a) Show that

$$\int \csc x\,dx = -\ln|\csc x + \cot x| + C$$

(b) Show that the result in part (a) can also be written as

$$\int \csc x\,dx = \ln|\csc x - \cot x| + C$$

and

$$\int \csc x\,dx = \ln\left|\tan \tfrac{1}{2}x\right| + C$$

62. Rewrite $\sin x + \cos x$ in the form

$$A\sin(x + \phi)$$

and use your result together with Exercise 61 to evaluate

$$\int \frac{dx}{\sin x + \cos x}$$

63. Use the method of Exercise 62 to evaluate

$$\int \frac{dx}{a\sin x + b\cos x} \quad (a, b \text{ not both zero})$$

64. (a) Use Formula (8) in Section 8.2 to show that

$$\int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx \quad (n \ge 2)$$

(b) Use this result to derive the ***Wallis sine formulas***:

$$\int_0^{\pi/2} \sin^n x\,dx = \frac{\pi}{2}\cdot\frac{1\cdot 3\cdot 5\cdots(n-1)}{2\cdot 4\cdot 6\cdots n} \quad \begin{pmatrix} n \text{ even} \\ \text{and} \ge 2 \end{pmatrix}$$

$$\int_0^{\pi/2} \sin^n x\,dx = \frac{2\cdot 4\cdot 6\cdots(n-1)}{3\cdot 5\cdot 7\cdots n} \quad \begin{pmatrix} n \text{ odd} \\ \text{and} \ge 3 \end{pmatrix}$$

65. Use the Wallis formulas in Exercise 64 to evaluate

(a) $\int_0^{\pi/2} \sin^3 x\,dx$ (b) $\int_0^{\pi/2} \sin^4 x\,dx$

(c) $\int_0^{\pi/2} \sin^5 x\,dx$ (d) $\int_0^{\pi/2} \sin^6 x\,dx$.

66. Use Formula (9) in Section 8.2 and the method of Exercise 64 to derive the ***Wallis cosine formulas***:

$$\int_0^{\pi/2} \cos^n x\,dx = \frac{\pi}{2}\cdot\frac{1\cdot 3\cdot 5\cdots(n-1)}{2\cdot 4\cdot 6\cdots n} \quad \begin{pmatrix} n \text{ even} \\ \text{and} \ge 2 \end{pmatrix}$$

$$\int_0^{\pi/2} \cos^n x\,dx = \frac{2\cdot 4\cdot 6\cdots(n-1)}{3\cdot 5\cdot 7\cdots n} \quad \begin{pmatrix} n \text{ odd} \\ \text{and} \ge 3 \end{pmatrix}$$

8.4 TRIGONOMETRIC SUBSTITUTIONS

In this section we will discuss a method for evaluating integrals containing radicals by making substitutions involving trigonometric functions. We will also show how integrals containing quadratic polynomials can sometimes be evaluated by completing the square.

THE METHOD OF TRIGONOMETRIC SUBSTITUTION

To start, we will be concerned with integrals that contain expressions of the form

$$\sqrt{a^2 - x^2}, \quad \sqrt{x^2 + a^2}, \quad \sqrt{x^2 - a^2}$$

in which a is a positive constant. The basic idea for evaluating such integrals is to make a substitution for x that will eliminate the radical. For example, to eliminate the radical in the expression $\sqrt{a^2 - x^2}$, we can make the substitution

$$x = a\sin\theta, \quad -\pi/2 \le \theta \le \pi/2 \tag{1}$$

which yields

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta} = \sqrt{a^2(1-\sin^2\theta)}$$

$$= a\sqrt{\cos^2\theta} = a|\cos\theta| = a\cos\theta \qquad \boxed{\cos\theta \geq 0 \text{ since } -\pi/2 \leq \theta \leq \pi/2}$$

The restriction on θ in (1) serves two purposes—it enables us to replace $|\cos\theta|$ by $\cos\theta$ to simplify the calculations, and it also ensures that the substitutions can be rewritten as $\theta = \sin^{-1}(x/a)$, if needed.

Example 1 Evaluate $\displaystyle\int \frac{dx}{x^2\sqrt{4-x^2}}$.

Solution. To eliminate the radical we make the substitution

$$x = 2\sin\theta, \qquad dx = 2\cos\theta\, d\theta$$

This yields

$$\int \frac{dx}{x^2\sqrt{4-x^2}} = \int \frac{2\cos\theta\, d\theta}{(2\sin\theta)^2\sqrt{4-4\sin^2\theta}}$$

$$= \int \frac{2\cos\theta\, d\theta}{(2\sin\theta)^2(2\cos\theta)} = \frac{1}{4}\int \frac{d\theta}{\sin^2\theta}$$

$$= \frac{1}{4}\int \csc^2\theta\, d\theta = -\frac{1}{4}\cot\theta + C \qquad (2)$$

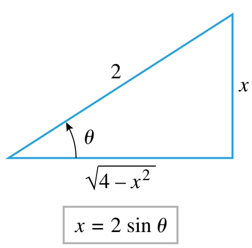


Figure 8.4.1

At this point we have completed the integration; however, because the original integral was expressed in terms of x, it is desirable to express $\cot\theta$ in terms of x as well. This can be done using trigonometric identities, but the expression can also be obtained by writing the substitution $x = 2\sin\theta$ as $\sin\theta = x/2$ and representing it geometrically as in Figure 8.4.1. From that figure we obtain

$$\cot\theta = \frac{\sqrt{4-x^2}}{x}$$

Substituting this in (2) yields

$$\int \frac{dx}{x^2\sqrt{4-x^2}} = -\frac{1}{4}\frac{\sqrt{4-x^2}}{x} + C$$

◀

Example 2 Evaluate $\displaystyle\int_1^{\sqrt{2}} \frac{dx}{x^2\sqrt{4-x^2}}$.

Solution. There are two possible approaches: we can make the substitution in the indefinite integral (as in Example 1) and then evaluate the definite integral using the x-limits of integration, or we can make the substitution in the definite integral and convert the x-limits to the corresponding θ-limits.

Method 1. Using the result from Example 1 with the x-limits of integration yields

$$\int_1^{\sqrt{2}} \frac{dx}{x^2\sqrt{4-x^2}} = -\frac{1}{4}\left[\frac{\sqrt{4-x^2}}{x}\right]_1^{\sqrt{2}} = -\frac{1}{4}[1-\sqrt{3}\,] = \frac{\sqrt{3}-1}{4}$$

Method 2. The substitution $x = 2\sin\theta$ can be expressed as $x/2 = \sin\theta$ or $\theta = \sin^{-1}(x/2)$, so the θ-limits that correspond to $x = 1$ and $x = \sqrt{2}$ are

$$x = 1: \qquad \theta = \sin^{-1}(1/2) = \pi/6$$

$$x = \sqrt{2}: \quad \theta = \sin^{-1}(\sqrt{2}/2) = \pi/4$$

Thus, from (2) in Example 1 we obtain

$$\int_1^{\sqrt{2}} \frac{dx}{x^2\sqrt{4-x^2}} = -\frac{1}{4}\left[\cot\theta\right]_{\pi/6}^{\pi/4} = -\frac{1}{4}[1-\sqrt{3}\,] = \frac{\sqrt{3}-1}{4}$$ ◀

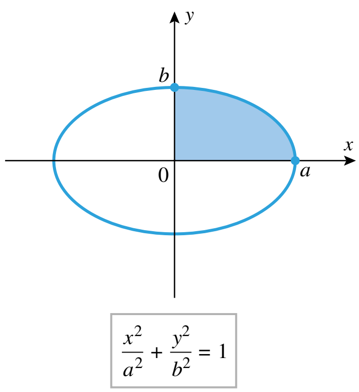


Figure 8.4.2

Example 3 Find the area of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Solution. Because the ellipse is symmetric about both axes, its area A is four times the area in the first quadrant (Figure 8.4.2). If we solve the equation of the ellipse for y in terms of x, we obtain

$$y = \pm\frac{b}{a}\sqrt{a^2-x^2}$$

where the positive square root gives the equation of the upper half. Thus, the area A is given by

$$A = 4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx = \frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$$

To evaluate this integral, we will make the substitution $x = a\sin\theta$ ($dx = a\cos\theta\,d\theta$) and convert the x-limits of integration to θ-limits. Since the substitution can be expressed as $\theta = \sin^{-1}(x/a)$, the θ-limits of integration are

$$x = 0\colon \quad \theta = \sin^{-1}(0) = 0$$

$$x = a\colon \quad \theta = \sin^{-1}(1) = \pi/2$$

Thus, we obtain

$$\begin{aligned} A &= \frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\,dx = \frac{4b}{a}\int_0^{\pi/2} a\cos\theta\cdot a\cos\theta\,d\theta \\ &= 4ab\int_0^{\pi/2}\cos^2\theta\,d\theta = 4ab\int_0^{\pi/2}\frac{1}{2}(1+\cos 2\theta)\,d\theta \\ &= 2ab\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/2} = 2ab\left[\frac{\pi}{2} - 0\right] = \pi ab \end{aligned}$$ ◀

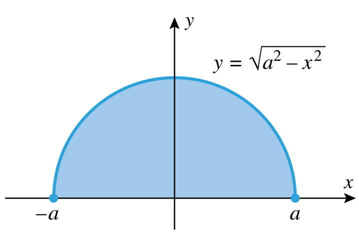


Figure 8.4.3

REMARK. In the special case where $a = b$, the ellipse becomes a circle of radius a, and the area formula becomes $A = \pi a^2$, as expected. It is worth noting that

$$\int_{-a}^{a} \sqrt{a^2-x^2}\,dx = \tfrac{1}{2}\pi a^2 \tag{3}$$

since this integral represents the area of the upper semicircle (Figure 8.4.3).

FOR THE READER. If you have a calculating utility with a numerical integration capability, use it and Formula (3) to approximate π to three decimal places.

Thus far, we have focused on using the substitution $x = a\sin\theta$ to evaluate integrals involving radicals of the form $\sqrt{a^2-x^2}$. Table 8.4.1 summarizes this method and describes some other substitutions of this type.

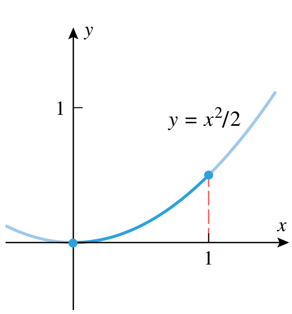


Figure 8.4.4

Example 4 Find the arc length of the curve $y = x^2/2$ from $x = 0$ to $x = 1$ (Figure 8.4.4).

Solution. From Formula (4) of Section 6.4 the arc length L of the curve is

$$L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_0^1 \sqrt{1+x^2}\,dx$$

The integrand involves a radical of the form $\sqrt{a^2+x^2}$ with $a = 1$, so from Table 8.4.1 we

Table 8.4.1

EXPRESSION IN THE INTEGRAND	SUBSTITUTION	RESTRICTION ON θ	SIMPLIFICATION
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$-\pi/2 \le \theta \le \pi/2$	$a^2 - x^2 = a^2 - a^2\sin^2\theta = a^2\cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$	$-\pi/2 < \theta < \pi/2$	$a^2 + x^2 = a^2 + a^2\tan^2\theta = a^2\sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$	$\begin{cases} 0 \le \theta < \pi/2 & (\text{if } x \ge a) \\ \pi/2 < \theta \le \pi & (\text{if } x \le -a) \end{cases}$	$x^2 - a^2 = a^2\sec^2\theta - a^2 = a^2\tan^2\theta$

make the substitution

$$x = \tan\theta, \quad -\pi/2 < \theta < \pi/2$$

$$\frac{dx}{d\theta} = \sec^2\theta \quad \text{or} \quad dx = \sec^2\theta\, d\theta$$

Since this substitution can be expressed as $\theta = \tan^{-1} x$, the θ-limits of integration that correspond to the x-limits, $x = 0$ and $x = 1$, are

$$x = 0: \quad \theta = \tan^{-1} 0 = 0$$

$$x = 1: \quad \theta = \tan^{-1} 1 = \pi/4$$

Thus,

$$\begin{aligned} L = \int_0^1 \sqrt{1 + x^2}\, dx &= \int_0^{\pi/4} \sqrt{1 + \tan^2\theta}\, \sec^2\theta\, d\theta \\ &= \int_0^{\pi/4} \sqrt{\sec^2\theta}\, \sec^2\theta\, d\theta \\ &= \int_0^{\pi/4} |\sec\theta| \sec^2\theta\, d\theta \\ &= \int_0^{\pi/4} \sec^3\theta\, d\theta \qquad \boxed{\sec\theta > 0 \text{ since } -\pi/2 < \theta < \pi/2} \\ &= \left[\tfrac{1}{2}\sec\theta\tan\theta + \tfrac{1}{2}\ln|\sec\theta + \tan\theta|\right]_0^{\pi/4} \qquad \boxed{\text{Formula (26) of Section 8.3}} \\ &= \tfrac{1}{2}[\sqrt{2} + \ln(\sqrt{2} + 1)] \approx 1.148 \end{aligned}$$

◀

Example 5 Evaluate $\displaystyle\int \frac{\sqrt{x^2 - 25}}{x}\, dx$, assuming that $x \ge 5$.

Solution. The integrand involves a radical of the form $\sqrt{x^2 - a^2}$ with $a = 5$, so from Table 8.4.1 we make the substitution

$$x = 5\sec\theta, \quad 0 \le \theta < \pi/2$$

$$\frac{dx}{d\theta} = 5\sec\theta\tan\theta \quad \text{or} \quad dx = 5\sec\theta\tan\theta\, d\theta$$

Thus,

$$\begin{aligned} \int \frac{\sqrt{x^2 - 25}}{x}\, dx &= \int \frac{\sqrt{25\sec^2\theta - 25}}{5\sec\theta}(5\sec\theta\tan\theta)\, d\theta \\ &= \int \frac{5|\tan\theta|}{5\sec\theta}(5\sec\theta\tan\theta)\, d\theta \\ &= 5\int \tan^2\theta\, d\theta \qquad \boxed{\tan\theta \ge 0 \text{ since } 0 \le \theta < \pi/2} \\ &= 5\int (\sec^2\theta - 1)\, d\theta = 5\tan\theta - 5\theta + C \end{aligned}$$

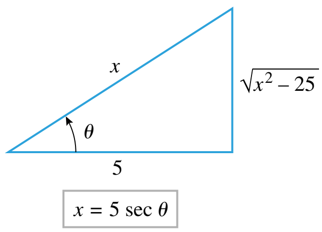


Figure 8.4.5

To express the solution in terms of x, we will represent the substitution $x = 5\sec\theta$ geometrically by the triangle in Figure 8.4.5, from which we obtain

$$\tan\theta = \frac{\sqrt{x^2-25}}{5}$$

From this and the fact that the substitution can be expressed as $\theta = \sec^{-1}(x/5)$, we obtain

$$\int \frac{\sqrt{x^2-25}}{x}\,dx = \sqrt{x^2-25} - 5\sec^{-1}\left(\frac{x}{5}\right) + C$$ ◀

INTEGRALS INVOLVING $ax^2 + bx + c$

Integrals that involve a quadratic expression $ax^2 + bx + c$, where $a \neq 0$ and $b \neq 0$, can often be evaluated by first completing the square, then making an appropriate substitution. The following examples illustrate this idea.

Example 6 Evaluate $\displaystyle\int \frac{x}{x^2-4x+8}\,dx$.

Solution. Completing the square yields

$$x^2 - 4x + 8 = (x^2 - 4x + 4) + 8 - 4 = (x-2)^2 + 4$$

Thus, the substitution

$$u = x - 2, \quad du = dx$$

yields

$$\begin{aligned}
\int \frac{x}{x^2-4x+8}\,dx &= \int \frac{x}{(x-2)^2+4}\,dx = \int \frac{u+2}{u^2+4}\,du \\
&= \int \frac{u}{u^2+4}\,du + 2\int \frac{du}{u^2+4} \\
&= \frac{1}{2}\int \frac{2u}{u^2+4}\,du + 2\int \frac{du}{u^2+4} \\
&= \frac{1}{2}\ln(u^2+4) + 2\left(\frac{1}{2}\right)\tan^{-1}\frac{u}{2} + C \\
&= \frac{1}{2}\ln[(x-2)^2+4] + \tan^{-1}\left(\frac{x-2}{2}\right) + C
\end{aligned}$$ ◀

Example 7 Evaluate $\displaystyle\int \frac{dx}{\sqrt{5-4x-2x^2}}$.

Solution. Completing the square yields

$$\begin{aligned}
5 - 4x - 2x^2 &= 5 - 2(x^2+2x) = 5 - 2(x^2+2x+1) + 2 \\
&= 5 - 2(x+1)^2 + 2 = 7 - 2(x+1)^2
\end{aligned}$$

Thus,

$$\begin{aligned}
\int \frac{dx}{\sqrt{5-4x-2x^2}} &= \int \frac{dx}{\sqrt{7-2(x+1)^2}} \\
&= \int \frac{du}{\sqrt{7-2u^2}} \qquad \boxed{u = x+1,\ du = dx} \\
&= \frac{1}{\sqrt{2}}\int \frac{du}{\sqrt{(7/2)-u^2}} \\
&= \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{u}{\sqrt{7/2}}\right) + C \qquad \boxed{\text{Formula 21, Section 8.1 with } a = \sqrt{7/2}} \\
&= \frac{1}{\sqrt{2}}\sin^{-1}(\sqrt{2/7}(x+1)) + C
\end{aligned}$$ ◀

EXERCISE SET 8.4 C CAS

In Exercises 1–26, evaluate the integral.

1. $\int \sqrt{4-x^2}\,dx$

2. $\int \sqrt{1-4x^2}\,dx$

3. $\int \frac{x^2}{\sqrt{9-x^2}}\,dx$

4. $\int \frac{dx}{x^2\sqrt{16-x^2}}$

5. $\int \frac{dx}{(4+x^2)^2}$

6. $\int \frac{x^2}{\sqrt{5+x^2}}\,dx$

7. $\int \frac{\sqrt{x^2-9}}{x}\,dx$

8. $\int \frac{dx}{x^2\sqrt{x^2-16}}$

9. $\int \frac{x^3}{\sqrt{2-x^2}}\,dx$

10. $\int x^3\sqrt{5-x^2}\,dx$

11. $\int \frac{dx}{x^2\sqrt{4x^2-9}}$

12. $\int \frac{\sqrt{1+t^2}}{t}\,dt$

13. $\int \frac{dx}{(1-x^2)^{3/2}}$

14. $\int \frac{dx}{x^2\sqrt{x^2+25}}$

15. $\int \frac{dx}{\sqrt{x^2-1}}$

16. $\int \frac{dx}{1+2x^2+x^4}$

17. $\int \frac{dx}{(9x^2-1)^{3/2}}$

18. $\int \frac{x^2}{\sqrt{x^2-25}}\,dx$

19. $\int e^x\sqrt{1-e^{2x}}\,dx$

20. $\int \frac{\cos\theta}{\sqrt{2-\sin^2\theta}}\,d\theta$

21. $\int_0^4 x^3\sqrt{16-x^2}\,dx$

22. $\int_0^{1/3} \frac{dx}{(4-9x^2)^2}$

23. $\int_{\sqrt{2}}^2 \frac{dx}{x^2\sqrt{x^2-1}}$

24. $\int_{\sqrt{2}}^2 \frac{\sqrt{2x^2-4}}{x}\,dx$

25. $\int_1^3 \frac{dx}{x^4\sqrt{x^2+3}}$

26. $\int_0^3 \frac{x^3}{(3+x^2)^{5/2}}\,dx$

27. The integral

$$\int \frac{x}{x^2+4}\,dx$$

can be evaluated either by a trigonometric substitution or by the substitution $u = x^2 + 4$. Do it both ways and show that the results are equivalent.

28. The integral

$$\int \frac{x^2}{x^2+4}\,dx$$

can be evaluated either by a trigonometric substitution or by algebraically rewriting the numerator of the integrand as $(x^2+4)-4$. Do it both ways and show that the results are equivalent.

29. Find the arc length of the curve $y = \ln x$ from $x = 1$ to $x = 2$.

30. Find the arc length of the curve $y = x^2$ from $x = 0$ to $x = 1$.

31. Find the area of the surface generated when the curve in Exercise 30 is revolved about the x-axis.

32. Find the volume of the solid generated when the region enclosed by $x = y(1-y^2)^{1/4}$, $y = 0$, $y = 1$, and $x = 0$ is revolved about the y-axis.

In Exercise 33, the trigonometric substitutions $x = a\sec\theta$ and $x = a\tan\theta$ lead to difficult integrals; for such integrals it is sometimes possible to use the ***hyperbolic substitutions***

$x = a\sinh u$ for integrals involving $\sqrt{x^2+a^2}$

$x = a\cosh u$ for integrals involving $\sqrt{x^2-a^2}$, $x \ge a$

These substitutions are useful because in each case the hyperbolic identity

$a^2\cosh^2 u - a^2\sinh^2 u = a^2$

removes the radical.

33. (a) Evaluate

$$\int \frac{dx}{\sqrt{x^2+9}}$$

using the hyperbolic substitution that is suggested above.

(b) Evaluate the integral in part (a) by a trigonometric substitution and show that the results in parts (a) and (b) agree.

(c) Use a hyperbolic substitution to evaluate

$$\int \sqrt{x^2-1}\,dx, \quad x \ge 1$$

34. In Example 3 we found the area of an ellipse by making the substitution $x = a\sin\theta$ in the required integral. Find the area by making the substitution $x = a\cos\theta$, and discuss any restrictions on θ that are needed.

In Exercises 35–46, evaluate the integral.

35. $\int \frac{dx}{x^2-4x+13}$

36. $\int \frac{dx}{\sqrt{2x-x^2}}$

37. $\int \frac{dx}{\sqrt{8+2x-x^2}}$

38. $\int \frac{dx}{16x^2+16x+5}$

39. $\int \frac{dx}{\sqrt{x^2-6x+10}}$

40. $\int \frac{x}{x^2+6x+10}\,dx$

41. $\int \sqrt{3-2x-x^2}\,dx$

42. $\int \frac{e^x}{\sqrt{1+e^x+e^{2x}}}\,dx$

43. $\int \frac{dx}{2x^2+4x+7}$

44. $\int \frac{2x+3}{4x^2+4x+5}\,dx$

45. $\int_1^2 \frac{dx}{\sqrt{4x-x^2}}$

46. $\int_0^1 \sqrt{x(4-x)}\,dx$

In Exercises 47 and 48, there is a good chance that your CAS will not be able to evaluate the integral as stated. If this is so, make a substitution that converts the integral into one that your CAS can evaluate.

C **47.** $\int \cos x \sin x \sqrt{1 - \sin^4 x}\, dx$

C **48.** $\int (x \cos x + \sin x)\sqrt{1 + x^2 \sin^2 x}\, dx$

8.5 INTEGRATING RATIONAL FUNCTIONS BY PARTIAL FRACTIONS

Recall that a rational function is a ratio of two polynomials. In this section we will give a general method for integrating rational functions that is based on the idea of decomposing a rational function into a sum of simple rational functions that can be integrated by the methods studied in earlier sections.

PARTIAL FRACTIONS

In algebra one learns to combine two or more fractions into a single fraction by finding a common denominator. For example,

$$\frac{2}{x-4} + \frac{3}{x+1} = \frac{2(x+1) + 3(x-4)}{(x-4)(x+1)} = \frac{5x-10}{x^2-3x-4} \tag{1}$$

However, for purposes of integration, the left side of (1) is preferable to the right side since each of the terms is easy to integrate:

$$\int \frac{5x-10}{x^2-3x-4}\, dx = \int \frac{2}{x-4}\, dx + \int \frac{3}{x+1}\, dx = 2\ln|x-4| + 3\ln|x+1| + C$$

Thus, it is desirable to have some method that will enable us to obtain the left side of (1), starting with the right side. To illustrate how this can be done, we begin by noting that on the left side the numerators are constants and the denominators are the factors of the denominator on the right side. Thus, to find the left side of (1), starting from the right side, we could factor the denominator of the right side and look for constants A and B such that

$$\frac{5x-10}{(x-4)(x+1)} = \frac{A}{x-4} + \frac{B}{x+1} \tag{2}$$

One way to find the constants A and B is to multiply (2) through by $(x-4)(x+1)$ to clear fractions. This yields

$$5x - 10 = A(x+1) + B(x-4) \tag{3}$$

This relationship holds for all x, so it holds in particular if $x = 4$ or $x = -1$. Substituting $x = 4$ in (3) makes the second term on the right drop out and yields the equation $10 = 5A$ or $A = 2$; and substituting $x = -1$ in (3) makes the first term on the right drop out and yields the equation $-15 = -5B$ or $B = 3$. Substituting these values in (2) we obtain

$$\frac{5x-10}{(x-4)(x+1)} = \frac{2}{x-4} + \frac{3}{x+1} \tag{4}$$

which agrees with (1).

A second method for finding the constants A and B is to multiply out the right side of (3) and collect like powers of x to obtain

$$5x - 10 = (A+B)x + (A - 4B)$$

Since the polynomials on the two sides are identical, their corresponding coefficients must be the same. Equating the corresponding coefficients on the two sides yields the following system of equations in the unknowns A and B:

$$\begin{aligned} A + B &= 5 \\ A - 4B &= -10 \end{aligned}$$

Solving this system yields $A = 2$ and $B = 3$ as before (verify).

The terms on the right side of (4) are called ***partial fractions*** of the expression on the left side because they each constitute *part* of that expression. To find those partial fractions we first had to make a guess about their form, and then we had to find the unknown constants. Our next objective is to extend this idea to general rational functions. For this purpose, suppose that $P(x)/Q(x)$ is a ***proper rational function***, by which we mean that the degree of the numerator is less than the degree of the denominator. There is a theorem in advanced algebra which states that every proper rational function can be expressed as a sum

$$\frac{P(x)}{Q(x)} = F_1(x) + F_2(x) + \cdots + F_n(x)$$

where $F_1(x), F_2(x), \ldots, F_n(x)$ are rational functions of the form

$$\frac{A}{(ax+b)^k} \quad \text{or} \quad \frac{Ax+B}{(ax^2+bx+c)^k}$$

in which the denominators are factors of $Q(x)$. The sum is called the ***partial fraction decomposition*** of $P(x)/Q(x)$, and the terms are called ***partial fractions***. As in our opening example, there are two parts to finding a partial fraction decomposition: determining the exact form of the decomposition and finding the unknown constants.

FINDING THE FORM OF A PARTIAL FRACTION DECOMPOSITION

The first step in finding the form of the partial fraction decomposition of a proper rational function $P(x)/Q(x)$ is to factor $Q(x)$ completely into linear and irreducible quadratic factors, and then collect all repeated factors so that $Q(x)$ is expressed as a product of *distinct* factors of the form

$$(ax+b)^m \quad \text{and} \quad (ax^2+bx+c)^m$$

From these factors we can determine the form of the partial fraction decomposition using two rules that we will now discuss.

LINEAR FACTORS

If all of the factors of $Q(x)$ are linear, then the partial fraction decomposition of $P(x)/Q(x)$ can be determined by using the following rule:

> LINEAR FACTOR RULE. For each factor of the form $(ax+b)^m$, the partial fraction decomposition contains the following sum of m partial fractions:
>
> $$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_m}{(ax+b)^m}$$
>
> where $A_1, A_2, \ldots, A_m$ are constants to be determined. In the case where $m = 1$, only the first term in the sum appears.

Example 1 Evaluate $\displaystyle\int \frac{dx}{x^2+x-2}$.

Solution. The integrand is a proper rational function that can be written as

$$\frac{1}{x^2+x-2} = \frac{1}{(x-1)(x+2)}$$

The factors $x - 1$ and $x + 2$ are both linear and appear to the first power, so each contributes one term to the partial fraction decomposition by the linear factor rule. Thus, the decomposition has the form

$$\frac{1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2} \tag{5}$$

where A and B are constants to be determined. Multiplying this expression through by $(x-1)(x+2)$ yields

$$1 = A(x+2) + B(x-1) \tag{6}$$

As discussed earlier, there are two methods for finding A and B: we can substitute values of x that are chosen to make terms on the right drop out, or we can multiply out on the right and equate corresponding coefficients on the two sides to obtain a system of equations that can be solved for A and B. We will use the first approach.

Setting $x = 1$ makes the second term in (6) drop out and yields $1 = 3A$ or $A = \frac{1}{3}$; and setting $x = -2$ makes the first term in (6) drop out and yields $1 = -3B$ or $B = -\frac{1}{3}$. Substituting these values in (5) yields the partial fraction decomposition

$$\frac{1}{(x-1)(x+2)} = \frac{\frac{1}{3}}{x-1} + \frac{-\frac{1}{3}}{x+2}$$

The integration can now be completed as follows:

$$\begin{aligned}\int \frac{dx}{(x-1)(x+2)} &= \frac{1}{3}\int \frac{dx}{x-1} - \frac{1}{3}\int \frac{dx}{x+2} \\ &= \frac{1}{3}\ln|x-1| - \frac{1}{3}\ln|x+2| + C = \frac{1}{3}\ln\left|\frac{x-1}{x+2}\right| + C\end{aligned}$$ ◀

If the factors of $Q(x)$ are linear and none are repeated, as in the last example, then the recommended method for finding the constants in the partial fraction decomposition is to substitute appropriate values of x to make terms drop out. However, if some of the linear factors are repeated, then it will not be possible to find all of the constants in this way. In this case the recommended procedure is to find as many constants as possible by substitution and then find the rest by equating coefficients. This is illustrated in the next example.

Example 2 Evaluate $\displaystyle\int \frac{2x+4}{x^3-2x^2}\,dx$.

Solution. The integrand can be rewritten as

$$\frac{2x+4}{x^3-2x^2} = \frac{2x+4}{x^2(x-2)}$$

Although x^2 is a quadratic factor, it is *not* irreducible since $x^2 = xx$. Thus, by the linear factor rule, x^2 introduces two terms (since $m = 2$) of the form

$$\frac{A}{x} + \frac{B}{x^2}$$

and the factor $x - 2$ introduces one term (since $m = 1$) of the form

$$\frac{C}{x-2}$$

so the partial fraction decomposition is

$$\frac{2x+4}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} \tag{7}$$

Multiplying by $x^2(x-2)$ yields

$$2x + 4 = Ax(x-2) + B(x-2) + Cx^2 \tag{8}$$

which, after multiplying out and collecting like powers of x, becomes

$$2x + 4 = (A + C)x^2 + (-2A + B)x - 2B \tag{9}$$

Setting $x = 0$ in (8) makes the first and third terms drop out and yields $B = -2$, and setting $x = 2$ in (8) makes the first and second terms drop out and yields $C = 2$ (verify). However, there is no substitution in (8) that produces A directly, so we look to Equation (9) to find this value. This can be done by equating the coefficients of x^2 on the two sides to obtain

$$A + C = 0 \quad \text{or} \quad A = -C = -2$$

Substituting the values $A = -2$, $B = -2$, and $C = 2$ in (7) yields the partial fraction decomposition

$$\frac{2x+4}{x^2(x-2)} = \frac{-2}{x} + \frac{-2}{x^2} + \frac{2}{x-2}$$

Thus,

$$\int \frac{2x+4}{x^2(x-2)}\,dx = -2\int \frac{dx}{x} - 2\int \frac{dx}{x^2} + 2\int \frac{dx}{x-2}$$

$$= -2\ln|x| + \frac{2}{x} + 2\ln|x-2| + C = 2\ln\left|\frac{x-2}{x}\right| + \frac{2}{x} + C \quad ◀$$

QUADRATIC FACTORS

If some of the factors of $Q(x)$ are irreducible quadratics, then the contribution of those factors to the partial fraction decomposition of $P(x)/Q(x)$ can be determined from the following rule:

QUADRATIC FACTOR RULE. For each factor of the form $(ax^2 + bx + c)^m$, the partial fraction decomposition contains the following sum of m partial fractions:

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_mx + B_m}{(ax^2 + bx + c)^m}$$

where $A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_m$ are constants to be determined. In the case where $m = 1$, only the first term in the sum appears.

Example 3 Evaluate $\displaystyle\int \frac{x^2 + x - 2}{3x^3 - x^2 + 3x - 1}\,dx$.

Solution. The denominator in the integrand can be factored by grouping:

$$\frac{x^2 + x - 2}{3x^3 - x^2 + 3x - 1} = \frac{x^2 + x - 2}{x^2(3x-1) + (3x-1)} = \frac{x^2 + x - 2}{(3x-1)(x^2+1)}$$

By the linear factor rule, the factor $3x - 1$ introduces one term; namely

$$\frac{A}{3x-1}$$

and by the quadratic factor rule, the factor $x^2 + 1$ introduces one term; namely

$$\frac{Bx + C}{x^2 + 1}$$

Thus, the partial fraction decomposition is

$$\frac{x^2 + x - 2}{(3x-1)(x^2+1)} = \frac{A}{3x-1} + \frac{Bx + C}{x^2+1} \tag{10}$$

Multiplying by $(3x - 1)(x^2 + 1)$ yields

$$x^2 + x - 2 = A(x^2 + 1) + (Bx + C)(3x - 1) \tag{11}$$

We could find A by substituting $x = \frac{1}{3}$ to make the last term drop out, and then find the rest of the constants by equating corresponding coefficients. However, in this case it is just as easy to find *all* of the constants by equating coefficients and solving the resulting system. For this purpose we multiply out the right side of (11) and collect like terms:

$$x^2 + x - 2 = (A + 3B)x^2 + (-B + 3C)x + (A - C)$$

Equating corresponding coefficients gives

$$\begin{aligned} A + 3B \qquad &= \ \ 1 \\ - \ B + 3C &= \ \ 1 \\ A \qquad - \ C &= -2 \end{aligned}$$

To solve this system, subtract the third equation from the first to eliminate A. Then use the resulting equation together with the second equation to solve for B and C. Finally, determine A from the first or third equation. This yields (verify)

$$A = -\tfrac{7}{5}, \quad B = \tfrac{4}{5}, \quad C = \tfrac{3}{5}$$

Thus, (10) becomes

$$\frac{x^2 + x - 2}{(3x - 1)(x^2 + 1)} = \frac{-\frac{7}{5}}{3x - 1} + \frac{\frac{4}{5}x + \frac{3}{5}}{x^2 + 1}$$

and

$$\begin{aligned} \int \frac{x^2 + x - 2}{(3x - 1)(x^2 + 1)}\,dx &= -\frac{7}{5}\int \frac{dx}{3x - 1} + \frac{4}{5}\int \frac{x}{x^2 + 1}\,dx + \frac{3}{5}\int \frac{dx}{x^2 + 1} \\ &= -\frac{7}{15}\ln|3x - 1| + \frac{2}{5}\ln(x^2 + 1) + \frac{3}{5}\tan^{-1} x + C \end{aligned}$$ ◀

FOR THE READER. Computer algebra systems have built-in capabilities for finding partial fraction decompositions. If you have a CAS, read the documentation on partial fraction decompositions, and use your CAS to find the decompositions in Examples 1, 2, and 3.

Example 4 Evaluate $\displaystyle\int \frac{3x^4 + 4x^3 + 16x^2 + 20x + 9}{(x + 2)(x^2 + 3)^2}\,dx.$

Solution. Observe that the integrand is a proper rational function since the numerator has degree 4 and the denominator has degree 5. Thus, the method of partial fractions is applicable. By the linear factor rule, the factor $x + 2$ introduces the single term

$$\frac{A}{x + 2}$$

and by the quadratic factor rule, the factor $(x^2 + 3)^2$ introduces two terms (since $m = 2$):

$$\frac{Bx + C}{x^2 + 3} + \frac{Dx + E}{(x^2 + 3)^2}$$

Thus, the partial fraction decomposition of the integrand is

$$\frac{3x^4 + 4x^3 + 16x^2 + 20x + 9}{(x + 2)(x^2 + 3)^2} = \frac{A}{x + 2} + \frac{Bx + C}{x^2 + 3} + \frac{Dx + E}{(x^2 + 3)^2} \tag{12}$$

Multiplying by $(x + 2)(x^2 + 3)^2$ yields

$$\begin{aligned} &3x^4 + 4x^3 + 16x^2 + 20x + 9 \\ &\qquad = A(x^2 + 3)^2 + (Bx + C)(x^2 + 3)(x + 2) + (Dx + E)(x + 2) \end{aligned} \tag{13}$$

which, after multiplying out and collecting like powers of x, becomes

$$\begin{aligned} &3x^4 + 4x^3 + 16x^2 + 20x + 9 \\ &\qquad = (A + B)x^4 + (2B + C)x^3 + (6A + 3B + 2C + D)x^2 \\ &\qquad\qquad + (6B + 3C + 2D + E)x + (9A + 6C + 2E) \end{aligned} \tag{14}$$

Equating corresponding coefficients in (14) yields the following system of five linear equations in five unknowns:

$$\begin{aligned} A + B &= 3 \\ 2B + C &= 4 \\ 6A + 3B + 2C + D &= 16 \\ 6B + 3C + 2D + E &= 20 \\ 9A + 6C + 2E &= 9 \end{aligned} \tag{15}$$

Efficient methods for solving systems of linear equations such as this are studied in a branch of mathematics called ***linear algebra***; those methods are outside the scope of this text. However, as a practical matter most linear systems of any size are solved by computer, and most computer algebra systems have commands that in many cases can solve linear systems exactly. In this particular case we can simplify the work by first substituting $x = -2$ in (13), which yields $A = 1$. Substituting this known value of A in (15) yields the simpler system

$$\begin{aligned} B &= 2 \\ 2B + C &= 4 \\ 3B + 2C + D &= 10 \\ 6B + 3C + 2D + E &= 20 \\ 6C + 2E &= 0 \end{aligned} \tag{16}$$

This system can be solved by starting at the top and working down, first substituting $B = 2$ in the second equation to get $C = 0$, then substituting the known values of B and C in the third equation to get $D = 4$, and so forth. This yields

$$A = 1, \quad B = 2, \quad C = 0, \quad D = 4, \quad E = 0$$

Thus, (12) becomes

$$\frac{3x^4 + 4x^3 + 16x^2 + 20x + 9}{(x+2)(x^2+3)^2} = \frac{1}{x+2} + \frac{2x}{x^2+3} + \frac{4x}{(x^2+3)^2}$$

and so

$$\begin{aligned} \int \frac{3x^4 + 4x^3 + 16x^2 + 20x + 9}{(x+2)(x^2+3)^2}\, dx \\ &= \int \frac{dx}{x+2} + \int \frac{2x}{x^2+3}\, dx + 4\int \frac{x}{(x^2+3)^2}\, dx \\ &= \ln|x+2| + \ln(x^2+3) - \frac{2}{x^2+3} + C \end{aligned}$$

◀

INTEGRATING IMPROPER RATIONAL FUNCTIONS

Although the method of partial fractions only applies to proper rational functions, an improper rational function can be integrated by performing a long division and expressing the function as the quotient plus the remainder over the divisor. The remainder over the divisor will be a proper rational function, which can then be decomposed into partial fractions. This idea is illustrated in the following example:

Example 5 Evaluate $\displaystyle\int \frac{3x^4 + 3x^3 - 5x^2 + x - 1}{x^2 + x - 2}\, dx$.

Solution. The integrand is an improper rational function since the numerator has degree 4 and the denominator has degree 2. Thus, we first perform the long division

$$\begin{array}{r|l} & 3x^2 \qquad\quad + 1 \\ \hline x^2 + x - 2 & 3x^4 + 3x^3 - 5x^2 + x - 1 \\ & \underline{3x^4 + 3x^3 - 6x^2} \\ & \qquad\qquad x^2 + x - 1 \\ & \qquad\qquad \underline{x^2 + x - 2} \\ & \qquad\qquad\qquad\quad 1 \end{array}$$

It follows that the integrand can be expressed as

$$\frac{3x^4 + 3x^3 - 5x^2 + x - 1}{x^2 + x - 2} = (3x^2 + 1) + \frac{1}{x^2 + x - 2}$$

and hence

$$\int \frac{3x^4 + 3x^3 - 5x^2 + x - 1}{x^2 + x - 2}\,dx = \int (3x^2 + 1)\,dx + \int \frac{dx}{x^2 + x - 2}$$

The second integral on the right now involves a proper rational function and can thus be evaluated by a partial fraction decomposition. Using the result of Example 1 we obtain

$$\int \frac{3x^4 + 3x^3 - 5x^2 + x - 1}{x^2 + x - 2}\,dx = x^3 + x + \frac{1}{3}\ln\left|\frac{x-1}{x+2}\right| + C$$ ◀

CONCLUDING REMARKS

There are some cases in which the method of partial fractions is inappropriate. For example, it would be illogical to use partial fractions to perform the integration

$$\int \frac{3x^2 + 2}{x^3 + 2x - 8}\,dx = \ln|x^3 + 2x - 8| + C$$

since the substitution $u = x^3 + 2x - 8$ is more direct. Similarly, the integration

$$\int \frac{2x - 1}{x^2 + 1}\,dx = \int \frac{2x}{x^2 + 1}\,dx - \int \frac{dx}{x^2 + 1} = \ln(x^2 + 1) - \tan^{-1} x + C$$

requires only a little algebra since the integrand is already in partial-fraction form.

EXERCISE SET 8.5 [C] CAS

In Exercises 1–8, write out the form of the partial fraction decomposition. (Do not find the numerical values of the coefficients.)

1. $\dfrac{3x - 1}{(x-2)(x+5)}$
2. $\dfrac{5}{x(x^2 - 9)}$
3. $\dfrac{2x - 3}{x^3 - x^2}$
4. $\dfrac{x^2}{(x+2)^3}$
5. $\dfrac{1 - 5x^2}{x^3(x^2 + 1)}$
6. $\dfrac{2x}{(x-1)(x^2+5)}$
7. $\dfrac{4x^3 - x}{(x^2 + 5)^2}$
8. $\dfrac{1 - 3x^4}{(x-2)(x^2+1)^2}$

In Exercises 9–32, evaluate the integral.

9. $\displaystyle\int \frac{dx}{x^2 + 3x - 4}$
10. $\displaystyle\int \frac{dx}{x^2 + 8x + 7}$
11. $\displaystyle\int \frac{11x + 17}{2x^2 + 7x - 4}\,dx$
12. $\displaystyle\int \frac{5x - 5}{3x^2 - 8x - 3}\,dx$
13. $\displaystyle\int \frac{2x^2 - 9x - 9}{x^3 - 9x}\,dx$
14. $\displaystyle\int \frac{dx}{x(x^2 - 1)}$
15. $\displaystyle\int \frac{x^2 + 2}{x + 2}\,dx$
16. $\displaystyle\int \frac{x^2 - 4}{x - 1}\,dx$
17. $\displaystyle\int \frac{3x^2 - 10}{x^2 - 4x + 4}\,dx$
18. $\displaystyle\int \frac{x^2}{x^2 - 3x + 2}\,dx$
19. $\displaystyle\int \frac{x^5 + 2x^2 + 1}{x^3 - x}\,dx$
20. $\displaystyle\int \frac{2x^5 - x^3 - 1}{x^3 - 4x}\,dx$
21. $\displaystyle\int \frac{2x^2 + 3}{x(x-1)^2}\,dx$
22. $\displaystyle\int \frac{3x^2 - x + 1}{x^3 - x^2}\,dx$
23. $\displaystyle\int \frac{x^2 + x - 16}{(x+1)(x-3)^2}\,dx$
24. $\displaystyle\int \frac{2x^2 - 2x - 1}{x^3 - x^2}\,dx$
25. $\displaystyle\int \frac{x^2}{(x+2)^3}\,dx$
26. $\displaystyle\int \frac{2x^2 + 3x + 3}{(x+1)^3}\,dx$
27. $\displaystyle\int \frac{2x^2 - 1}{(4x-1)(x^2+1)}\,dx$
28. $\displaystyle\int \frac{dx}{x^3 + x}$
29. $\displaystyle\int \frac{x^3 + 3x^2 + x + 9}{(x^2+1)(x^2+3)}\,dx$
30. $\displaystyle\int \frac{x^3 + x^2 + x + 2}{(x^2+1)(x^2+2)}\,dx$
31. $\displaystyle\int \frac{x^3 - 3x^2 + 2x - 3}{x^2 + 1}\,dx$
32. $\displaystyle\int \frac{x^4 + 6x^3 + 10x^2 + x}{x^2 + 6x + 10}\,dx$

In Exercises 33 and 34, evaluate the integral by making a substitution that converts the integrand to a rational function.

33. $\displaystyle\int \frac{\cos\theta}{\sin^2\theta + 4\sin\theta - 5}\,d\theta$
34. $\displaystyle\int \frac{e^t}{e^{2t} - 4}\,dt$

35. Find the volume of the solid generated when the region enclosed by $y = x^2/(9 - x^2)$, $y = 0$, $x = 0$, and $x = 2$ is revolved about the x-axis.

36. Find the area of the region under the curve $y = 1/(1 + e^x)$, over the interval $[-\ln 5, \ln 5]$. [*Hint:* Make a substitution that converts the integrand to a rational function.]

In Exercises 37 and 38, use a CAS to evaluate the integral in two ways: (i) integrate directly; (ii) use the CAS to find the partial fraction decomposition and integrate the decomposition. Integrate by hand to check the results.

C **37.** $\displaystyle\int \frac{x^2 + 1}{(x^2 + 2x + 3)^2}\, dx$

C **38.** $\displaystyle\int \frac{x^5 + x^4 + 4x^3 + 4x^2 + 4x + 4}{(x^2 + 2)^3}\, dx$

In Exercises 39 and 40, integrate by hand and check your answers using a CAS.

C **39.** $\displaystyle\int \frac{dx}{x^4 - 3x^3 - 7x^2 + 27x - 18}$

C **40.** $\displaystyle\int \frac{dx}{16x^3 - 4x^2 + 4x - 1}$

41. Show that

$$\int_0^1 \frac{x}{x^4 + 1}\, dx = \frac{\pi}{8}$$

42. Use partial fractions to derive the integration formula

$$\int \frac{1}{a^2 - x^2}\, dx = \frac{1}{2a} \ln \left| \frac{a + x}{a - x} \right| + C$$

8.6 USING TABLES OF INTEGRALS AND COMPUTER ALGEBRA SYSTEMS

In this section we will discuss how to integrate using tables, and we will address some of the issues that relate to using computer algebra systems for integration. Readers who are not using computer algebra systems can skip that material with no problem.

INTEGRAL TABLES

Tables of integrals are useful for eliminating tedious hand computation. The endpapers of this text contain a relatively brief table of integrals that we will refer to as the ***Endpaper Integral Table***; more comprehensive tables are published in standard reference books such as the *CRC Standard Mathematical Tables and Formulae*, CRC Press, Inc., 1996.

All integral tables have their own scheme for classifying integrals according to the form of the integrand. For example, the Endpaper Integral Table classifies the integrals into 15 categories; *Basic Functions*, *Reciprocals of Basic Functions*, *Powers of Trigonometric Functions*, *Products of Trigonometric Functions*, and so forth. The first step in working with tables is to read through the classifications so that you understand the classification scheme and know where to look in the table for integrals of different types.

PERFECT MATCHES

If you are lucky, the integral you are attempting to evaluate will match up perfectly with one of the forms in the table. However, when looking for matches you may have to make an adjustment for the variable of integration. For example, the integral

$$\int x^2 \sin x\, dx$$

is a perfect match with Formula (46) in the Endpaper Integral Table, except for the letter used for the variable of integration. Thus, to apply Formula (46) to the given integral we need to change the variable of integration in the formula from u to x. With that minor modification we obtain

$$\int x^2 \sin x\, dx = 2x \sin x + (2 - x^2) \cos x + C$$

Here are some more examples of perfect matches:

Example 1 Use the Endpaper Integral Table to evaluate

(a) $\int \sin 7x \cos 2x\, dx$ (b) $\int x^2\sqrt{7+3x}\, dx$

(c) $\int \frac{\sqrt{2-x^2}}{x}\, dx$ (d) $\int (x^3+7x+1)\sin \pi x\, dx$

Solution (a). The integrand can be classified as a product of trigonometric functions. Thus, from Formula (40) with $m = 7$ and $n = 2$ we obtain

$$\int \sin 7x \cos 2x\, dx = -\frac{\cos 9x}{18} - \frac{\cos 5x}{10} + C$$

Solution (b). The integrand can be classified as a power of x multiplying $\sqrt{a+bx}$. Thus, from Formula (103) with $a = 7$ and $b = 3$ we obtain

$$\int x^2\sqrt{7+3x}\, dx = \frac{2}{2835}(135x^2 - 252x + 392)(7+3x)^{3/2} + C$$

Solution (c). The integrand can be classified as a power of x dividing $\sqrt{a^2-x^2}$. Thus, from Formula (79) with $a = \sqrt{2}$ we obtain

$$\int \frac{\sqrt{2-x^2}}{x}\, dx = \sqrt{2-x^2} - \sqrt{2}\ln\left|\frac{\sqrt{2}+\sqrt{2-x^2}}{x}\right| + C$$

Solution (d). The integrand can be classified as a polynomial multiplying a trigonometric function. Thus, we apply Formula (58) with $p(x) = x^3+7x+1$ and $a = \pi$. The successive nonzero derivatives of $p(x)$ are

$$p'(x) = 3x^2 + 7, \quad p''(x) = 6x, \quad p'''(x) = 6$$

and hence

$$\int (x^3+7x+1)\sin \pi x\, dx$$

$$= -\frac{x^3+7x+1}{\pi}\cos \pi x + \frac{3x^2+7}{\pi^2}\sin \pi x + \frac{6x}{\pi^3}\cos \pi x - \frac{6}{\pi^4}\sin \pi x + C \quad ◀$$

MATCHES REQUIRING SUBSTITUTIONS

Sometimes an integral that does not match any table entry can be made to match by making an appropriate substitution. Here are some examples.

Example 2 Use the Endpaper Integral Table to evaluate $\int \sqrt{x - 4x^2}\, dx$.

Solution. The integrand does not match any of the forms in the table precisely. It comes closest to matching Formula (112), but it misses because of the factor of 4 multiplying x^2 inside the radical. However, if we make the substitution

$$u = 2x, \quad du = 2\, dx$$

then the $4x^2$ will become a u^2, and the transformed integral will be

$$\int \sqrt{x-4x^2}\, dx = \frac{1}{2}\int \sqrt{\tfrac{1}{2}u - u^2}\, du$$

which matches Formula (112) with $a = \frac{1}{4}$. Thus, we obtain

$$\int \sqrt{x-4x^2}\, dx = \frac{1}{2}\left[\frac{u-\frac{1}{4}}{2}\sqrt{\tfrac{1}{2}u - u^2} + \frac{1}{32}\sin^{-1}\left(\frac{u-\frac{1}{4}}{\frac{1}{4}}\right)\right] + C$$

$$= \frac{1}{2}\left[\frac{2x-\frac{1}{4}}{2}\sqrt{x-4x^2} + \frac{1}{32}\sin^{-1}\left(\frac{2x-\frac{1}{4}}{\frac{1}{4}}\right)\right] + C$$

$$= \frac{8x-1}{16}\sqrt{x-4x^2} + \frac{1}{64}\sin^{-1}(8x-1) + C \quad ◀$$

Example 3 Use the Endpaper Integral Table to evaluate

(a) $\int e^{\pi x} \sin^{-1}(e^{\pi x})\, dx$ (b) $\int x\sqrt{x^2 - 4x + 5}\, dx$

Solution (a). The integrand does not even come close to matching any of the forms in the table. However, a little thought suggests the substitution

$$u = e^{\pi x}, \quad du = \pi e^{\pi x}\, dx$$

from which we obtain

$$\int e^{\pi x} \sin^{-1}(e^{\pi x})\, dx = \frac{1}{\pi}\int \sin^{-1} u\, du$$

The integrand is now a basic function, and Formula (7) yields

$$\begin{aligned}\int e^{\pi x} \sin^{-1}(e^{\pi x})\, dx &= \frac{1}{\pi}[u \sin^{-1} u + \sqrt{1-u^2}] + C \\ &= \frac{1}{\pi}[e^{\pi x} \sin^{-1}(e^{\pi x}) + \sqrt{1-e^{2\pi x}}] + C\end{aligned}$$

Solution (b). Again, the integrand does not closely match any of the forms in the table. However, a little thought suggests that it may be possible to bring the integrand closer to the form $x\sqrt{x^2+a^2}$ by completing the square to eliminate the term involving x inside the radical. Doing this yields

$$\int x\sqrt{x^2-4x+5}\, dx = \int x\sqrt{(x^2-4x+4)+1}\, dx = \int x\sqrt{(x-2)^2+1}\, dx \tag{1}$$

At this point we are closer to the form $x\sqrt{x^2+a^2}$, but we are not quite there because of the $(x-2)^2$ rather than x^2 inside the radical. However, we can resolve that problem with the substitution

$$u = x - 2, \quad du = dx$$

With this substitution we have $x = u + 2$, so (1) can be expressed in terms of u as

$$\int x\sqrt{x^2-4x+5}\, dx = \int (u+2)\sqrt{u^2+1}\, du = \int u\sqrt{u^2+1}\, du + 2\int \sqrt{u^2+1}\, du$$

The first integral on the right is now a perfect match with Formula (84) with $a = 1$, and the second is a perfect match with Formula (72) with $a = 1$. Thus, applying these formulas and dropping the unnecessary absolute value signs we obtain

$$\int x\sqrt{x^2-4x+5}\, dx = \left[\frac{1}{3}(u^2+1)^{3/2}\right] + 2\left[\frac{u}{2}\sqrt{u^2+1} + \frac{1}{2}\ln(u+\sqrt{u^2+1})\right] + C$$

If we now replace u by $x - 2$ (in which case $u^2 + 1 = x^2 - 4x + 5$), we obtain

$$\begin{aligned}\int x\sqrt{x^2-4x+5}\, dx = \tfrac{1}{3}(x^2-4x+5)^{3/2} + (x-2)\sqrt{x^2-4x+5} \\ + \ln(x-2+\sqrt{x^2-4x+5}) + C\end{aligned}$$

Although correct, this form of the answer has an unnecessary mixture of radicals and fractional exponents. If desired, we can "clean up" the answer by writing

$$(x^2-4x+5)^{3/2} = (x^2-4x+5)\sqrt{x^2-4x+5}$$

from which it follows that (verify)

$$\begin{aligned}\int x\sqrt{x^2-4x+5}\, dx = \tfrac{1}{3}(x^2-x-1)\sqrt{x^2-4x+5} \\ + \ln(x-2+\sqrt{x^2-4x+5}) + C\end{aligned}$$ ◀

MATCHES REQUIRING REDUCTION FORMULAS

In cases where the entry in an integral table is a reduction formula, that formula will have to be applied first to reduce the given integral to a form in which it can be evaluated.

Example 4 Use the Endpaper Integral Table to evaluate $\int \frac{x^3}{\sqrt{1+x}}\,dx$.

Solution. The integrand can be classified as a power of x multiplying the reciprocal of $\sqrt{a+bx}$. Thus, from Formula (107) with $a = 1$, $b = 1$, and $n = 3$, followed by Formula (106), we obtain

$$\begin{aligned}\int \frac{x^3}{\sqrt{1+x}}\,dx &= \frac{2x^3\sqrt{1+x}}{7} - \frac{6}{7}\int \frac{x^2}{\sqrt{1+x}}\,dx \\ &= \frac{2x^3\sqrt{1+x}}{7} - \frac{6}{7}\left[\frac{2}{15}(3x^2 - 4x + 8)\sqrt{1+x}\right] + C \\ &= \left(\frac{2x^3}{7} - \frac{12x^2}{35} + \frac{16x}{35} - \frac{32}{35}\right)\sqrt{1+x} + C\end{aligned}$$ ◀

MATCHES REQUIRING SPECIAL SUBSTITUTIONS

The Endpaper Integral Table has numerous entries involving an exponent of 3/2 or involving square roots (exponent 1/2), but it has no entries with other fractional exponents. However, integrals involving fractional powers of x can often be simplified by making the substitution $u = x^{1/n}$ in which n is the least common multiple of the denominators of the exponents. Here are some examples.

Example 5 Evaluate

(a) $\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}\,dx$ (b) $\int \frac{dx}{2+2\sqrt{x}}$ (c) $\int \sqrt{1+e^x}\,dx$

Solution (a). The integrand contains $x^{1/2}$ and $x^{1/3}$, so we make the substitution $u = x^{1/6}$, from which we obtain

$$x = u^6, \quad dx = 6u^5\,du$$

Thus,

$$\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}\,dx = \int \frac{(u^6)^{1/2}}{1+(u^6)^{1/3}}(6u^5)\,du = 6\int \frac{u^8}{1+u^2}\,du$$

By long division

$$\frac{u^8}{1+u^2} = u^6 - u^4 + u^2 - 1 + \frac{1}{1+u^2}$$

from which it follows that

$$\begin{aligned}\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}\,dx &= 6\int \left(u^6 - u^4 + u^2 - 1 + \frac{1}{1+u^2}\right)du \\ &= \tfrac{6}{7}u^7 - \tfrac{6}{5}u^5 + 2u^3 - 6u + 6\tan^{-1}u + C \\ &= \tfrac{6}{7}x^{7/6} - \tfrac{6}{5}x^{5/6} + 2x^{1/2} - 6x^{1/6} + 6\tan^{-1}(x^{1/6}) + C\end{aligned}$$

Solution (b). The integrand contains $x^{1/2}$ but does not match any of the forms in the Endpaper Integral Table. Thus, we make the substitution $u = x^{1/2}$, from which we obtain

$$x = u^2, \quad dx = 2u\,du$$

Making this substitution yields

$$\begin{aligned}\int \frac{dx}{2+2\sqrt{x}} &= \int \frac{2u}{2+2u}\,du \\ &= \int \left(1 - \frac{1}{1+u}\right)du \quad \text{Long division} \\ &= u - \ln|1+u| + C \\ &= \sqrt{x} - \ln(1+\sqrt{x}) + C \quad \text{Absolute value not needed}\end{aligned}$$

Solution (c). Again, the integral does not match any of the forms in the Endpaper Integral Table. However, the integrand contains $(1+e^x)^{1/2}$, which is analogous to the situation in part (b), except that here it is $1+e^x$ rather than x that is raised to the $1/2$ power. This suggests the substitution $u=(1+e^x)^{1/2}$, from which we obtain (verify)

$$x=\ln(u^2-1), \quad dx=\frac{2u}{u^2-1}\,du$$

Thus,

$$\begin{aligned}\int\sqrt{1+e^x}\,dx &= \int u\left(\frac{2u}{u^2-1}\right)du\\ &= \int\frac{2u^2}{u^2-1}\,du\\ &= \int\left(2+\frac{2}{u^2-1}\right)du \qquad \text{Long division}\\ &= 2u+\int\left(\frac{1}{u-1}-\frac{1}{u+1}\right)du \qquad \text{Partial fractions}\\ &= 2u+\ln|u-1|-\ln|u+1|+C\\ &= 2u+\ln\left|\frac{u-1}{u+1}\right|+C\\ &= 2\sqrt{1+e^x}+\ln\left[\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right]+C \qquad \text{Absolute value not needed}\end{aligned}$$

◀

Functions that consist of finitely many sums, differences, quotients, and products of $\sin x$ and $\cos x$ are called ***rational functions of sin x and cos x***. Some examples are

$$\frac{\sin x+3\cos^2 x}{\cos x+4\sin x}, \quad \frac{\sin x}{1+\cos x-\cos^2 x}, \quad \frac{3\sin^5 x}{1+4\sin x}$$

The Endpaper Integral Table gives a few formulas for integrating rational functions of $\sin x$ and $\cos x$ under the heading *Reciprocals of Basic Functions*. For example, it follows from Formula (18) that

$$\int\frac{1}{1+\sin x}\,dx=\tan x-\sec x+C \tag{2}$$

However, since the integrand is a rational function of $\sin x$, it may be desirable in a particular application to express the value of the integral in terms of $\sin x$ and $\cos x$ and rewrite (2) as

$$\int\frac{1}{1+\sin x}\,dx=\frac{\sin x-1}{\cos x}+C$$

Many rational functions of $\sin x$ and $\cos x$ can be evaluated by an ingenious method that was discovered by the mathematician Karl Weierstrass (see p. 140). The idea is to make the substitution

$$u=\tan(x/2), \quad -\pi/2<x/2<\pi/2$$

from which it follows that

$$x=2\tan^{-1}u, \quad dx=\frac{2}{1+u^2}\,du$$

To implement this substitution we need to express $\sin x$ and $\cos x$ in terms of u. For this purpose we will use the identities

$$\sin x = 2\sin(x/2)\cos(x/2) \tag{3}$$

$$\cos x = \cos^2(x/2)-\sin^2(x/2) \tag{4}$$

and the following relationships suggested by Figure 8.6.1:

$$\sin(x/2)=\frac{u}{\sqrt{1+u^2}} \quad\text{and}\quad \cos(x/2)=\frac{1}{\sqrt{1+u^2}}$$

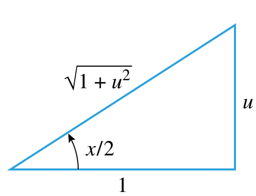


Figure 8.6.1

Substituting these expressions in (3) and (4) yields

$$\sin x = 2\left(\frac{u}{\sqrt{1+u^2}}\right)\left(\frac{1}{\sqrt{1+u^2}}\right) = \frac{2u}{1+u^2}$$

$$\cos x = \left(\frac{1}{\sqrt{1+u^2}}\right)^2 - \left(\frac{u}{\sqrt{1+u^2}}\right)^2 = \frac{1-u^2}{1+u^2}$$

In summary, we have shown that the substitution $u = \tan(x/2)$ can be implemented in a rational function of $\sin x$ and $\cos x$ by letting

$$\sin x = \frac{2u}{1+u^2}, \quad \cos x = \frac{1-u^2}{1+u^2}, \quad dx = \frac{2}{1+u^2}\,du \tag{5}$$

Example 6 Evaluate $\displaystyle\int \frac{dx}{1-\sin x+\cos x}$.

Solution. The integrand is a rational function of $\sin x$ and $\cos x$ that does not match any of the formulas in the Endpaper Integral Table, so we make the substitution $u = \tan(x/2)$. Thus, from (5) we obtain

$$\begin{aligned}\int \frac{dx}{1-\sin x+\cos x} &= \int \frac{\dfrac{2\,du}{1+u^2}}{1-\left(\dfrac{2u}{1+u^2}\right)+\left(\dfrac{1-u^2}{1+u^2}\right)} \\ &= \int \frac{2\,du}{(1+u^2)-2u+(1-u^2)} \\ &= \int \frac{du}{1-u} = -\ln|1-u|+C = -\ln|1-\tan(x/2)|+C \quad ◀\end{aligned}$$

REMARK. The substitution $u = \tan(x/2)$ will convert any rational function of $\sin x$ and $\cos x$ to an ordinary rational function of u. However, the method can lead to cumbersome partial fraction decompositions, so it may be worthwhile to explore the existence of simpler methods when hand computation is to be used.

INTEGRATING WITH COMPUTER ALGEBRA SYSTEMS

Integration tables are rapidly giving way to computerized integration using computer algebra systems. However, as with many powerful tools, a knowledgeable operator is an important component of the system.

Sometimes computer algebra systems do not produce the most general form of the indefinite integral. For example, the integral formula

$$\int \frac{dx}{x-1} = \ln|x-1| + C$$

which can be obtained by inspection or by using the substitution $u = x-1$, is valid for $x > 1$ or for $x < 1$. However, *Mathematica*, *Maple*, *Derive*, and the computer algebra systems used by the Texas Instruments TI-89 and Hewlett-Packard HP-49 calculators evaluate this integral as*

$\ln(-1+x)$,	$\ln(x-1)$,	$\ln(x-1)$,	$\ln(\lvert x-1\rvert)$,	$\ln(x-1)$
Mathematica	*Maple*	*Derive*	TI-89	HP-49

Observe that none of the systems include the constant of integration—the answer produced is a particular antiderivative and not the most general antiderivative (indefinite integral).

*Results produced by *Mathematica*, *Maple*, *Derive*, the TI-89, and the HP-49 may vary depending on the version of the software that is used.

Observe also that only the TI-89 includes the absolute value signs; consequently, the antiderivatives produced in this instance by the other systems are valid only for $x > 1$. All systems, however, are able to recover to correctly calculate the definite integral

$$\int_0^{1/2} \frac{dx}{x-1} = -\ln 2$$

Now let us examine how these systems handle the integral

$$\int x\sqrt{x^2-4x+5}\,dx = \tfrac{1}{3}(x^2-x-1)\sqrt{x^2-4x+5} + \ln(x-2+\sqrt{x^2-4x+5}\,) \tag{6}$$

which we obtained in Example 3(b) (with the constant of integration included). *Derive*, the TI-89, and the HP-49 produce this result in slightly different algebraic forms, but *Maple* produces the result

$$\int x\sqrt{x^2-4x+5}\,dx = \tfrac{1}{3}(x^2-4x+5)^{3/2} + \tfrac{1}{2}(2x-4)\sqrt{x^2-4x+5} + \sinh^{-1}(x-2)$$

This can be rewritten as (6) by expressing the fractional exponent in radical form and expressing $\sinh^{-1}(x-2)$ in logarithmic form using Theorem 7.8.4 (verify). *Mathematica* produces the result

$$\int x\sqrt{x^2-4x+5}\,dx = \tfrac{1}{3}(x^2-x-1)\sqrt{x^2-4x+5} - \sinh^{-1}(2-x)$$

which can be rewritten in form (6) by using Theorem 7.8.4 together with the identity $\sinh^{-1}(-x) = -\sinh^{-1}x$ (verify).

Computer algebra systems can sometimes produce inconvenient or unnatural answers to integration problems. For example, the systems mentioned above produce the following results when asked to integrate $(x+1)^7$:

$\dfrac{(x+1)^8}{8}$,	$\tfrac{1}{8}x^8 + x^7 + \tfrac{7}{2}x^6 + 7x^5 + \tfrac{35}{4}x^4 + 7x^3 + \tfrac{7}{2}x^2 + x$
Mathematica, *Maple*, *Derive*, TI-89	HP-49

The answers produced by the majority of these systems are in keeping with the hand computation

$$\int (x+1)^7\,dx = \frac{(x+1)^8}{8} + C$$

that uses the substitution $u = x+1$, whereas the answer produced by the HP-49 appears to be based on expanding $(x+1)^7$ and integrating term by term.

FOR THE READER. If you expand the expression $\frac{1}{8}(x+1)^8$, you will discover that it contains a summand $\frac{1}{8}$ that does not appear in the HP-49 result. What is the explanation?

In Example 2(a) of Section 8.3 we showed that

$$\int \sin^4 x\cos^5 x\,dx = \tfrac{1}{5}\sin^5 x - \tfrac{2}{7}\sin^7 x + \tfrac{1}{9}\sin^9 x + C$$

This is the answer produced by the HP-49. In contrast, *Mathematica* integrates this as

$$\tfrac{3}{128}\sin x - \tfrac{1}{192}\sin 3x - \tfrac{1}{320}\sin 5x + \tfrac{1}{1792}\sin 7x + \tfrac{1}{2304}\sin 9x$$

and *Maple*, *Derive*, and the TI-89 essentially integrate it as

$$-\tfrac{1}{9}\sin^3 x\cos^6 x - \tfrac{1}{21}\sin x\cos^6 x + \tfrac{1}{105}\cos^4 x\sin x + \tfrac{4}{315}\cos^2 x\sin x + \tfrac{8}{315}\sin x$$

Although these three results look quite different, they can be obtained from one another using appropriate trigonometric identities.

COMPUTER ALGEBRA SYSTEMS HAVE LIMITATIONS

A computer algebra system combines a set of integration rules (such as substitution) with a library of functions that it can use to construct antiderivatives. Such libraries contain elementary functions, such as polynomials, rational functions, trigonometric functions, as well as various nonelementary functions that arise in engineering, physics, and other applied fields. Just as our Endpaper Integral Table has only 121 indefinite integrals, these libraries are not exhaustive of all possible integrands. If the system cannot manipulate the integrand to a form matching one in its library, the program will give some indication that it cannot evaluate the integral. For example, when asked to evaluate the integral

$$\int (1+\ln x)\sqrt{1+(x\ln x)^2}\,dx \tag{7}$$

all of the systems mentioned above respond by displaying some form of the unevaluated integral as an answer, indicating that they could not perform the integration.

FOR THE READER. Sometimes integrals that cannot be evaluated by a CAS in their given form can be evaluated by first rewriting them in a different form or by making a substitution. Make a u-substitution in (7) that will enable you to evaluate the integral with your CAS.

Sometimes computer algebra systems respond by expressing an integral in terms of another integral. For example, if you try to integrate e^{x^2} using *Mathematica*, *Maple*, or *Derive*, you will obtain an expression involving erf (which stands for ***error function***). The function erf(x) is defined as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$$

so all three programs essentially rewrite the given integral in terms of a closely related integral. Indeed, this is what we did in integrating $1/x$, since the natural logarithm function is (formally) defined as

$$\ln x = \int_1^x \frac{1}{t}\,dt$$

(see Section 7.5).

Example 7 A particle moves along an x-axis in such a way that its velocity $v(t)$ at time t is

$$v(t) = 30\cos^7 t\sin^4 t \quad (t \geq 0)$$

Graph the position versus time curve for the particle, given that the particle is at $x = 1$ when $t = 0$.

Solution. Since $dx/dt = v(t)$ and $x = 1$ when $t = 0$, the position function $x(t)$ is given by

$$x(t) = 1 + \int_0^t v(s)\,ds$$

Many computer algebra systems will allow us to enter this expression directly into a command for plotting functions, but it is often more efficient to perform the integration first. Using the HP-49 to perform the integration (the other systems mentioned above produce equivalent results), and including the constant of integration, yields

$$x = \int 30\cos^7 t\sin^4 t\,dt$$

$$= -\tfrac{30}{11}\sin^{11} t + 10\sin^9 t - \tfrac{90}{7}\sin^7 t + 6\sin^5 t + C$$

Using the initial condition $x(0) = 1$, we substitute the values $x = 1$ and $t = 0$ into this equation to find that $C = 1$, so

$$x(t) = -\tfrac{30}{11}\sin^{11} t + 10\sin^9 t - \tfrac{90}{7}\sin^7 t + 6\sin^5 t + 1 \quad (t \geq 0)$$

The graph of x versus t is shown in Figure 8.6.2. ◄

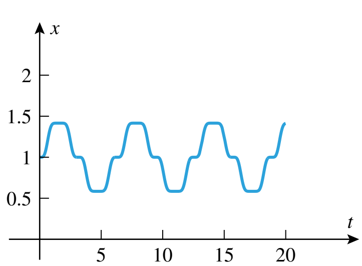


Figure 8.6.2

EXERCISE SET 8.6 C CAS

In Exercises 1–24:
(a) Use the Endpaper Integral Table to evaluate the integral.
(b) If you have a CAS, use it to evaluate the integral, and then confirm that the result is equivalent to the one that you found in part (a).

1. $\int \frac{3x}{4x-1}\,dx$

2. $\int \frac{x}{(2-3x)^2}\,dx$

3. $\int \frac{1}{x(2x+5)}\,dx$

4. $\int \frac{1}{x^2(1-5x)}\,dx$

5. $\int x\sqrt{2x-3}\,dx$

6. $\int \frac{x}{\sqrt{2-x}}\,dx$

7. $\int \frac{1}{x\sqrt{4-3x}}\,dx$

8. $\int \frac{1}{x\sqrt{3x-4}}\,dx$

9. $\int \frac{1}{5-x^2}\,dx$

10. $\int \frac{1}{x^2-9}\,dx$

11. $\int \sqrt{x^2-3}\,dx$

12. $\int \frac{\sqrt{x^2+5}}{x^2}\,dx$

13. $\int \frac{x^2}{\sqrt{x^2+4}}\,dx$

14. $\int \frac{1}{x^2\sqrt{x^2-2}}\,dx$

15. $\int \sqrt{9-x^2}\,dx$

16. $\int \frac{\sqrt{4-x^2}}{x^2}\,dx$

17. $\int \frac{\sqrt{3-x^2}}{x}\,dx$

18. $\int \frac{1}{x\sqrt{6x-x^2}}\,dx$

19. $\int \sin 3x \sin 2x\,dx$

20. $\int \sin 2x \cos 5x\,dx$

21. $\int x^3 \ln x\,dx$

22. $\int \frac{\ln x}{\sqrt{x}}\,dx$

23. $\int e^{-2x}\sin 3x\,dx$

24. $\int e^x \cos 2x\,dx$

In Exercises 25–36:
(a) Make the indicated u-substitution, and then use the Endpaper Integral Table to evaluate the integral.
(b) If you have a CAS, use it to evaluate the integral, and then confirm that the result is equivalent to the one that you found in part (a).

25. $\int \frac{e^{4x}}{(4-3e^{2x})^2}\,dx,\ u = e^{2x}$

26. $\int \frac{\cos 2x}{(\sin 2x)(3-\sin 2x)}\,dx,\ u = \sin 2x$

27. $\int \frac{1}{\sqrt{x}(9x+4)}\,dx,\ u = 3\sqrt{x}$

28. $\int \frac{\cos 4x}{9+\sin^2 4x}\,dx,\ u = \sin 4x$

29. $\int \frac{1}{\sqrt{9x^2-4}}\,dx,\ u = 3x$

30. $\int x\sqrt{2x^4+3}\,dx,\ u = \sqrt{2}x^2$

31. $\int \frac{x^5}{\sqrt{5-9x^4}}\,dx,\ u = 3x^2$

32. $\int \frac{1}{x^2\sqrt{3-4x^2}}\,dx,\ u = 2x$

33. $\int \frac{\sin^2(\ln x)}{x}\,dx,\ u = \ln x$

34. $\int e^{-2x}\cos^2(e^{-2x})\,dx,\ u = e^{-2x}$

35. $\int xe^{-2x}\,dx,\ u = -2x$

36. $\int \ln(5x-1)\,dx,\ u = 5x-1$

In Exercises 37–48:
(a) Make an appropriate u-substitution, and then use the Endpaper Integral Table to evaluate the integral.
(b) If you have a CAS, use it to evaluate the integral (no substitution), and then confirm that the result is equivalent to that in part (a).

37. $\int \frac{\sin 3x}{(\cos 3x)(\cos 3x+1)^2}\,dx$

38. $\int \frac{\ln x}{x\sqrt{4\ln x-1}}\,dx$

39. $\int \frac{x}{16x^4-1}\,dx$

40. $\int \frac{e^x}{3-4e^{2x}}\,dx$

41. $\int e^x\sqrt{3-4e^{2x}}\,dx$

42. $\int \frac{\sqrt{4-9x^2}}{x^2}\,dx$

43. $\int \sqrt{5x-9x^2}\,dx$

44. $\int \frac{1}{x\sqrt{x-5x^2}}\,dx$

45. $\int x\sin 3x\,dx$

46. $\int \cos\sqrt{x}\,dx$

47. $\int e^{-\sqrt{x}}\,dx$

48. $\int x\ln(2-3x^2)\,dx$

In Exercises 49–52:
(a) Complete the square, make an appropriate u-substitution, and then use the Endpaper Integral Table to evaluate the integral.
(b) If you have a CAS, use it to evaluate the integral (no substitution or square completion), and then confirm that the result is equivalent to that in part (a).

49. $\int \frac{1}{x^2+4x-5}\,dx$

50. $\int \sqrt{3-2x-x^2}\,dx$

51. $\int \frac{x}{\sqrt{5+4x-x^2}}\,dx$

52. $\int \frac{x}{x^2+6x+13}\,dx$

In Exercises 53–66:
(a) Make an appropriate u-substitution of the form $u = x^{1/n}$, $u = (x+a)^{1/n}$, or $u = x^n$, and then use the Endpaper Integral Table to evaluate the integral.
(b) If you have a CAS, use it to evaluate the integral, and then confirm that the result is equivalent to the one that you found in part (a).

53. $\int x\sqrt{x-2}\,dx$

54. $\int \frac{x}{\sqrt{x+1}}\,dx$

55. $\int x^5\sqrt{x^3+1}\,dx$

56. $\int \frac{1}{x\sqrt{x^3-1}}\,dx$

57. $\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}}$

58. $\int \frac{dx}{x-x^{3/5}}$

59. $\int \frac{dx}{x(1-x^{1/4})}$

60. $\int \frac{x^{2/3}}{x+1}\,dx$

61. $\int \frac{dx}{x^{1/2}-x^{1/3}}$

62. $\int \frac{1+\sqrt{x}}{1-\sqrt{x}}\,dx$

63. $\int \frac{x^3}{\sqrt{1+x^2}}\,dx$

64. $\int \frac{x}{(x+3)^{1/5}}\,dx$

65. $\int \sin\sqrt{x}\,dx$

66. $\int e^{\sqrt{x}}\,dx$

In Exercises 67–72:
(a) Make u-substitution (5) to convert the integrand to a rational function of u, and then use the Endpaper Integral Table to evaluate the integral.
(b) If you have a CAS, use it to evaluate the integral (no substitution), and then confirm that the result is equivalent to that in part (a).

67. $\int \frac{dx}{1+\sin x+\cos x}$

68. $\int \frac{dx}{2+\sin x}$

69. $\int \frac{d\theta}{1-\cos\theta}$

70. $\int \frac{dx}{4\sin x-3\cos x}$

71. $\int \frac{\cos x}{2-\cos x}\,dx$

72. $\int \frac{dx}{\sin x+\tan x}$

In Exercises 73 and 74, use any method to solve for x.

73. $\int_2^x \frac{1}{t(4-t)}\,dt = 0.5,\ 2 < x < 4$

74. $\int_1^x \frac{1}{t\sqrt{2t-1}}\,dt = 1,\ x > \frac{1}{2}$

In Exercises 75–78, use any method to find the area of the region enclosed by the curves.

75. $y = \sqrt{25-x^2},\ y = 0,\ x = 0,\ x = 4$

76. $y = \sqrt{9x^2-4},\ y = 0,\ x = 2$

77. $y = \frac{1}{25-16x^2},\ y = 0,\ x = 0,\ x = 1$

78. $y = \sqrt{x}\ln x,\ y = 0,\ x = 4$

In Exercises 79–82, use any method to find the volume of the solid generated when the region enclosed by the curves is revolved about the y-axis.

79. $y = \cos x,\ y = 0,\ x = 0,\ x = \pi/2$

80. $y = \sqrt{x-4},\ y = 0,\ x = 8$

81. $y = e^{-x},\ y = 0,\ x = 0,\ x = 3$

82. $y = \ln x,\ y = 0,\ x = 5$

In Exercises 83 and 84, use any method to find the arc length of the curve.

83. $y = 2x^2,\ 0 \le x \le 2$

84. $y = 3\ln x,\ 1 \le x \le 3$

In Exercises 85 and 86, use any method to find the area of the surface generated by revolving the curve about the x-axis.

85. $y = \sin x,\ 0 \le x \le \pi$

86. $y = 1/x,\ 1 \le x \le 4$

In Exercises 87 and 88, information is given about the motion of a particle moving along a coordinate line.
(a) Use a CAS to find the position function of the particle for $t \ge 0$. You may approximate the constants of integration, where necessary.
(b) Graph the position versus time curve.

C **87.** $v(t) = 20\cos^6 t\sin^3 t,\ s(0) = 2$

C **88.** $a(t) = e^{-t}\sin 2t\sin 4t,\ v(0) = 0,\ s(0) = 10$

89. (a) Use the substitution $u = \tan(x/2)$ to show that

$$\int \sec x\,dx = \ln\left|\frac{1+\tan(x/2)}{1-\tan(x/2)}\right| + C$$

and confirm that this is consistent with Formula (22) of Section 8.3.

(b) Use the result in part (a) to show that

$$\int \sec x\,dx = \ln\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right| + C$$

90. Use the substitution $u = \tan(x/2)$ to show that

$$\int \csc x\,dx = \frac{1}{2}\ln\left[\frac{1-\cos x}{1+\cos x}\right] + C$$

and confirm that this is consistent with the result in Exercise 61(a) of Section 8.3.

91. Find a substitution that can be used to integrate rational functions of $\sinh x$ and $\cosh x$ and use your substitution to evaluate

$$\int \frac{dx}{2\cosh x+\sinh x}$$

without expressing the integrand in terms of e^x and e^{-x}.

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8.7 NUMERICAL INTEGRATION; SIMPSON'S RULE

Our usual procedure for evaluating a definite integral is to find an antiderivative of the integrand and apply the Fundamental Theorem of Calculus. However, if an antiderivative of the integrand cannot be found, then we must settle for a numerical approximation of the integral. In Section 5.4 we discussed three procedures for approximating areas using Riemann sums—left endpoint approximation, right endpoint approximation, and midpoint approximation. In this section we will adapt those ideas to approximating general definite integrals, and we will discuss some new approximation methods that often provide more accuracy with less computation.

A REVIEW OF RIEMANN SUM APPROXIMATIONS

Recall from Section 5.5 that the definite integral of a continuous function f over an interval $[a, b]$ may be computed as

$$\int_a^b f(x)\,dx = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

where the sum that appears on the right side is called a Riemann sum. In this formula, the interval $[a, b]$ is divided into n subintervals of width $\Delta x = (b-a)/n$, and x_k^* denotes an arbitrary point in the kth subinterval. It follows that as n increases the Riemann sum will eventually be a good approximation to the integral, which we denote by writing

$$\int_a^b f(x)\,dx \approx \sum_{k=1}^{n} f(x_k^*)\Delta x$$

or, equivalently,

$$\int_a^b f(x)\,dx \approx \Delta x\left[f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)\right]$$

In this section we will denote the values of f at the endpoints of the subintervals by

$$y_0 = f(a),\quad y_1 = f(x_1),\quad y_2 = f(x_2), \ldots,\quad y_{n-1} = f(x_{n-1}),\quad y_n = f(b)$$

and we will denote the values of f at the midpoints of the subintervals by

$$y_{m_1}, y_{m_2}, \ldots, y_{m_n}$$

(Figure 8.7.1).

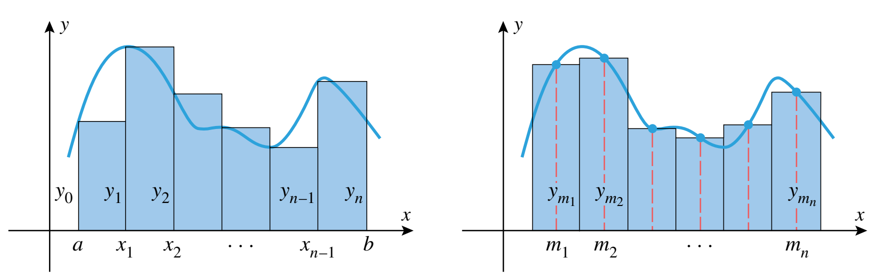


Figure 8.7.1

With this notation the left endpoint, right endpoint, and midpoint approximations discussed in Section 5.4 can be expressed as shown in Table 8.7.1.

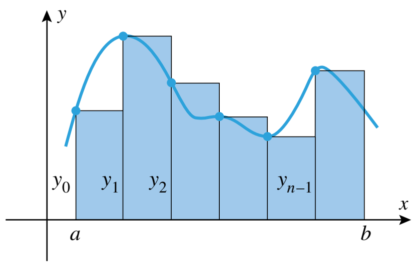
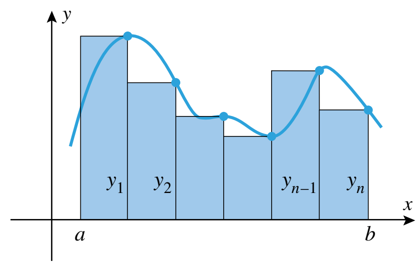
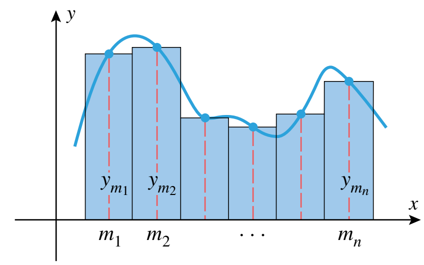
TRAPEZOIDAL APPROXIMATION

The left-hand and right-hand endpoint approximations are rarely used in applications; however, if we take the average of the left-hand and right-hand endpoint approximations, we obtain a result, called the ***trapezoidal approximation***, which is commonly used:

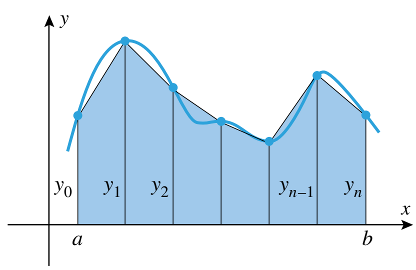
Trapezoidal Approximation

$$\int_a^b f(x)\,dx \approx \left(\frac{b-a}{2n}\right)[y_0 + 2y_1 + \cdots + 2y_{n-1} + y_n] \qquad (1)$$

Table 8.7.1

LEFT ENDPOINT APPROXIMATION	RIGHT ENDPOINT APPROXIMATION	MIDPOINT APPROXIMATION
$\int_a^b f(x)\,dx \approx \left(\frac{b-a}{n}\right)[y_0 + y_1 + \cdots + y_{n-1}]$	$\int_a^b f(x)\,dx \approx \left(\frac{b-a}{n}\right)[y_1 + y_2 + \cdots + y_n]$	$\int_a^b f(x)\,dx \approx \left(\frac{b-a}{n}\right)[y_{m_1} + y_{m_2} + \cdots + y_{m_n}]$
		

The name *trapezoidal approximation* can be explained by considering the case in which $f(x) \geq 0$ on $[a, b]$, so that $\int_a^b f(x)\,dx$ represents the area under $f(x)$ over $[a, b]$. Geometrically, the trapezoidal approximation formula results if we approximate this area by the sum of the trapezoidal areas shown in Figure 8.7.2 (Exercise 43).



Trapezoidal approximation

Figure 8.7.2

Example 1 In Table 8.7.2 we have approximated

$$\ln 2 = \int_1^2 \frac{1}{x}\,dx$$

using the midpoint approximation and the trapezoidal approximation. In each case we used $n = 10$ subdivisions of the interval $[1, 2]$, so that

$$\underbrace{\frac{b-a}{n} = \frac{2-1}{10} = 0.1}_{\text{Midpoint}} \quad \text{and} \quad \underbrace{\frac{b-a}{2n} = \frac{2-1}{20} = 0.05}_{\text{Trapezoidal}}$$ ◀

REMARK. In Example 1 we rounded the numerical values to nine places to the right of the decimal point; we will follow this procedure throughout this section. If your calculator cannot produce this many places, then you will have to make the appropriate adjustments. What is important here is that you understand the principles involved.

COMPARISON OF THE MIDPOINT AND TRAPEZOIDAL APPROXIMATIONS

The value of ln 2 rounded to nine decimal places is

$$\ln 2 = \int_1^2 \frac{1}{x}\,dx \approx 0.693147181 \tag{2}$$

so that the midpoint approximation in Example 1 produced a more accurate result than the

Table 8.7.2

Midpoint Approximation

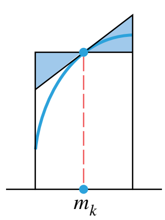
i	MIDPOINT m_i	$y_{m_i} = f(m_i) = 1/m_i$
1	1.05	0.952380952
2	1.15	0.869565217
3	1.25	0.800000000
4	1.35	0.740740741
5	1.45	0.689655172
6	1.55	0.645161290
7	1.65	0.606060606
8	1.75	0.571428571
9	1.85	0.540540541
10	1.95	0.512820513
		6.928353603

$$\int_1^2 \frac{1}{x}\,dx \approx (0.1)(6.928353603) \approx 0.692835360$$

Trapezoidal Approximation

i	ENDPOINT x_i	$y_i = f(x_i) = 1/x_i$	MULTIPLIER w_i	$w_i y_i$
0	1.0	1.000000000	1	1.000000000
1	1.1	0.909090909	2	1.818181818
2	1.2	0.833333333	2	1.666666667
3	1.3	0.769230769	2	1.538461538
4	1.4	0.714285714	2	1.428571429
5	1.5	0.666666667	2	1.333333333
6	1.6	0.625000000	2	1.250000000
7	1.7	0.588235294	2	1.176470588
8	1.8	0.555555556	2	1.111111111
9	1.9	0.526315789	2	1.052631579
10	2.0	0.500000000	1	0.500000000
				13.875428063

$$\int_1^2 \frac{1}{x}\,dx \approx (0.05)(13.875428063) \approx 0.693771403$$



The shaded triangles have equal areas.

Figure 8.7.3

trapezoidal approximation (verify). To see why this should be so, we need to look at the midpoint approximation from another viewpoint. [For simplicity in the explanations, we will assume that $f(x) \geq 0$, but the conclusions will be true without this assumption.] For differentiable functions, the midpoint approximation is sometimes called the *tangent line approximation* because over each subinterval the area of the rectangle used in the midpoint approximation is equal to the area of the trapezoid whose upper boundary is the tangent line to $y = f(x)$ at the midpoint of the interval (Figure 8.7.3). The equality of these areas follows from the fact that the shaded triangles in Figure 8.7.3 are congruent.

In this section we will denote the midpoint and trapezoidal approximations of $\int_a^b f(x)\,dx$ with n subintervals by M_n and T_n, respectively, and we will denote the errors in these approximations by

$$|E_M| = \left|\int_a^b f(x)\,dx - M_n\right| \quad \text{and} \quad |E_T| = \left|\int_a^b f(x)\,dx - T_n\right|$$

In Figure 8.7.4*a* we have isolated a subinterval of $[a, b]$ on which the graph of a function f is concave down, and we have shaded the areas that represent the errors in the midpoint and trapezoidal approximations over the subinterval. In Figure 8.7.4*b* we show a succession of four illustrations which make it evident that the error from the midpoint approximation is less than that from the trapezoidal approximation. If the graph of f were concave up, analogous figures would lead to the same conclusion. (This argument, due to Frank Buck, appeared in *The College Mathematics Journal*, Vol. 16, No. 1, 1985.)

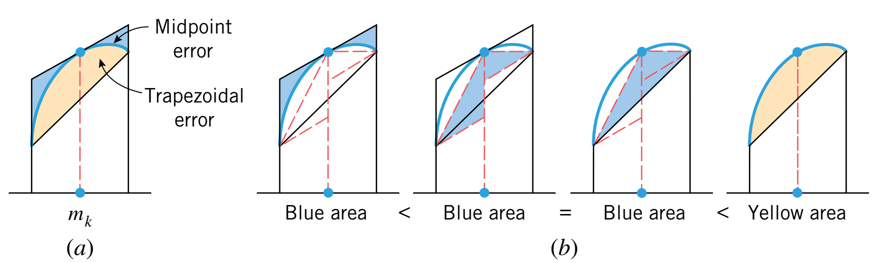


Figure 8.7.4

Figure 8.7.4*a* also suggests that on a subinterval where the graph is concave down, the midpoint approximation is larger than the value of the integral and the trapezoidal approximation is smaller. On an interval where the graph is concave up it is the other way around. In summary, we have the following result, which we state without formal proof:

8.7.1 THEOREM. *Let f be continuous on $[a, b]$, and let $|E_M|$ and $|E_T|$ be the absolute errors that result from the midpoint and trapezoidal approximations of $\int_a^b f(x)\,dx$ using n subintervals.*

(*a*) *If the graph of f is either concave up or concave down on (a, b), then $|E_M| < |E_T|$, that is, the error from the midpoint approximation is less than that from the trapezoidal approximation.*

(*b*) *If the graph of f is concave down on (a, b), then*

$$T_n < \int_a^b f(x)\,dx < M_n$$

(*c*) *If the graph of f is concave up on (a, b), then*

$$M_n < \int_a^b f(x)\,dx < T_n$$

Example 2 As observed earlier and illustrated in Table 8.7.3, the midpoint approximation of

$$\int_1^2 \frac{1}{x}\,dx = \ln 2$$

in Example 1 is more accurate than the trapezoidal approximation when partitioning [1, 2] into $n = 10$ subintervals. This is consistent with part (*a*) of Theorem 8.7.1, since $f(x) = 1/x$ is continuous on [1, 2] and concave up on (1, 2). Moreover, $M_{10} < \ln 2 < T_{10}$, as predicted by part (*c*) of Theorem 8.7.1. ◀

Table 8.7.3

ln 2 (NINE DECIMAL PLACES)	APPROXIMATION	DIFFERENCE
0.693147181	$T_{10} \approx 0.693771403$	$E_T = \ln 2 - T_{10} \approx -0.000624222$
0.693147181	$M_{10} \approx 0.692835360$	$E_M = \ln 2 - M_{10} \approx 0.000311821$

Example 3 In Table 8.7.4 we have approximated

$$\sin 1 = \int_0^1 \cos x\,dx$$

using the midpoint and trapezoidal approximations with $n = 5$ subdivisions of the interval [0, 1]. (As before, the numerical values are rounded to nine decimal places.) Note that $f(x) = \cos x$ is continuous on [0, 1] and concave down on (0, 1). Thus, Theorem 8.7.1(*a*) guarantees that $|E_M| < |E_T|$, as shown in Table 8.7.4. Also, $T_5 < \sin 1 < M_5$, as predicted

Table 8.7.4

sin 1 (NINE DECIMAL PLACES)	APPROXIMATION	DIFFERENCE
0.841470985	$T_5 \approx 0.838664210$	$E_T = \sin 1 - T_5 \approx 0.002806775$
0.841470985	$M_5 \approx 0.842875074$	$E_M = \sin 1 - M_5 \approx -0.001404089$

by Theorem 8.7.1(*b*).

Table 8.7.5 shows approximations for

$$\sin 3 = \int_0^3 \cos x\, dx$$

using the midpoint and trapezoidal approximations with $n = 10$ subdivisions of the interval [0, 3]. Note that $|E_M| < |E_T|$ and $T_{10} < \sin 3 < M_{10}$, although these results are not guaranteed by Theorem 8.7.1 since $f(x) = \cos x$ changes concavity on the interval (0, 3). ◀

Table 8.7.5

sin 3 (NINE DECIMAL PLACES)	APPROXIMATION	DIFFERENCE
0.141120008	$T_{10} \approx 0.140060017$	$E_T = \sin 3 - T_{10} \approx 0.001059991$
0.141120008	$M_{10} \approx 0.141650601$	$E_M = \sin 3 - M_{10} \approx -0.000530592$

WARNING. Do not conclude that the midpoint approximation is always better than the trapezoidal approximation; for some values of n, the trapezoidal approximation can be more accurate over an interval on which the function changes concavity.

SIMPSON'S RULE

Over an interval on which the integrand does not change concavity, Theorem 8.7.1 guarantees that a definite integral is better approximated by the midpoint approximation than by the trapezoidal approximation and that the value of the definite integral lies between these two approximations. The numerical evidence in Tables 8.7.3 and 8.7.4 (and even in Table 8.7.5, despite the change in concavity of the integrand over the interval) reveals that $E_T \approx -2E_M$ in these instances. This suggests that

$$\begin{aligned} 3\int_a^b f(x)\,dx &= 2\int_a^b f(x)\,dx + \int_a^b f(x)\,dx \\ &= 2(M_n + E_M) + (T_n + E_T) \\ &= (2M_n + T_n) + (2E_M + E_T) \approx 2M_n + T_n \end{aligned}$$

That is,

$$\int_a^b f(x)\,dx \approx \tfrac{1}{3}(2M_n + T_n)$$

Table 8.7.6 displays the approximations $\frac{1}{3}(2M_n + T_n)$ corresponding to the data in Tables 8.7.3 to 8.7.5. Thus, with little extra effort, we have much improved approximations for these definite integrals.

Table 8.7.6

CALCULATOR VALUE (NINE DECIMAL PLACES)	DEFINITE INTEGRAL APPROXIMATION	DIFFERENCE
$\ln 2 \approx 0.693147181$	$\int_1^2 (1/x)\,dx \approx \frac{1}{3}(2M_{10} + T_{10}) \approx 0.693147375$	-0.000000194
$\sin 1 \approx 0.841470985$	$\int_0^1 \cos x\,dx \approx \frac{1}{3}(2M_5 + T_5) \approx 0.841471453$	-0.000000468
$\sin 3 \approx 0.141120008$	$\int_0^3 \cos x\,dx \approx \frac{1}{3}(2M_{10} + T_{10}) \approx 0.141120406$	-0.000000398

Using the midpoint and trapezoidal approximation formulas in Table 8.7.1 and Formula (1), we can derive a similar formula for this approximation. For convenience, we partition the interval $[a, b]$ into $2n$ subintervals, each of length $(b - a)/(2n)$. As before, label the endpoints of these subintervals by $a = x_0, x_1, x_2, \ldots, x_{2n} = b$. Then $x_0, x_2, x_4, \ldots, x_{2n}$

define a partition of $[a, b]$ into n equal subintervals, and the midpoints of these subintervals are $x_1, x_3, x_5, \ldots, x_{2n-1}$, respectively. Using $y_i = f(x_i)$, we have

$$M_n = \left(\frac{b-a}{n}\right)[y_1 + y_3 + \cdots + y_{2n-1}] = \left(\frac{b-a}{2n}\right)[2y_1 + 2y_3 + \cdots + 2y_{2n-1}]$$

$$T_n = \left(\frac{b-a}{2n}\right)[y_0 + 2y_2 + 2y_4 + \cdots + 2y_{2n-2} + y_{2n}]$$

Now define S_{2n} by

$$\begin{aligned} S_{2n} &= \frac{1}{3}(2M_n + T_n) \\ &= \frac{1}{3}\left(\frac{b-a}{2n}\right)[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{2n-2} + 4y_{2n-1} + y_{2n}] \end{aligned} \tag{3}$$

The approximation

$$\int_a^b f(x)\,dx \approx S_{2n}$$

as given in (3) is known as ***Simpson's** rule***. We denote the absolute error in this approximation by

$$|E_S| = \left|\int_a^b f(x)\,dx - S_{2n}\right|$$

Example 4 Table 8.7.6 shows the Simpson's rule approximations

$$S_{20} = \tfrac{1}{3}(2M_{10} + T_{10}), \quad S_{10} = \tfrac{1}{3}(2M_5 + T_5), \quad \text{and} \quad S_{20} = \tfrac{1}{3}(2M_{10} + T_{10})$$

for the definite integrals

$$\int_1^2 \frac{1}{x}\,dx, \quad \int_0^1 \cos x\,dx, \quad \text{and} \quad \int_0^3 \cos x\,dx$$

respectively.

In Table 8.7.7 we have approximated

$$\ln 2 = \int_1^2 \frac{1}{x}\,dx$$

using (3) for Simpson's rule, where the interval $[1, 2]$ is partitioned into $2n = 10$ subintervals. Thus,

$$\frac{1}{3}\left(\frac{b-a}{2n}\right) = \frac{1}{3}\left(\frac{2-1}{10}\right) = \frac{1}{30}$$

* THOMAS SIMPSON (1710–1761). English mathematician. Simpson was the son of a weaver. He was trained to follow in his father's footsteps and had little formal education in his early life. His interest in science and mathematics was aroused in 1724, when he witnessed an eclipse of the Sun and received two books from a peddler, one on astrology and the other on arithmetic. Simpson quickly absorbed their contents and soon became a successful local fortune teller. His improved financial situation enabled him to give up weaving and marry his landlady, an older woman. Then in 1733 some mysterious "unfortunate incident" forced him to move. He settled in Derby, where he taught in an evening school and worked at weaving during the day. In 1736 he moved to London and published his first mathematical work in a periodical called the *Ladies' Diary* (of which he later became the editor). In 1737 he published a successful calculus textbook that enabled him to give up weaving completely and concentrate on textbook writing and teaching. His fortunes improved further in 1740 when one Robert Heath accused him of plagiarism. The publicity was marvelous, and Simpson proceeded to dash off a succession of best-selling textbooks: *Algebra* (ten editions plus translations), *Geometry* (twelve editions plus translations), *Trigonometry* (five editions plus translations), and numerous others. It is interesting to note that Simpson did not discover the rule that bears his name. It was a well-known result by Simpson's time.

Table 8.7.7 Simpson's Rule

i	ENDPOINT x_i	$y_i = f(x_i) = 1/x_i$	MULTIPLIER w_i	$w_i y_i$
0	1.0	1.000000000	1	1.000000000
1	1.1	0.909090909	4	3.636363636
2	1.2	0.833333333	2	1.666666667
3	1.3	0.769230769	4	3.076923077
4	1.4	0.714285714	2	1.428571429
5	1.5	0.666666667	4	2.666666667
6	1.6	0.625000000	2	1.250000000
7	1.7	0.588235294	4	2.352941176
8	1.8	0.555555556	2	1.111111111
9	1.9	0.526315789	4	2.105263158
10	2.0	0.500000000	1	0.500000000
				20.794506921

$$\int_1^2 \frac{1}{x}\,dx \approx \left(\tfrac{1}{30}\right)(20.794506921) \approx 0.693150231$$

Then

$$\begin{aligned}|E_S| &= \left|\int_1^2 \frac{1}{x}\,dx - S_{10}\right| \\ &= |\ln 2 - S_{10}| \approx |0.693147181 - 0.693150231| = 0.000003050\end{aligned}$$

By contrast, $M_5 \approx 0.691907886$ and $T_5 \approx 0.695634921$ have absolute errors

$$|E_M| \approx 0.001239295 \quad \text{and} \quad |E_T| \approx 0.002487740$$

respectively, so S_{10} is a much more accurate approximation of $\ln 2$ than either M_5 or T_5. ◀

GEOMETRIC INTERPRETATION OF SIMPSON'S RULE

Both the midpoint and trapezoidal approximations for a definite integral are obtained by approximating a segment of the curve $y = f(x)$ by a linear segment (Figure 8.7.5). Formula (3) for Simpson's rule can be obtained by approximating a segment of the curve $y = f(x)$ by a segment of a quadratic function $y = Ax^2 + Bx + C$, thus capturing some sense of the concavity of the function.

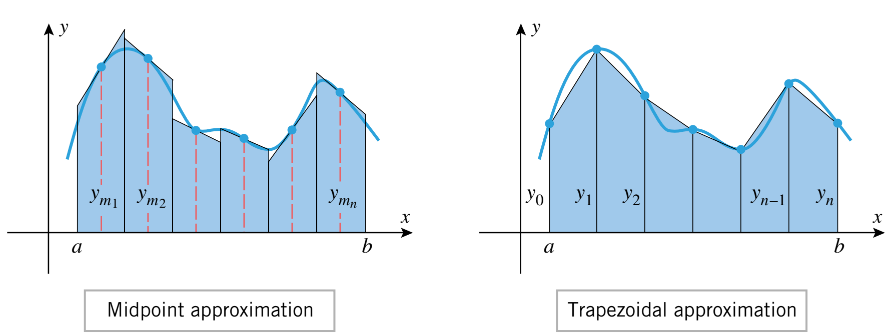


Figure 8.7.5

For this interpretation of Simpson's rule we start with the observation that for

$a \le X_0 < X_2 \le b$

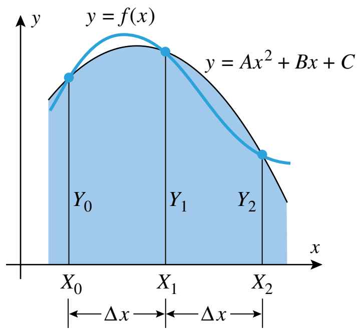


Figure 8.7.6

there is a unique function $g(x)$ of the form

$$g(x) = Ax^2 + Bx + C$$

such that

$$g(X_0) = f(X_0), \quad g(X_2) = f(X_2), \quad \text{and} \quad g(X_1) = f(X_1)$$

where $X_1 = (X_0 + X_2)/2$ (Figure 8.7.6). That is, we approximate $f(x)$ on $[X_0, X_2]$ by fitting a polynomial $g(x)$ of degree at most 2 to the points on the graph of $y = f(x)$ corresponding to $x = X_0$, X_1, and X_2. We then use $\int_{X_0}^{X_2} g(x)\,dx$ to approximate $\int_{X_0}^{X_2} f(x)\,dx$.

Setting

$$\Delta x = \frac{X_2 - X_0}{2}$$

so that

$$X_2 = X_0 + 2\Delta x, \quad Y_0 = f(X_0), \quad Y_1 = f(X_1), \quad \text{and} \quad Y_2 = f(X_2)$$

the key result to establish is

$$\int_{X_0}^{X_2} g(x)\,dx = \int_{X_0}^{X_2} (Ax^2 + Bx + C)\,dx = \frac{\Delta x}{3}[Y_0 + 4Y_1 + Y_2] \tag{4}$$

We verify (4) by working from both ends to arrive at a common middle. Starting with the expression $Y_0 + 4Y_1 + Y_2$ on the right side of Equation (4),

$$\begin{aligned}
Y_0 + 4Y_1 + Y_2 & \\
&= g(X_0) + 4g(X_1) + g(X_2) \\
&= A[X_0^2 + 4X_1^2 + X_2^2] + B[X_0 + 4X_1 + X_2] + C[1 + 4 + 1] \\
&= A\left[X_0^2 + 4\left(\frac{X_0 + X_2}{2}\right)^2 + X_2^2\right] + B\left[X_0 + 4\left(\frac{X_0 + X_2}{2}\right) + X_2\right] + 6C \\
&= A[X_0^2 + (X_0 + X_2)^2 + X_2^2] + B[3X_0 + 3X_2] + 6C \\
&= 2A[X_0^2 + X_0X_2 + X_2^2] + 3B[X_0 + X_2] + 6C
\end{aligned} \tag{5}$$

Furthermore,

$$\begin{aligned}
\int_{X_0}^{X_2} g(x)\,dx = \int_{X_0}^{X_2} (Ax^2 + Bx + C)\,dx &= \frac{A}{3}x^3 + \frac{B}{2}x^2 + Cx\Big]_{X_0}^{X_2} \\
&= \frac{A}{3}(X_2^3 - X_0^3) + \frac{B}{2}(X_2^2 - X_0^2) + C(X_2 - X_0) \\
&= \left(\frac{X_2 - X_0}{3}\right)\left[A(X_2^2 + X_2X_0 + X_0^2) + \frac{3B}{2}(X_2 + X_0) + 3C\right] \\
&= \left(\frac{2\Delta x}{3}\right)\left[A(X_2^2 + X_2X_0 + X_0^2) + \frac{3B}{2}(X_2 + X_0) + 3C\right] \\
&= \frac{\Delta x}{3}[2A(X_2^2 + X_2X_0 + X_0^2) + 3B(X_2 + X_0) + 6C]
\end{aligned} \tag{6}$$

Substituting (5) into (6) gives (4).

Using the partition $a = x_0, x_1, x_2, \ldots, x_{2n} = b$ of the interval $[a, b]$ into $2n$ subintervals, each of width

$$\Delta x = \frac{b - a}{2n}$$

and applying (4) to the subintervals $[x_0, x_2], [x_2, x_4], \ldots, [x_{2n-2}, x_{2n}]$, we can now derive

Simpson's rule in (3) as the integral of a piecewise-quadratic approximation to $f(x)$:

$$\begin{aligned}\int_{a=x_0}^{b=x_{2n}} f(x)\,dx &= \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \cdots + \int_{x_{2n-2}}^{x_{2n}} f(x)\,dx \\ &\approx \frac{\Delta x}{3}[y_0 + 4y_1 + y_2] + \frac{\Delta x}{3}[y_2 + 4y_3 + y_4] + \cdots + \frac{\Delta x}{3}[y_{2n-2} + 4y_{2n-1} + y_{2n}] \\ &= \frac{\Delta x}{3}[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{2n-2} + 4y_{2n-1} + y_{2n}] \\ &= S_{2n}\end{aligned}$$

ERROR ESTIMATES

With all the methods studied in this section, there are two sources of error: the *intrinsic* or *truncation error* due to the approximation formula, and the *roundoff error* introduced in the calculations. In general, increasing n reduces the truncation error but increases the roundoff error, since more computations are required for larger n. In practical applications, it is important to know how large n must be taken to ensure that a specified degree of accuracy is obtained. The analysis of roundoff error is complicated and will not be considered here. However, the following theorems, which are proved in books on numerical analysis, provide upper bounds on the truncation errors in the midpoint, trapezoidal, and Simpson's rule approximations.

8.7.2 THEOREM (*Midpoint and Trapezoidal Error Estimates*). *If f'' is continuous on $[a, b]$ and if K_2 is the maximum value of $|f''(x)|$ on $[a, b]$, then for n subintervals of $[a, b]$*

(a) $$|E_M| = \left|\int_a^b f(x)\,dx - M_n\right| \le \frac{(b-a)^3 K_2}{24n^2} \tag{7}$$

(b) $$|E_T| = \left|\int_a^b f(x)\,dx - T_n\right| \le \frac{(b-a)^3 K_2}{12n^2} \tag{8}$$

8.7.3 THEOREM (*Simpson Error Estimate*). *If $f^{(4)}$ is continuous on $[a, b]$ and if K_4 is the maximum value of $|f^{(4)}(x)|$ on $[a, b]$, then for $2n$ subintervals of $[a, b]$*

$$|E_S| = \left|\int_a^b f(x)\,dx - S_{2n}\right| \le \frac{(b-a)^5 K_4}{180(2n)^4} \tag{9}$$

Example 5 Find an upper bound on the absolute error that results from approximating

$$\ln 2 = \int_1^2 \frac{1}{x}\,dx$$

using (a) the midpoint approximation M_{10} with $n = 10$ subintervals, (b) the trapezoidal approximation T_{10} with $n = 10$ subintervals, and (c) Simpson's rule S_{10} with $2n = 10$ subintervals.

Solution. We will apply Formulas (7), (8), and (9) with

$$f(x) = \frac{1}{x}, \quad a = 1, \quad \text{and} \quad b = 2$$

For (7) and (8) we use $n = 10$; for (9) we use $2n = 10$, or $n = 5$. We have

$$f'(x) = -\frac{1}{x^2}, \quad f''(x) = \frac{2}{x^3}, \quad f'''(x) = -\frac{6}{x^4}, \quad f^{(4)}(x) = \frac{24}{x^5}$$

Thus,

$$|f''(x)| = \left|\frac{2}{x^3}\right| = \frac{2}{x^3}, \quad |f^{(4)}(x)| = \left|\frac{24}{x^5}\right| = \frac{24}{x^5} \qquad (10\text{–}11)$$

where we have dropped the absolute values because $f''(x)$ and $f^{(4)}(x)$ have positive values for $1 \le x \le 2$. Since (10) and (11) are continuous and decreasing on $[1, 2]$, both functions have their maximum values at $x = 1$; for (10) this maximum value is 2 and for (11) the maximum value is 24. Thus we can take $K_2 = 2$ in (7) and (8), and $K_4 = 24$ in (9). This yields

$$|E_M| \le \frac{(b-a)^3 K_2}{24n^2} = \frac{1^3 \cdot 2}{24 \cdot 10^2} \approx 0.000833333$$

$$|E_T| \le \frac{(b-a)^3 K_2}{12n^2} = \frac{1^3 \cdot 2}{12 \cdot 10^2} \approx 0.001666667$$

$$|E_S| \le \frac{(b-a)^5 K_4}{180(2n)^4} = \frac{1^5 \cdot 24}{180 \cdot 10^4} \approx 0.000013333$$ ◀

Note that the error bounds calculated in the preceding example are consistent with the values of E_M, E_T, and E_S calculated in Examples 2 and 4. In fact, these errors are considerably smaller in absolute value than the upper bounds of Example 5. It is quite common that the actual errors in the approximations M_n, T_n, and S_{2n} are substantially smaller than the upper bounds given in Theorems 8.7.2 and 8.7.3.

Example 6 How many subintervals should be used in approximating

$$\ln 2 = \int_1^2 \frac{1}{x}\,dx$$

by Simpson's rule for five decimal-place accuracy?

Solution. To obtain five decimal-place accuracy, we must choose the number of subintervals so that

$$|E_S| \le 0.000005 = 5 \times 10^{-6}$$

From (9), this can be achieved by taking $2n$ in Simpson's rule to satisfy

$$\frac{(b-a)^5 K_4}{180(2n)^4} \le 5 \times 10^{-6}$$

Taking $a = 1$, $b = 2$, and $K_4 = 24$ (found in Example 5) in this inequality yields

$$\frac{1^5 \cdot 24}{180 \cdot (2n)^4} \le 5 \times 10^{-6}$$

which, on taking reciprocals, can be rewritten as

$$(2n)^4 \ge \frac{2 \times 10^6}{75} \quad \text{or} \quad n^4 \ge \frac{10^4}{6}$$

Thus,

$$n \ge \frac{10}{\sqrt[4]{6}} \approx 6.389$$

Since n must be an integer, the smallest value of n that satisfies this requirement is $n = 7$, or $2n = 14$. Thus, the approximation S_{14} using 14 subintervals will produce five decimal-place accuracy. ◀

REMARK. In cases where it is difficult to find the values of K_2 and K_4 in Formulas (7), (8), and (9), these constants may be replaced by any larger constants. For example, suppose that a constant K can be easily found with the certainty that $|f''(x)| < K$ on the interval. Then $K_2 \leq K$ and

$$|E_T| \leq \frac{(b-a)^3 K_2}{12n^2} \leq \frac{(b-a)^3 K}{12n^2} \tag{12}$$

so the right side of (12) is also an upper bound on the value of $|E_T|$. Using K, however, will likely increase the computed value of n needed for a given error tolerance. Many applications involve the resolution of competing practical issues, here illustrated through the trade-off between the convenience of finding a crude bound for $|f''(x)|$ versus the efficiency of using the smallest possible n for a desired accuracy.

Example 7 How many subintervals should be used in approximating

$$\int_0^1 \cos(x^2)\, dx$$

by the midpoint approximation for three decimal-place accuracy?

Solution. To obtain three decimal-place accuracy, we must choose n so that

$$|E_M| \leq 0.0005 = 5 \times 10^{-4} \tag{13}$$

From (7) with $f(x) = \cos(x^2)$, $a = 0$, and $b = 1$, an upper bound on $|E_M|$ is given by

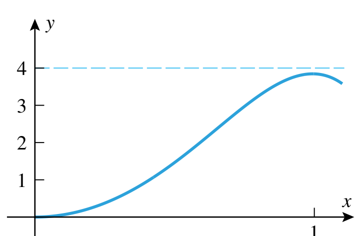
$$|E_M| \leq \frac{K_2}{24n^2} \tag{14}$$

where $|K_2|$ is the maximum value of $|f''(x)|$ on the interval [0, 1]. But,

$$f'(x) = -2x\sin(x^2)$$
$$f''(x) = -4x^2\cos(x^2) - 2\sin(x^2) = -[4x^2\cos(x^2) + 2\sin(x^2)]$$

so that

$$|f''(x)| = |4x^2\cos(x^2) + 2\sin(x^2)| \tag{15}$$



$y = |f''(x)| = |4x^2\cos(x^2) + 2\sin(x^2)|$

Figure 8.7.7

It would be tedious to look for the maximum value of this function on the interval [0, 1]. For x in [0, 1], it is easy to see that each of the expressions x^2, $\cos(x^2)$, and $\sin(x^2)$ is bounded in absolute value by 1, so $|4x^2\cos(x^2) + 2\sin(x^2)| \leq 4 + 2 = 6$ on [0, 1]. We can improve on this by using a graphing utility to sketch $|f''(x)|$, as shown in Figure 8.7.7. It is evident from the graph that

$$|f''(x)| < 4 \quad \text{for} \quad 0 \leq x \leq 1$$

Thus, it follows from (14) that

$$|E_M| \leq \frac{K_2}{24n^2} < \frac{4}{24n^2} = \frac{1}{6n^2}$$

and hence we can satisfy (13) by choosing n so that

$$\frac{1}{6n^2} < 5 \times 10^{-4}$$

which, on taking reciprocals, can be written as

$$n^2 > \frac{10^4}{30} \quad \text{or} \quad n > \frac{10^2}{\sqrt{30}} \approx 18.257$$

The smallest integer value of n satisfying this inequality is $n = 19$. Thus, the midpoint approximation M_{19} using 19 subintervals will produce three decimal-place accuracy. ◀

A COMPARISON OF THE THREE METHODS

Of the three methods studied in this section, Simpson's rule generally produces more accurate results than the midpoint or trapezoidal approximations for an equivalent amount of effort. To make this plausible, let us express (7), (8), and (9) in terms of the subinterval width

$$\Delta x = \frac{b-a}{n} \quad \text{for } M_n \text{ and } T_n$$

and

$$\Delta x = \frac{b-a}{2n} \quad \text{for } S_{2n}$$

We obtain

$$|E_M| \le \frac{1}{24} K_2(b-a)(\Delta x)^2 \tag{16}$$

$$|E_T| \le \frac{1}{12} K_2(b-a)(\Delta x)^2 \tag{17}$$

$$|E_S| \le \frac{1}{180} K_4(b-a)(\Delta x)^4 \tag{18}$$

(verify). Thus, for Simpson's rule the upper bound on the absolute error is proportional to $(\Delta x)^4$, whereas the upper bound on the absolute error for the midpoint and trapezoidal approximations is proportional to $(\Delta x)^2$. Thus, reducing the interval width by a factor of 10, for example, reduces the error bound by a factor of 100 for the midpoint and trapezoidal approximations but reduces the error bound by a factor of 10,000 for Simpson's rule. This suggests that, as n increases, the accuracy of Simpson's rule improves much more rapidly than that of the other approximations.

As a final note, observe that if $f(x)$ is a polynomial of degree 3 or less, then we have $f^{(4)}(x) = 0$ for all x, so $K_4 = 0$ in (9) and consequently $|E_S| = 0$. Thus, Simpson's rule gives exact results for polynomials of degree 3 or less. Similarly, the midpoint and trapezoidal approximations give exact results for polynomials of degree 1 or less. (You should also be able to see that this is so geometrically.)

EXERCISE SET 8.7 C CAS

In Exercises 1–6, use $n = 10$ subintervals to approximate the integral by (a) the midpoint approximation, (b) the trapezoidal approximation, and use $2n = 10$ subintervals to approximate the integral by (c) Simpson's rule. In each case, find the exact value of the integral and approximate the absolute error. Express your answers to at least four decimal places.

1. $\int_0^3 \sqrt{x+1}\,dx$ **2.** $\int_1^4 \frac{1}{\sqrt{x}}\,dx$ **3.** $\int_0^\pi \sin x\,dx$

4. $\int_0^1 \cos x\,dx$ **5.** $\int_1^3 e^{-x}\,dx$ **6.** $\int_{-1}^1 \frac{1}{2x+3}\,dx$

In Exercises 7–12, use inequalities (7), (8), and (9) to find upper bounds on the errors in parts (a), (b), and (c) of the indicated exercise.

7. Exercise 1 **8.** Exercise 2 **9.** Exercise 3

10. Exercise 4 **11.** Exercise 5 **12.** Exercise 6

In Exercises 13–18, use inequalities (7), (8), and (9) to find a number n of subintervals for (a) the midpoint approximation and (b) the trapezoidal approximation to ensure that the absolute error of the approximation will be less than the given value. Also, (c) find a number $2n$ of subintervals to ensure that the absolute error for the Simpson's rule approximation will be less than the given value.

13. Exercise 1; 5×10^{-4} **14.** Exercise 2; 5×10^{-4}

15. Exercise 3; 10^{-3} **16.** Exercise 4; 10^{-3}

17. Exercise 5; 10^{-6} **18.** Exercise 6; 10^{-6}

In Exercises 19 and 20, find a function $g(x)$ of the form

$$g(x) = Ax^2 + Bx + C$$

whose graph contains the points $(X_0, f(X_0))$, $(X_1, f(X_1))$, and $(X_2, f(X_2))$, for the given function $f(x)$ and the given values X_0, X_1, and X_2. Verify that

$$\int_{X_0}^{X_2} g(x)\,dx = \frac{\Delta x}{3}[f(X_0) + 4f(X_1) + f(X_2)]$$

where $\Delta x = (X_2 - X_0)/2$ as asserted with Formula (4).

19. $f(x) = \dfrac{1}{x}$; $X_0 = 2, X_1 = 3, X_2 = 4$

20. $f(x) = \cos^2(\pi x)$; $X_0 = 0, X_1 = \frac{1}{6}, X_2 = \frac{1}{3}$

In Exercises 21–26, approximate the integral using Simpson's rule with $2n = 10$ subintervals, and compare your answer to that produced by a calculating utility with a numerical integration capability. Express your answers to at least four decimal places.

21. $\displaystyle\int_0^1 e^{-x^2}\,dx$

22. $\displaystyle\int_0^2 \frac{x}{\sqrt{1+x^3}}\,dx$

23. $\displaystyle\int_1^2 \sqrt{1+x^3}\,dx$

24. $\displaystyle\int_0^\pi \frac{1}{2-\sin x}\,dx$

25. $\displaystyle\int_0^2 \sin(x^2)\,dx$

26. $\displaystyle\int_1^3 \sqrt{\ln x}\,dx$

In Exercises 27 and 28, the exact value of the integral is π (verify). Use $n = 10$ subintervals to approximate the integral by (a) the midpoint approximation and (b) the trapezoidal approximation, and use $2n = 10$ subintervals to approximate the integral by (c) Simpson's rule. Estimate the absolute error, and express your answers to at least four decimal places.

27. $\displaystyle\int_0^1 \frac{4}{1+x^2}\,dx$

28. $\displaystyle\int_0^2 \sqrt{4-x^2}\,dx$

29. In Example 6 we showed that taking $2n = 14$ subdivisions ensures that the approximation of

$$\ln 2 = \int_1^2 \frac{1}{x}\,dx$$

by Simpson's rule is accurate to five decimal places. Confirm this by comparing the approximation of ln 2 produced by Simpson's rule with $2n = 14$ to the value produced directly by your calculating utility.

30. In parts (a) and (b), determine whether an approximation of the integral by the trapezoidal rule would be less than or would be greater than the exact value of the integral.

(a) $\displaystyle\int_1^2 e^{-x^2}\,dx$ (b) $\displaystyle\int_0^{0.5} e^{-x^2}\,dx$

In Exercises 31 and 32, find a value for n to ensure that the absolute error in approximating the integral by the midpoint rule will be less than 10^{-4}.

31. $\displaystyle\int_0^2 x\sin x\,dx$

32. $\displaystyle\int_0^1 e^{\cos x}\,dx$

In Exercises 33 and 34, show that inequalities (7) and (8) are of no value in finding an upper bound on the absolute error that results from approximating the integral using either the midpoint approximation or the trapezoidal approximation.

33. $\displaystyle\int_0^1 \sqrt{x}\,dx$

34. $\displaystyle\int_0^1 \sin\sqrt{x}\,dx$

In Exercises 35 and 36, use Simpson's rule with $2n = 10$ subintervals to approximate the length of the curve. Express your answers to at least four decimal places.

35. $y = \sin x,\ 0 \le x \le \pi$

36. $y = 1/x,\ 1 \le x \le 3$

Numerical integration methods can be used in problems where only measured or experimentally determined values of the integrand are available. In Exercises 37–42, use Simpson's rule to estimate the value of the relevant integral.

37. A graph of the speed v versus time t for a test run of an Infiniti G20 automobile is shown in the accompanying figure. Estimate the speeds at $t = 0, 5, 10, 15$, and 20 s from the graph, convert to ft/s using 1 mi/h = 22/15 ft/s, and use these speeds to approximate the number of feet traveled during the first 20 s. Round your answer to the nearest foot. [*Hint:* Distance traveled = $\int_0^{20} v(t)\,dt$.] [Data from *Road and Track*, October 1990.]

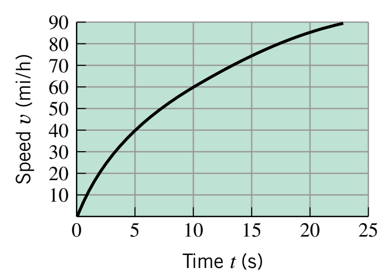


Figure Ex-37

38. A graph of the acceleration a versus time t for an object moving on a straight line is shown in the accompanying figure. Estimate the accelerations at $t = 0, 1, 2, \ldots, 8$ s from the graph and use them to approximate the change in velocity from $t = 0$ to $t = 8$ s. Round your answer to the nearest tenth cm/s. [*Hint:* Change in velocity = $\int_0^8 a(t)\,dt$.]

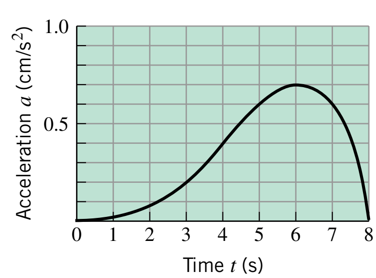


Figure Ex-38

39. The accompanying table gives the speeds, in miles per second, at various times for a test rocket that was fired upward from the surface of the Earth. Use these values to approximate the number of miles traveled during the first 180 s. Round your answer to the nearest tenth of a mile.
[*Hint:* Distance traveled = $\int_0^{180} v(t)\,dt$.]

40. The accompanying table gives the speeds of a bullet at various distances from the muzzle of a rifle. Use these values to approximate the number of seconds for the bullet to travel 1800 ft. Express your answer to the nearest hundredth of a second. [*Hint:* If v is the speed of the bullet and x is the distance traveled, then $v = dx/dt$ so that $dt/dx = 1/v$ and $t = \int_0^{1800} (1/v)\, dx$.]

TIME t (s)	SPEED v (mi/s)
0	0.00
30	0.03
60	0.08
90	0.16
120	0.27
150	0.42
180	0.65

Table Ex-39

DISTANCE x (ft)	SPEED v (ft/s)
0	3100
300	2908
600	2725
900	2549
1200	2379
1500	2216
1800	2059

Table Ex-40

41. Measurements of a pottery shard recovered from an archaeological dig reveal that the shard came from a pot with a flat bottom and circular cross sections (see the accompanying figure). The figure shows interior radius measurements of the shard made every 4 cm from the bottom of the pot to the top. Use those values to approximate the interior volume of the pot to the nearest tenth of a liter (1 L = 1000 cm^3). [*Hint:* Use 6.2.3 (volume by cross sections) to set up an appropriate integral for the volume.]

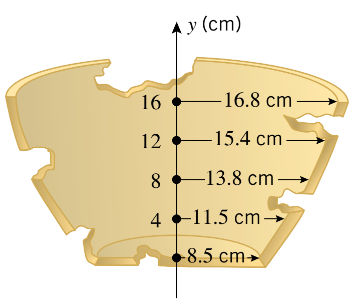


Figure Ex-41

42. Engineers want to construct a straight and level road 600 ft long and 75 ft wide by making a vertical cut through an intervening hill (see the accompanying figure). Heights of the hill above the centerline of the proposed road, as obtained at various points from a contour map of the region, are shown in the accompanying figure. To estimate the construction costs, the engineers need to know the volume of earth that must be removed. Approximate this volume, rounded to the nearest cubic foot. [*Hint:* First, set up an integral for the cross-sectional area of the cut along the centerline of the road, then assume that the height of the hill does not vary between the centerline and edges of the road.]

HORIZONTAL DISTANCE x (ft)	HEIGHT h (ft)
0	0
100	7
200	16
300	24
400	25
500	16
600	0

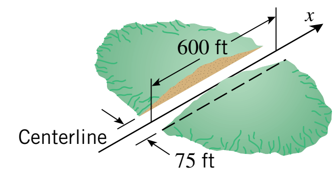


Figure Ex-42

43. Derive the trapezoidal rule by summing the areas of the trapezoids in Figure 8.7.2.

44. Let f be a function that is positive, continuous, decreasing, and concave down on the interval $[a, b]$. Assuming that $[a, b]$ is subdivided into n equal subintervals, arrange the following approximations of $\int_a^b f(x)\, dx$ in order of increasing value: left endpoint, right endpoint, midpoint, and trapezoidal.

C **45.** Let $f(x) = \cos(x^2)$.
(a) Use a CAS to approximate the maximum value of $|f''(x)|$ on the interval [0, 1].
(b) How large must n be in the midpoint approximation of $\int_0^1 f(x)\, dx$ to ensure that the absolute error is less than 5×10^{-4}? Compare your result with that obtained in Example 7.
(c) Estimate the integral using the midpoint approximation with the value of n obtained in part (b).

C **46.** Let $f(x) = \sqrt{1 + x^3}$.
(a) Use a CAS to approximate the maximum value of $|f''(x)|$ on the interval [0, 1].
(b) How large must n be in the trapezoidal approximation of $\int_0^1 f(x)\, dx$ to ensure that the absolute error is less than 10^{-3}?
(c) Estimate the integral using the trapezoidal approximation with the value of n obtained in part (b).

C **47.** Let $f(x) = \cos(x^2)$.
(a) Use a CAS to approximate the maximum value of $|f^{(4)}(x)|$ on the interval [0, 1].
(b) How large must the value of n be in the approximation of $\int_0^1 f(x)\, dx$ by Simpson's rule to ensure that the absolute error is less than 10^{-4}?
(c) Estimate the integral using Simpson's rule with the value of n obtained in part (b).

C **48.** Let $f(x) = \sqrt{1 + x^3}$.
(a) Use a CAS to approximate the maximum value of $|f^{(4)}(x)|$ on the interval [0, 1].
(b) How large must the value of n be in the approximation of $\int_0^1 f(x)\, dx$ by Simpson's rule to ensure that the absolute error is less than 10^{-5}?
(c) Estimate the integral using Simpson's rule with the value of n obtained in part (b).

8.8 IMPROPER INTEGRALS

Up to now we have focused on definite integrals with continuous integrands and finite intervals of integration. In this section we will extend the concept of a definite integral to include infinite intervals of integration and integrands that become infinite within the interval of integration.

IMPROPER INTEGRALS

It is assumed in the definition of the definite integral

$$\int_a^b f(x)\,dx$$

that $[a, b]$ is a finite interval and that the limit that defines the integral exists; that is, the function f is integrable. We observed in Theorems 5.5.2 and 5.5.8 that continuous functions are integrable, as are bounded functions with finitely many points of discontinuity. We also observed in Theorem 5.5.8 that functions that are not bounded on the interval of integration are not integrable. Thus, for example, a function with a vertical asymptote within the interval of integration would not be integrable.

Our main objective in this section is to extend the concept of a definite integral to allow for infinite intervals of integration and integrands with vertical asymptotes within the interval of integration. We will call the vertical asymptotes ***infinite discontinuities***, and we will call integrals with infinite intervals of integration or infinite discontinuities within the interval of integration ***improper integrals***. Here are some examples:

- Improper integrals with infinite intervals of integration:

$$\int_1^{+\infty} \frac{dx}{x^2}, \quad \int_{-\infty}^0 e^x\,dx, \quad \int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$$

- Improper integrals with infinite discontinuities in the interval of integration:

$$\int_{-3}^3 \frac{dx}{x^2}, \quad \int_1^2 \frac{dx}{x-1}, \quad \int_0^\pi \tan x\,dx$$

- Improper integrals with infinite discontinuities and infinite intervals of integration:

$$\int_0^{+\infty} \frac{dx}{\sqrt{x}}, \quad \int_{-\infty}^{+\infty} \frac{dx}{x^2-9}, \quad \int_1^{+\infty} \sec x\,dx$$

INTEGRALS OVER INFINITE INTERVALS

To motivate a reasonable definition for improper integrals of the form

$$\int_a^{+\infty} f(x)\,dx$$

let us begin with the case where f is continuous and nonnegative on $[a, +\infty)$, so we can think of the integral as the area under the curve $y = f(x)$ over the interval $[a, +\infty)$ (Figure 8.8.1). At first, you might be inclined to argue that this area is infinite because the region has infinite extent. However, such an argument would be based on vague intuition rather than precise mathematical logic, since the concept of area has only been defined over intervals of *finite extent*. Thus, before we can make any reasonable statements about the area of the region in Figure 8.8.1, we need to begin by defining what we mean by the area of this region. For that purpose, it will help to focus on a specific example.

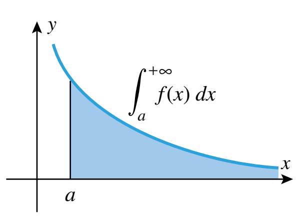


Figure 8.8.1

Suppose we are interested in the area A of the region that lies below the curve $y = 1/x^2$ and above the interval $[1, +\infty)$ on the x-axis. Instead of trying to find the entire area at once, let us begin by calculating the portion of the area that lies above a finite interval $[1, \ell]$,

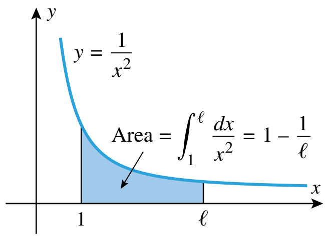


Figure 8.8.2

where $\ell > 1$ is arbitrary. That area is

$$\int_1^\ell \frac{dx}{x^2} = -\frac{1}{x}\bigg]_1^\ell = 1 - \frac{1}{\ell}$$

(Figure 8.8.2). If we now allow ℓ to increase so that $\ell \to +\infty$, then the portion of the area over the interval $[1, \ell]$ will begin to fill out the area over the entire interval $[1, +\infty)$ (Figure 8.8.3), and hence we can reasonably define the area A under $y = 1/x^2$ over the interval $[1, +\infty)$ to be

$$A = \int_1^{+\infty} \frac{dx}{x^2} = \lim_{\ell \to +\infty} \int_1^\ell \frac{dx}{x^2} = \lim_{\ell \to +\infty} \left(1 - \frac{1}{\ell}\right) = 1 \tag{1}$$

Thus, the area has a finite value of 1 and is not infinite as we first conjectured.

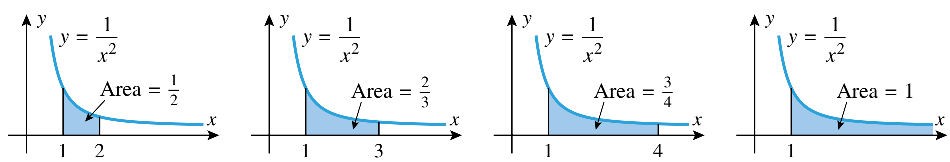


Figure 8.8.3

With the preceding discussion as our guide, we make the following definition (which is applicable to functions with both positive and negative values):

8.8.1 DEFINITION. The ***improper integral of f over the interval [a, +∞)*** is defined as

$$\int_a^{+\infty} f(x)\,dx = \lim_{\ell \to +\infty} \int_a^\ell f(x)\,dx$$

In the case where the limit exists, the improper integral is said to ***converge***, and the limit is defined to be the value of the integral. In the case where the limit does not exist, the improper integral is said to ***diverge***, and it is not assigned a value.

If f is nonnegative on $[a, +\infty)$ and the improper integral converges, then the value of the integral is regarded to be the area under the graph of f over the interval $[a, +\infty)$; and if the integral diverges, then the area under the graph of f over the interval $[a, +\infty)$ is regarded to be infinite.

Example 1 Evaluate

(a) $\int_1^{+\infty} \frac{dx}{x^3}$ (b) $\int_1^{+\infty} \frac{dx}{x}$

Solution (a). Following the definition, we replace the infinite upper limit by a finite upper limit ℓ, and then take the limit of the resulting integral. This yields

$$\int_1^{+\infty} \frac{dx}{x^3} = \lim_{\ell \to +\infty} \int_1^\ell \frac{dx}{x^3} = \lim_{\ell \to +\infty} \left[-\frac{1}{2x^2}\right]_1^\ell = \lim_{\ell \to +\infty} \left(\frac{1}{2} - \frac{1}{2\ell^2}\right) = \frac{1}{2}$$

Solution (b).

$$\int_1^{+\infty} \frac{dx}{x} = \lim_{\ell \to +\infty} \int_1^\ell \frac{dx}{x} = \lim_{\ell \to +\infty} \big[\ln x\big]_1^\ell = \lim_{\ell \to +\infty} \ln \ell = +\infty$$

In this case the integral diverges and hence has no value. ◀

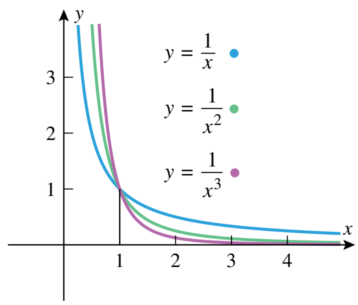


Figure 8.8.4

Because the functions $1/x^3$, $1/x^2$, and $1/x$ are nonnegative over the interval $[1, +\infty)$, it follows from (1) and the last example that over this interval the area under $y = 1/x^3$ is $\frac{1}{2}$, the area under $y = 1/x^2$ is 1, and the area under $y = 1/x$ is infinite. However, on the surface the graphs of the three functions seem very much alike (Figure 8.8.4), and there is nothing to suggest why one of the areas should be infinite and the other two finite. One explanation is that $1/x^3$ and $1/x^2$ approach zero more rapidly than $1/x$ as $x \to +\infty$, so that the area over the interval $[1, \ell]$ accumulates less rapidly under the curves $y = 1/x^3$ and $y = 1/x^2$ than under $y = 1/x$ as $\ell \to +\infty$, and the difference is just enough that the first two areas are finite and the third is infinite.

Example 2 For what values of p does the integral $\int_1^{+\infty} \frac{dx}{x^p}$ converge?

Solution. We know from the preceding example that the integral diverges if $p = 1$, so let us assume that $p \neq 1$. In this case we have

$$\int_1^{+\infty} \frac{dx}{x^p} = \lim_{\ell \to +\infty} \int_1^{\ell} x^{-p}\, dx = \lim_{\ell \to +\infty} \frac{x^{1-p}}{1-p}\bigg]_1^{\ell} = \lim_{\ell \to +\infty} \left[\frac{\ell^{1-p}}{1-p} - \frac{1}{1-p}\right]$$

If $p > 1$, then the exponent $1 - p$ is negative and $\ell^{1-p} \to 0$ as $\ell \to +\infty$; and if $p < 1$, then the exponent $1 - p$ is positive and $\ell^{1-p} \to +\infty$ as $\ell \to +\infty$. Thus, the integral converges if $p > 1$ and diverges otherwise. In the convergent case the value of the integral is

$$\int_1^{+\infty} \frac{dx}{x^p} = \left[0 - \frac{1}{1-p}\right] = \frac{1}{p-1} \quad (p > 1)$$ ◀

The following theorem summarizes this result:

8.8.2 THEOREM.

$$\int_1^{+\infty} \frac{dx}{x^p} = \begin{cases} \dfrac{1}{p-1} & \textit{if} \quad p > 1 \\ \textit{diverges} & \textit{if} \quad p \le 1 \end{cases}$$

Example 3 Evaluate $\int_0^{+\infty} (1 - x)e^{-x}\, dx$.

Solution. Integrating by parts with $u = 1 - x$ and $dv = e^{-x}\, dx$ yields

$$\int (1 - x)e^{-x}\, dx = -e^{-x}(1 - x) - \int e^{-x}\, dx = -e^{-x} + xe^{-x} + e^{-x} + C = xe^{-x} + C$$

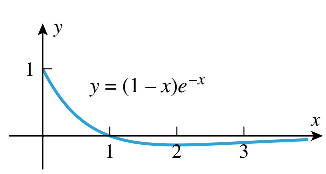
Thus,

$$\int_0^{+\infty} (1 - x)e^{-x}\, dx = \lim_{\ell \to +\infty} \left[xe^{-x}\right]_0^{\ell} = \lim_{\ell \to +\infty} \frac{\ell}{e^{\ell}}$$

The limit is an indeterminate form of type ∞/∞, so we will apply L'Hôpital's rule by differentiating the numerator and denominator with respect to ℓ. This yields

$$\int_0^{+\infty} (1 - x)e^{-x}\, dx = \lim_{\ell \to +\infty} \frac{1}{e^{\ell}} = 0$$

An explanation of why this integral is zero can be obtained by interpreting the integral as the net signed area between the graph of $y = (1 - x)e^{-x}$ and the interval $[0, +\infty)$ (Figure 8.8.5). ◀



The net signed area between the graph and the interval $[0, +\infty)$ is zero.

Figure 8.8.5

We also make the following definition:

8.8.3 DEFINITION. The ***improper integral of f over the interval*** **$(-\infty, b]$** is defined as

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{k\to-\infty}\int_{k}^{b} f(x)\,dx \tag{2}$$

The integral is said to ***converge*** if the limit exists and ***diverge*** if it does not. The ***improper integral of f over the interval*** **$(-\infty, +\infty)$** is defined as

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{+\infty} f(x)\,dx \tag{3}$$

where c is any real number. The improper integral is said to ***converge*** if both terms converge and ***diverge*** if either term diverges.

REMARK. In this definition, if f is nonnegative on the interval of integration, then the improper integral is regarded to be the area under the graph of f over that interval; the area has a finite value if the integral converges and is infinite if it diverges. We also note that in (3) it is usual to choose $c = 0$, but the choice does not matter; it can be proved that neither the convergence nor the value of the integral depends on the choice of c.

Example 4 Evaluate $\displaystyle\int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$.

Solution. We will evaluate the integral by choosing $c = 0$ in (3). With this value for c we obtain

$$\int_{0}^{+\infty} \frac{dx}{1+x^2} = \lim_{\ell\to+\infty}\int_{0}^{\ell} \frac{dx}{1+x^2} = \lim_{\ell\to+\infty}\left[\tan^{-1} x\right]_0^{\ell} = \lim_{\ell\to+\infty}(\tan^{-1}\ell) = \frac{\pi}{2}$$

$$\int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{k\to-\infty}\int_{k}^{0} \frac{dx}{1+x^2} = \lim_{k\to-\infty}\left[\tan^{-1} x\right]_k^{0} = \lim_{k\to-\infty}(-\tan^{-1} k) = \frac{\pi}{2}$$

Thus, the integral converges and its value is

$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{+\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

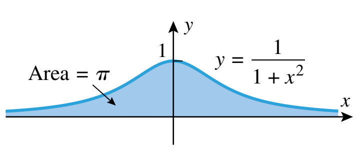


Figure 8.8.6

Since the integrand is nonnegative on the interval $(-\infty, +\infty)$, the integral represents the area of the region shown in Figure 8.8.6. ◀

INTEGRALS WHOSE INTEGRANDS HAVE INFINITE DISCONTINUITIES

Next we will consider improper integrals whose integrands have infinite discontinuities. We will start with the case where the interval of integration is a finite interval $[a, b]$ and the infinite discontinuity occurs at the right-hand endpoint.

To motivate an appropriate definition for such an integral let us consider the case where f is nonnegative on $[a, b]$, so we can interpret the improper integral $\int_a^b f(x)\,dx$ as the area of the region in Figure 8.8.7*a*. The problem of finding the area of this region is complicated by the fact that it extends indefinitely in the positive y-direction. However, instead of trying to find the entire area at once, we can proceed indirectly by calculating the portion of the area over the interval $[a, \ell]$ and then letting ℓ approach b to fill out the area of the entire region (Figure 8.8.7*b*). Motivated by this idea, we make the following definition:

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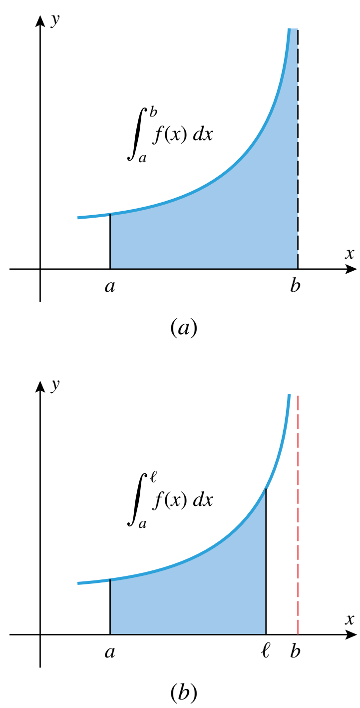

Figure 8.8.7

8.8.4 DEFINITION. If f is continuous on the interval $[a, b]$, except for an infinite discontinuity at b, then the ***improper integral of f over the interval* [*a, b*]** is defined as

$$\int_a^b f(x)\,dx = \lim_{\ell\to b^-}\int_a^\ell f(x)\,dx \tag{4}$$

In the case where the limit exists, the improper integral is said to ***converge***, and the limit is defined to be the value of the integral. In the case where the limit does not exist, the improper integral is said to ***diverge***, and it is not assigned a value.

Example 5 Evaluate $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x}}$.

Solution. The integral is improper because the integrand approaches $+\infty$ as x approaches the upper limit 1 from the left. From (4),

$$\int_0^1 \frac{dx}{\sqrt{1-x}} = \lim_{\ell\to 1^-}\int_0^\ell \frac{dx}{\sqrt{1-x}} = \lim_{\ell\to 1^-}\Big[-2\sqrt{1-x}\Big]_0^\ell$$
$$= \lim_{\ell\to 1^-}[-2\sqrt{1-\ell}+2] = 2$$ ◀

Improper integrals with an infinite discontinuity at the left-hand endpoint or inside the interval of integration are defined as follows.

8.8.5 DEFINITION. If f is continuous on the interval $[a, b]$, except for an infinite discontinuity at a, then the ***improper integral of f over the interval* [*a, b*]** is defined as

$$\int_a^b f(x)\,dx = \lim_{k\to a^+}\int_k^b f(x)\,dx \tag{5}$$

The integral is said to ***converge*** if the limit exists and ***diverge*** if it does not. If f is continuous on the interval $[a, b]$, except for an infinite discontinuity at a point c in (a, b), then the ***improper integral of f over the interval* [*a, b*]** is defined as

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \tag{6}$$

The improper integral is said to ***converge*** if both terms converge and ***diverge*** if either term diverges (Figure 8.8.8).

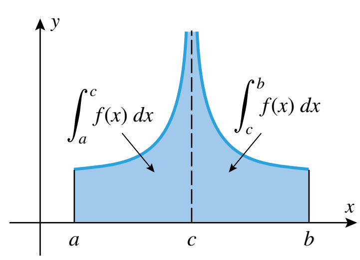

Figure 8.8.8

Example 6 Evaluate

(a) $\displaystyle\int_1^2 \frac{dx}{1-x}$ (b) $\displaystyle\int_1^4 \frac{dx}{(x-2)^{2/3}}$ (c) $\displaystyle\int_0^{+\infty} \frac{dx}{\sqrt{x}(x+1)}$

Solution (a). The integral is improper because the integrand approaches $-\infty$ as x approaches the lower limit 1 from the right (Figure 8.8.9). From Definition 8.8.5 we obtain

$$\int_1^2 \frac{dx}{1-x} = \lim_{k\to 1^+}\int_k^2 \frac{dx}{1-x} = \lim_{k\to 1^+}\Big[-\ln|1-x|\Big]_k^2$$
$$= \lim_{k\to 1^+}\Big[-\ln|-1| + \ln|1-k|\Big] = \lim_{k\to 1^+}\ln|1-k| = -\infty$$

so the integral diverges.

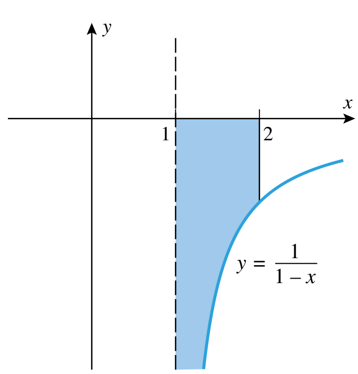

Figure 8.8.9

Solution (b). The integral is improper because the integrand approaches $+\infty$ at the point $x = 2$, which is inside the interval of integration. From Definition 8.8.5 we obtain

$$\int_1^4 \frac{dx}{(x-2)^{2/3}} = \int_1^2 \frac{dx}{(x-2)^{2/3}} + \int_2^4 \frac{dx}{(x-2)^{2/3}} \tag{7}$$

But

$$\int_1^2 \frac{dx}{(x-2)^{2/3}} = \lim_{\ell \to 2^-} \int_1^\ell \frac{dx}{(x-2)^{2/3}} = \lim_{\ell \to 2^-} [3(\ell-2)^{1/3} - 3(1-2)^{1/3}] = 3$$

$$\int_2^4 \frac{dx}{(x-2)^{2/3}} = \lim_{k \to 2^+} \int_k^4 \frac{dx}{(x-2)^{2/3}} = \lim_{k \to 2^+} [3(4-2)^{1/3} - 3(k-2)^{1/3}] = 3\sqrt[3]{2}$$

Thus, from (7)

$$\int_1^4 \frac{dx}{(x-2)^{2/3}} = 3 + 3\sqrt[3]{2}$$

Solution (c). This integral is improper for two reasons—the interval of integration is infinite, and there is an infinite discontinuity at $x = 0$. To evaluate this integral we will split the interval of integration at a convenient point, say $x = 1$, and write

$$\int_0^{+\infty} \frac{dx}{\sqrt{x}(x+1)} = \int_0^1 \frac{dx}{\sqrt{x}(x+1)} + \int_1^{+\infty} \frac{dx}{\sqrt{x}(x+1)}$$

The integrand in these two improper integrals does not match any of the forms in the Endpaper Integral Table, but the radical suggests the substitution $x = u^2$, $dx = 2u\,du$, from which we obtain

$$\int \frac{dx}{\sqrt{x}(x+1)} = \int \frac{2u\,du}{u(u^2+1)} = 2\int \frac{du}{u^2+1}$$
$$= 2\tan^{-1} u + C = 2\tan^{-1}\sqrt{x} + C$$

Thus,

$$\int_0^{+\infty} \frac{dx}{\sqrt{x}(x+1)} = 2 \lim_{k \to 0^+} \left[\tan^{-1}\sqrt{x}\right]_k^1 + 2 \lim_{\ell \to +\infty} \left[\tan^{-1}\sqrt{x}\right]_1^\ell$$
$$= 2\left[\frac{\pi}{4} - 0\right] + 2\left[\frac{\pi}{2} - \frac{\pi}{4}\right] = \pi$$ ◀

WARNING. It is sometimes tempting to apply the Fundamental Theorem of Calculus directly to an improper integral without taking the appropriate limits. To illustrate what can go wrong with this procedure, suppose we ignore the fact that the integral

$$\int_0^2 \frac{dx}{(x-1)^2} \tag{8}$$

is improper and write

$$\int_0^2 \frac{dx}{(x-1)^2} = -\frac{1}{x-1}\bigg]_0^2 = -1 - (1) = -2$$

This result is clearly nonsense because the integrand is never negative and consequently the integral cannot be negative! To evaluate (8) correctly we should write

$$\int_0^2 \frac{dx}{(x-1)^2} = \int_0^1 \frac{dx}{(x-1)^2} + \int_1^2 \frac{dx}{(x-1)^2}$$

But

$$\int_0^1 \frac{dx}{(x-1)^2} = \lim_{\ell \to 1^-} \int_0^\ell \frac{dx}{(x-1)^2} = \lim_{\ell \to 1^-} \left[-\frac{1}{\ell-1} - 1\right] = +\infty$$

so that (8) diverges.

THE APPLICATION OF IMPROPER INTEGRALS TO ARC LENGTH AND SURFACE AREA

In Definitions 6.4.2 and 6.5.2 for arc length and surface area we required the function f to be smooth (continuous first derivative) to ensure the integrability in the resulting formula. However, smoothness is overly restrictive since some of the most basic formulas in geometry involve functions that are not smooth but lead to convergent improper integrals. Accordingly, let us agree to extend the definitions of arc length and surface area to allow functions that are not smooth, but for which the resulting integral in the formula converges.

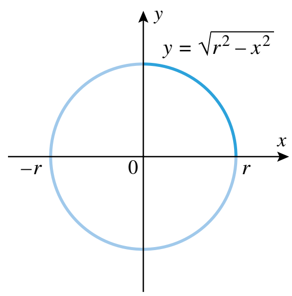

Figure 8.8.10

Example 7 Derive the formula for the circumference of a circle of radius r.

Solution. For convenience, let us assume that the circle is centered at the origin, in which case its equation is $x^2 + y^2 = r^2$. We will find the arc length of the portion of the circle that lies in the first quadrant and then multiply by 4 to obtain the total circumference (Figure 8.8.10).

Since the equation of the upper semicircle is $y = \sqrt{r^2 - x^2}$, it follows from Formula (4) of Section 6.4 that the circumference C is

$$C = 4\int_0^r \sqrt{1 + (dy/dx)^2}\,dx = 4\int_0^r \sqrt{1 + \left(-\frac{x}{\sqrt{r^2 - x^2}}\right)^2}\,dx$$
$$= 4r\int_0^r \frac{dx}{\sqrt{r^2 - x^2}}$$

This integral is improper because of the infinite discontinuity at $x = r$, and hence we evaluate it by writing

$$C = 4r \lim_{\ell \to r^-} \int_0^{\ell} \frac{dx}{\sqrt{r^2 - x^2}}$$
$$= 4r \lim_{\ell \to r^-} \left[\sin^{-1}\left(\frac{x}{r}\right)\right]_0^{\ell}$$

Formula (77) in the Endpaper Integral Table

$$= 4r \lim_{\ell \to r^-} \left[\sin^{-1}\left(\frac{\ell}{r}\right) - \sin^{-1} 0\right]$$
$$= 4r[\sin^{-1} 1 - \sin^{-1} 0] = 4r\left(\frac{\pi}{2} - 0\right) = 2\pi r$$ ◀

EXERCISE SET 8.8

Graphing Utility   C CAS

1. In each part, determine whether the integral is improper, and if so, explain why.

(a) $\displaystyle\int_1^5 \frac{dx}{x-3}$ (b) $\displaystyle\int_1^5 \frac{dx}{x+3}$ (c) $\displaystyle\int_0^1 \ln x\,dx$

(d) $\displaystyle\int_1^{+\infty} e^{-x}\,dx$ (e) $\displaystyle\int_{-\infty}^{+\infty} \frac{dx}{\sqrt[3]{x-1}}$ (f) $\displaystyle\int_0^{\pi/4} \tan x\,dx$

2. In each part, determine all values of p for which the integral is improper.

(a) $\displaystyle\int_0^1 \frac{dx}{x^p}$ (b) $\displaystyle\int_1^2 \frac{dx}{x-p}$ (c) $\displaystyle\int_0^1 e^{-px}\,dx$

In Exercises 3–30, evaluate the integrals that converge.

3. $\displaystyle\int_0^{+\infty} e^{-x}\,dx$

4. $\displaystyle\int_{-1}^{+\infty} \frac{x}{1+x^2}\,dx$

5. $\displaystyle\int_4^{+\infty} \frac{2}{x^2-1}\,dx$

6. $\displaystyle\int_0^{+\infty} xe^{-x^2}\,dx$

7. $\displaystyle\int_e^{+\infty} \frac{1}{x\ln^3 x}\,dx$

8. $\displaystyle\int_2^{+\infty} \frac{1}{x\sqrt{\ln x}}\,dx$

9. $\displaystyle\int_{-\infty}^0 \frac{dx}{(2x-1)^3}$

10. $\displaystyle\int_{-\infty}^2 \frac{dx}{x^2+4}$

11. $\displaystyle\int_{-\infty}^0 e^{3x}\,dx$

12. $\displaystyle\int_{-\infty}^0 \frac{e^x\,dx}{3-2e^x}$

13. $\displaystyle\int_{-\infty}^{+\infty} x^3\,dx$

14. $\displaystyle\int_{-\infty}^{+\infty} \frac{x}{\sqrt{x^2+2}}\,dx$

15. $\displaystyle\int_{-\infty}^{+\infty} \frac{x}{(x^2+3)^2}\,dx$

16. $\displaystyle\int_{-\infty}^{+\infty} \frac{e^{-t}}{1+e^{-2t}}\,dt$

17. $\displaystyle\int_3^4 \frac{dx}{(x-3)^2}$

18. $\displaystyle\int_0^8 \frac{dx}{\sqrt[3]{x}}$

19. $\displaystyle\int_0^{\pi/2} \tan x\,dx$

20. $\displaystyle\int_0^{9} \frac{dx}{\sqrt{9-x}}$

21. $\displaystyle\int_0^{1} \frac{dx}{\sqrt{1-x^2}}$

22. $\displaystyle\int_{-3}^{1} \frac{x\,dx}{\sqrt{9-x^2}}$

23. $\displaystyle\int_0^{\pi/6} \frac{\cos x}{\sqrt{1-2\sin x}}\,dx$

24. $\displaystyle\int_0^{\pi/4} \frac{\sec^2 x}{1-\tan x}\,dx$

25. $\displaystyle\int_0^{3} \frac{dx}{x-2}$

26. $\displaystyle\int_{-2}^{2} \frac{dx}{x^2}$

27. $\displaystyle\int_{-1}^{8} x^{-1/3}\,dx$

28. $\displaystyle\int_0^{4} \frac{dx}{(x-2)^{2/3}}$

29. $\displaystyle\int_0^{+\infty} \frac{1}{x^2}\,dx$

30. $\displaystyle\int_1^{+\infty} \frac{dx}{x\sqrt{x^2-1}}$

In Exercises 31–34, make the u-substitution and evaluate the resulting definite integral.

31. $\displaystyle\int_0^{+\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx;\ u=\sqrt{x}$ [*Note:* $u\to+\infty$ as $x\to+\infty$.]

32. $\displaystyle\int_0^{+\infty} \frac{dx}{\sqrt{x}(x+4)};\ u=\sqrt{x}$

33. $\displaystyle\int_0^{+\infty} \frac{e^{-x}}{\sqrt{1-e^{-x}}}\,dx;\ u=1-e^{-x}$

[*Note:* $u\to 1$ as $x\to+\infty$.]

34. $\displaystyle\int_0^{+\infty} \frac{e^{-x}}{\sqrt{1-e^{-2x}}}\,dx;\ u=e^{-x}$

In Exercises 35 and 36, express the improper integral as a limit, and then evaluate that limit with a CAS. Confirm the answer by evaluating the integral directly with the CAS.

35. $\displaystyle\int_0^{+\infty} e^{-x}\cos x\,dx$

36. $\displaystyle\int_0^{+\infty} xe^{-3x}\,dx$

37. In each part, try to evaluate the integral exactly with a CAS. If your result is not a simple numerical answer, then use the CAS to find a numerical approximation of the integral.

(a) $\displaystyle\int_{-\infty}^{+\infty} \frac{1}{x^8+x+1}\,dx$ (b) $\displaystyle\int_0^{+\infty} \frac{1}{\sqrt{1+x^3}}\,dx$

(c) $\displaystyle\int_1^{+\infty} \frac{\ln x}{e^x}\,dx$ (d) $\displaystyle\int_1^{+\infty} \frac{\sin x}{x^2}\,dx$

38. In each part, confirm the result with a CAS.

(a) $\displaystyle\int_0^{+\infty} \frac{\sin x}{\sqrt{x}}\,dx=\sqrt{\frac{\pi}{2}}$ (b) $\displaystyle\int_{-\infty}^{+\infty} e^{-x^2}\,dx=\sqrt{\pi}$

(c) $\displaystyle\int_0^{1} \frac{\ln x}{1+x}\,dx=-\frac{\pi^2}{12}$

39. Find the length of the curve $y=(4-x^{2/3})^{3/2}$ over the interval $[0,8]$.

40. Find the length of the curve $y=\sqrt{9-x^2}$ over the interval $[0,3]$.

In Exercises 41 and 42, use L'Hôpital's rule to help evaluate the improper integral.

41. $\displaystyle\int_0^{1} \ln x\,dx$

42. $\displaystyle\int_1^{+\infty} \frac{\ln x}{x^2}\,dx$

43. Find the area of the region between the x-axis and the curve $y=e^{-3x}$ for $x\ge 0$.

44. Find the area of the region between the x-axis and the curve $y=8/(x^2-4)$ for $x\ge 3$.

45. Suppose that the region between the x-axis and the curve $y=e^{-x}$ for $x\ge 0$ is revolved about the x-axis.
(a) Find the volume of the solid that is generated.
(b) Find the surface area of the solid.

46. Suppose that f and g are continuous functions and that

$$0\le f(x)\le g(x)$$

if $x\ge a$. Give a reasonable informal argument using areas to explain why the following results are true.
(a) If $\int_a^{+\infty} f(x)\,dx$ diverges, then $\int_a^{+\infty} g(x)\,dx$ diverges.
(b) If $\int_a^{+\infty} g(x)\,dx$ converges, then $\int_a^{+\infty} f(x)\,dx$ converges and $\int_a^{+\infty} f(x)\,dx\le\int_a^{+\infty} g(x)\,dx$.
[*Note:* The results in this exercise are sometimes called ***comparison tests*** for improper integrals.]

In Exercises 47–51, use the results in Exercise 46.

47. (a) Confirm graphically and algebraically that $e^{-x^2}\le e^{-x}$ if $x\ge 1$.
(b) Evaluate the integral

$$\int_1^{+\infty} e^{-x}\,dx$$

(c) What does the result obtained in part (b) tell you about the integral

$$\int_1^{+\infty} e^{-x^2}\,dx?$$

48. (a) Confirm graphically and algebraically that

$$\frac{1}{2x+1}\le\frac{e^x}{2x+1}\quad(x\ge 0)$$

(b) Evaluate the integral

$$\int_0^{+\infty} \frac{dx}{2x+1}$$

(c) What does the result obtained in part (b) tell you about the integral

$$\int_0^{+\infty} \frac{e^x}{2x+1}\,dx?$$

49. Let R be the region to the right of $x=1$ that is bounded by the x-axis and the curve $y=1/x$. When this region is revolved about the x-axis it generates a solid whose surface is known as ***Gabriel's Horn*** (for reasons that should be clear from the accompanying figure). Show that the solid has a finite volume but its surface has an infinite area. [*Note:* It has been suggested that if one could saturate the interior of the solid with paint and allow it to seep through to the surface, then one could paint an infinite surface with a finite amount of paint! What do you think?]

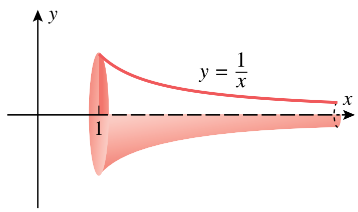

Figure Ex-49

50. In each part, use Exercise 46 to determine whether the integral converges or diverges. If it converges, then use part (b) of that exercise to find an upper bound on the value of the integral.

(a) $\displaystyle\int_2^{+\infty} \frac{\sqrt{x^3+1}}{x}\,dx$ (b) $\displaystyle\int_2^{+\infty} \frac{x}{x^5+1}\,dx$

(c) $\displaystyle\int_0^{+\infty} \frac{xe^x}{2x+1}\,dx$

51. Show that

$$\lim_{x\to+\infty} \frac{\int_0^{2x} \sqrt{1+t^3}\,dt}{x^{5/2}}$$

is an indeterminate form of type ∞/∞, and then use L'Hôpital's rule to find the limit.

52. (a) Give a reasonable informal argument, based on areas, that explains why the integrals

$$\int_0^{+\infty} \sin x\,dx \quad \text{and} \quad \int_0^{+\infty} \cos x\,dx$$

diverge.

(b) Show that $\displaystyle\int_0^{+\infty} \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$ diverges.

53. In electromagnetic theory, the magnetic potential at a point on the axis of a circular coil is given by

$$u = \frac{2\pi NIr}{k} \int_a^{+\infty} \frac{dx}{(r^2+x^2)^{3/2}}$$

where N, I, r, k, and a are constants. Find u.

C **54.** The *average speed*, $\bar{v}$, of the molecules of an ideal gas is given by

$$\bar{v} = \frac{4}{\sqrt{\pi}}\left(\frac{M}{2RT}\right)^{3/2} \int_0^{+\infty} v^3 e^{-Mv^2/(2RT)}\,dv$$

and the *root-mean-square speed*, v_{rms}, by

$$v_{\text{rms}}^2 = \frac{4}{\sqrt{\pi}}\left(\frac{M}{2RT}\right)^{3/2} \int_0^{+\infty} v^4 e^{-Mv^2/(2RT)}\,dv$$

where v is the molecular speed, T is the gas temperature, M is the molecular weight of the gas, and R is the gas constant.

(a) Use a CAS to show that

$$\int_0^{+\infty} x^3 e^{-a^2x^2}\,dx = \frac{1}{2a^4}, \quad a > 0$$

and use this result to show that $\bar{v} = \sqrt{8RT/\pi M}$.

(b) Use a CAS to show that

$$\int_0^{+\infty} x^4 e^{-a^2x^2}\,dx = \frac{3\sqrt{\pi}}{8a^5}, \quad a > 0$$

and use this result to show that $v_{\text{rms}} = \sqrt{3RT/M}$.

55. In Exercise 19 of Section 6.6, we determined the work required to lift a 6000-lb satellite to an orbital position that is 1000 mi above the Earth's surface. The ideas discussed in that exercise will be needed here.

(a) Find a definite integral that represents the work required to lift a 6000-lb satellite to a position ℓ miles above the Earth's surface.

(b) Find a definite integral that represents the work required to lift a 6000-lb satellite an "infinite distance" above the Earth's surface. Evaluate the integral. [*Note:* The result obtained here is sometimes called the work required to "escape" the Earth's gravity.]

A ***transform*** is a formula that converts or "transforms" one function into another. Transforms are used in applications to convert a difficult problem into an easier problem whose solution can then be used to solve the original difficult problem. The ***Laplace transform*** of a function $f(t)$, which plays an important role in the study of differential equations, is denoted by $\mathscr{L}\{f(t)\}$ and is defined by

$$\mathscr{L}\{f(t)\} = \int_0^{+\infty} e^{-st} f(t)\,dt$$

In this formula s is treated as a constant in the integration process; thus, the Laplace transform has the effect of transforming $f(t)$ into a function of s. Use this formula in Exercises 56 and 57.

56. Show that

(a) $\mathscr{L}\{1\} = \dfrac{1}{s},\ s > 0$ (b) $\mathscr{L}\{e^{2t}\} = \dfrac{1}{s-2},\ s > 2$

(c) $\mathscr{L}\{\sin t\} = \dfrac{1}{s^2+1},\ s > 0$

(d) $\mathscr{L}\{\cos t\} = \dfrac{s}{s^2+1},\ s > 0.$

57. In each part, find the Laplace transform.

(a) $f(t) = t,\ s > 0$ (b) $f(t) = t^2,\ s > 0$

(c) $f(t) = \begin{cases} 0, & t < 3 \\ 1, & t \geq 3 \end{cases},\quad s > 0$

C **58.** Later in the text, we will show that

$$\int_0^{+\infty} e^{-x^2}\,dx = \tfrac{1}{2}\sqrt{\pi}$$

Confirm that this is reasonable by using a CAS or a calculator with a numerical integration capability.

59. Use the result in Exercise 58 to show that

(a) $\displaystyle\int_{-\infty}^{+\infty} e^{-ax^2}\,dx = \sqrt{\frac{\pi}{a}},\ a > 0$

(b) $\displaystyle\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{+\infty} e^{-x^2/2\sigma^2}\,dx = 1,\ \sigma > 0.$

A convergent improper integral over an infinite interval can be approximated by first replacing the infinite limit(s) of integration by finite limit(s), then using a numerical integration technique, such as Simpson's rule, to approximate the integral with finite limit(s). This technique is illustrated in Exercises 60 and 61.

60. Suppose that the integral in Exercise 58 is approximated by first writing it as

$$\int_0^{+\infty} e^{-x^2}\,dx = \int_0^K e^{-x^2}\,dx + \int_K^{+\infty} e^{-x^2}\,dx$$

then dropping the second term, and then applying Simpson's rule to the integral

$$\int_0^K e^{-x^2}\,dx$$

The resulting approximation has two sources of error: the error from Simpson's rule and the error

$$E = \int_K^{+\infty} e^{-x^2}\,dx$$

that results from discarding the second term. We call E the ***truncation error***.

(a) Approximate the integral in Exercise 58 by applying Simpson's rule with $2n = 10$ subdivisions to the integral

$$\int_0^3 e^{-x^2}\,dx$$

Round your answer to four decimal places and compare it to $\frac{1}{2}\sqrt{\pi}$ rounded to four decimal places.

(b) Use the result that you obtained in Exercise 46 and the fact that $e^{-x^2} \le \frac{1}{3}xe^{-x^2}$ for $x \ge 3$ to show that the truncation error for the approximation in part (a) satisfies $0 < E < 2.1 \times 10^{-5}$.

61. (a) It can be shown that

$$\int_0^{+\infty} \frac{1}{x^6+1}\,dx = \frac{\pi}{3}$$

Approximate this integral by applying Simpson's rule with $2n = 20$ subdivisions to the integral

$$\int_0^4 \frac{1}{x^6+1}\,dx$$

Round your answer to three decimal places and compare it to $\pi/3$ rounded to three decimal places.

(b) Use the result that you obtained in Exercise 46 and the fact that $1/(x^6+1) < 1/x^6$ for $x \ge 4$ to show that the truncation error for the approximation in part (a) satisfies $0 < E < 2 \times 10^{-4}$.

62. For what values of p does $\int_0^{+\infty} e^{px}\,dx$ converge?

63. Show that $\int_0^1 \frac{dx}{x^p}$ converges if $p < 1$ and diverges if $p \ge 1$.

C **64.** It is sometimes possible to convert an improper integral into a "proper" integral having the same value by making an appropriate substitution. Evaluate the following integral by making the indicated substitution, and investigate what happens if you evaluate the integral directly using a CAS.

$$\int_0^1 \sqrt{\frac{1+x}{1-x}}\,dx; \quad u = \sqrt{1-x}$$

In Exercises 65 and 66, transform the given improper integral into a proper integral by making the stated u-substitution, then approximate the proper integral by Simpson's rule with $2n = 10$ subdivisions. Round your answer to three decimal places.

65. $\displaystyle\int_0^1 \frac{\cos x}{\sqrt{x}}\,dx; \quad u = \sqrt{x}$

66. $\displaystyle\int_0^1 \frac{\sin x}{\sqrt{1-x}}\,dx; \quad u = \sqrt{1-x}$

SUPPLEMENTARY EXERCISES

C CAS

1. Consider the following methods for evaluating integrals: u-substitution, integration by parts, partial fractions, reduction formulas, and trigonometric substitutions. In each part, state the approach that you would try first to evaluate the integral. If none of them seems appropriate, then say so. You need not evaluate the integral.

(a) $\int x\sin x\,dx$ (b) $\int \cos x\sin x\,dx$

(c) $\int \tan^7 x\,dx$ (d) $\int \tan^7 x\sec^2 x\,dx$

(e) $\int \frac{3x^2}{x^3+1}\,dx$ (f) $\int \frac{3x^2}{(x+1)^3}\,dx$

(g) $\int \tan^{-1} x\,dx$ (h) $\int \sqrt{4-x^2}\,dx$

(i) $\int x\sqrt{4-x^2}\,dx$

2. Consider the following trigonometric substitutions:

$$x = 3\sin\theta, \quad x = 3\tan\theta, \quad x = 3\sec\theta$$

In each part, state the substitution that you would try first to evaluate the integral. If none seems appropriate, then state a trigonometric substitution that you would use. You need not evaluate the integral.

(a) $\int \sqrt{9+x^2}\,dx$ (b) $\int \sqrt{9-x^2}\,dx$

(c) $\int \sqrt{1-9x^2}\,dx$ (d) $\int \sqrt{x^2-9}\,dx$

(e) $\int \sqrt{9+3x^2}\,dx$ (f) $\int \sqrt{1+(9x)^2}\,dx$

3. (a) What condition must a rational function satisfy for the method of partial fractions to be applicable directly?
(b) If the condition in part (a) is not satisfied, what must you do if you want to use partial fractions?

4. What is an improper integral?

5. In each part, find the number of the formula in the Endpaper Integral Table that you would apply to evaluate the integral. You need not evaluate the integral.

(a) $\int \sin 7x \cos 9x\,dx$ (b) $\int (x^7-x^5)e^{9x}\,dx$

(c) $\int x\sqrt{x-x^2}\,dx$ (d) $\int \frac{dx}{x\sqrt{4x+3}}$

(e) $\int x^9\pi^x\,dx$ (f) $\int \frac{3x-1}{2+x^2}\,dx$

6. Evaluate the integral $\int_0^1 \frac{x^3}{\sqrt{x^2+1}}\,dx$ using
(a) integration by parts
(b) the substitution $u=\sqrt{x^2+1}$.

7. In each part, evaluate the integral by making an appropriate substitution and applying a reduction formula.

(a) $\int \sin^4 2x\,dx$ (b) $\int x\cos^5(x^2)\,dx$

8. Consider the integral $\int \frac{1}{x^3-x}\,dx$.
(a) Evaluate the integral using the substitution $x=\sec\theta$. For what values of x is your result valid?
(b) Evaluate the integral using the substitution $x=\sin\theta$. For what values of x is your result valid?
(c) Evaluate the integral using the method of partial fractions. For what values of x is your result valid?

9. (a) Evaluate the integral

$$\int \frac{1}{\sqrt{2x-x^2}}\,dx$$

three ways: using the substitution $u=\sqrt{x}$, using the substitution $u=\sqrt{2-x}$, and completing the square.
(b) Show that the answers in part (a) are equivalent.

10. Find the area of the region that is enclosed by the curves $y=(x-3)/(x^3+x^2)$, $y=0$, $x=1$, and $x=2$.

11. Sketch the region whose area is $\int_0^{+\infty}\frac{dx}{1+x^2}$, and use your sketch to show that

$$\int_0^{+\infty}\frac{dx}{1+x^2}=\int_0^1\sqrt{\frac{1-y}{y}}\,dy$$

12. Find the area that is enclosed between the x-axis and the curve $y=(\ln x-1)/x^2$ for $x\ge e$.

13. Find the volume of the solid that is generated when the region between the x-axis and the curve $y=e^{-x}$ for $x\ge 0$ is revolved about the y-axis.

14. Find a positive value of a that satisfies the equation

$$\int_0^{+\infty}\frac{1}{x^2+a^2}\,dx=1$$

In Exercises 15–30, evaluate the integral.

15. $\int \sqrt{\cos\theta}\sin\theta\,d\theta$ **16.** $\int_0^{\pi/4}\tan^7\theta\,d\theta$

17. $\int x\tan^2(x^2)\sec^2(x^2)\,dx$ **18.** $\int_{-1/\sqrt{2}}^{1/\sqrt{2}}(1-2x^2)^{3/2}\,dx$

19. $\int \frac{dx}{(3+x^2)^{3/2}}$

20. $\int \frac{\cos\theta}{\sin^2\theta-6\sin\theta+12}\,d\theta$

21. $\int \frac{x+3}{\sqrt{x^2+2x+2}}\,dx$ **22.** $\int \frac{\sec^2\theta}{\tan^3\theta-\tan^2\theta}\,d\theta$

23. $\int \frac{dx}{(x-1)(x+2)(x-3)}$ **24.** $\int \frac{dx}{x(x^2+x+1)}$

25. $\int_4^8 \frac{\sqrt{x-4}}{x}\,dx$ **26.** $\int_0^9 \frac{\sqrt{x}}{x+9}\,dx$

27. $\int \frac{1}{\sqrt{e^x+1}}\,dx$ **28.** $\int_0^{\ln 2}\sqrt{e^x-1}\,dx$

29. $\int_a^{+\infty}\frac{x\,dx}{(x^2+1)^2}$

30. $\int_0^{+\infty}\frac{dx}{a^2+b^2x^2},\quad a,b>0$

Some integrals that can be evaluated by hand cannot be evaluated by all computer algebra systems. In Exercises 31–34, evaluate the integral by hand, and determine if it can be evaluated on your CAS.

C **31.** $\int \frac{x^3}{\sqrt{1-x^8}}\,dx$

C **32.** $\int (\cos^{32}x\sin^{30}x-\cos^{30}x\sin^{32}x)\,dx$

C **33.** $\int \sqrt{x-\sqrt{x^2-4}}\,dx$. [*Hint:* $\frac{1}{2}(\sqrt{x+2}-\sqrt{x-2})^2=?$]

C **34.** $\int \frac{1}{x^{10}+x}\,dx$. [*Hint:* Rewrite the denominator as $x^{10}(1+x^{-9})$.]

C **35.** Let

$$f(x)=\frac{-2x^5+26x^4+15x^3+6x^2+20x+43}{x^6-x^5-18x^4-2x^3-39x^2-x-20}$$

(a) Use a CAS to factor the denominator, and then write down the form of the partial fraction decomposition. You need not find the values of the constants.
(b) Check your answer in part (a) by using the CAS to find the partial fraction decomposition of f.
(c) Integrate f by hand, and then check your answer by integrating with the CAS.

36. The ***Gamma function***, $\Gamma(x)$, is defined as

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t}\, dt$$

It can be shown that this improper integral converges if and only if $x > 0$.

(a) Find $\Gamma(1)$.

(b) Prove: $\Gamma(x+1) = x\Gamma(x)$ for all $x > 0$. [*Hint:* Use integration by parts.]

(c) Use the results in parts (a) and (b) to find $\Gamma(2)$, $\Gamma(3)$, and $\Gamma(4)$; and then make a conjecture about $\Gamma(n)$ for positive integer values of n.

(d) Show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. [*Hint:* See Exercise 58 of Section 8.8.]

(e) Use the results obtained in parts (b) and (d) to show that $\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}$ and $\Gamma\left(\frac{5}{2}\right) = \frac{3}{4}\sqrt{\pi}$.

37. Refer to the Gamma function defined in Exercise 36 to show that

(a) $\displaystyle\int_0^1 (\ln x)^n\, dx = (-1)^n \Gamma(n+1), \quad n > 0.$

[*Hint:* Let $t = -\ln x$.]

(b) $\displaystyle\int_0^{+\infty} e^{-x^n}\, dx = \Gamma\left(\frac{n+1}{n}\right), \quad n > 0.$

[*Hint:* Let $t = x^n$. Use the result in Exercise 36(b).]

C **38.** A ***simple pendulum*** consists of a mass that swings in a vertical plane at the end of a massless rod of length L, as shown in the accompanying figure. Suppose that a simple pendulum is displaced through an angle θ_0 and released from rest. It can be shown that in the absence of friction, the time T required for the pendulum to make one complete back-and-forth swing, called the ***period***, is given by

$$T = \sqrt{\frac{8L}{g}} \int_0^{\theta_0} \frac{1}{\sqrt{\cos\theta - \cos\theta_0}}\, d\theta \qquad (1)$$

where $\theta = \theta(t)$ is the angle the pendulum makes with the vertical at time t. The improper integral in (1) is difficult to evaluate numerically. By a substitution outlined below it can be shown that the period can be expressed as

$$T = 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2 \sin^2\phi}}\, d\phi \qquad (2)$$

where $k = \sin(\theta_0/2)$. The integral in (2) is called a ***complete elliptic integral of the first kind*** and is more easily evaluated by numerical methods.

(a) Obtain (2) from (1) by substituting

$$\cos\theta = 1 - 2\sin^2(\theta/2)$$
$$\cos\theta_0 = 1 - 2\sin^2(\theta_0/2)$$
$$k = \sin(\theta_0/2)$$

and then making the change of variable

$$\sin\phi = \sin(\theta/2)/\sin(\theta_0/2) = \sin(\theta/2)/k$$

(b) Use (2) and the numerical integration capability of your CAS to estimate the period of a simple pendulum for which $L = 1.5$ ft, $\theta_0 = 20^\circ$, and $g = 32$ ft/s^2.

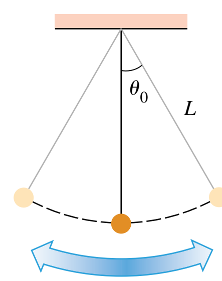

Figure Ex-38

EXPANDING THE CALCULUS HORIZON

Railroad Design

Your company has a contract to construct a track bed for a railroad line between towns A and B shown on the contour map in Figure 1. The bed can be created by cutting trenches through the surface or by using some combination of trenches and tunnels. As chief engineer, your assignment is to analyze the costs of trenches and tunnels and to propose a design strategy for minimizing the total construction cost.

Engineering Requirements

The Transportation Board submits the following engineering requirements to your company:

- The track bed is to be straight and 10 m wide. The grade is to increase at a constant rate from the existing elevation of 100 m at town A to an elevation of 110 m at point M and then decrease at a constant rate to the existing elevation of 88 m at town B.

- From town A to point M and from point N to town B the track bed is to be created by excavating a trench whose vertical cross sections are trapezoids with the dimensions shown in Figure 2.
- Between points M and N your company must decide whether to excavate a trench of the type in Figure 2 or to excavate a tunnel whose vertical cross sections have the dimensions shown in Figure 3.

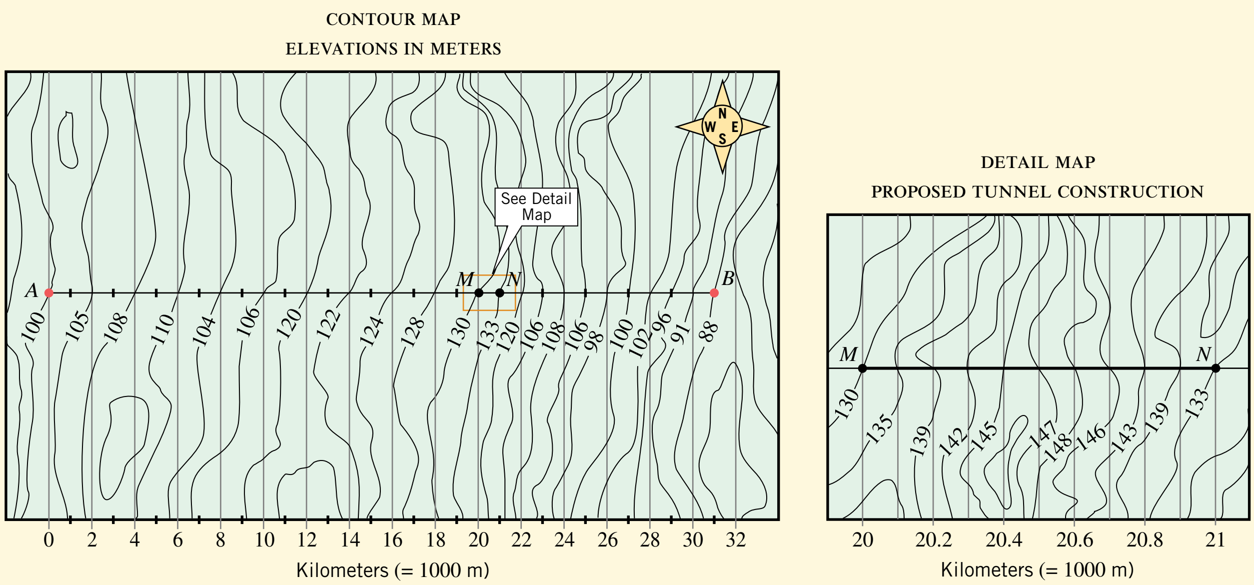

Figure 1

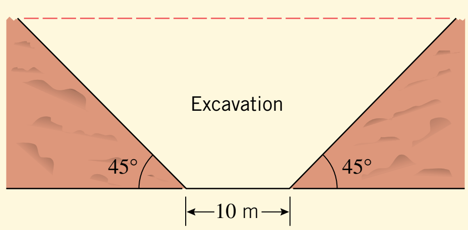

Figure 2

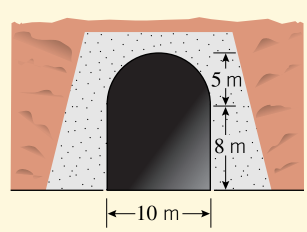

Figure 3

Cost Factors

Surface excavation of railbeds is performed using bulldozers, hydraulic excavators (backhoes), loading tractors, and other specialized equipment. Typically, the excavated dirt is piled at the side of the tracks to form sloped embankments, and the excavation cost is estimated from the volume of dirt to be removed and piled.

Tunnels in rock are often excavated by drilling shafts and inserting boring machines (called *moles*) to loosen and remove rock and dirt. Tunnels in soft ground are often excavated by starting at the tunnel face and using bucket or rotary excavators housed inside of shields. As the excavator progresses, tunnel liners are inserted behind it to support the earth and prevent cave-ins. Dirt removal is performed using conveyors or sometimes using railcars (called *muck cars*) that run on specially constructed tracks. Ventilation and air compression are other factors that add to the excavation cost of tunnels. In general, the excavation cost for a tunnel can be estimated from two components, the total volume of dirt to be removed and a cost that increases with the distance to the tunnel opening.

Make the following cost assumptions:

- The excavation and dirt-piling cost for a trench is \$4.00 per cubic meter.
- The drilling and dirt-piling cost for a tunnel is \$8.00 per cubic meter, and the costs involved in moving a load of dirt inside the tunnel a distance of 1 m toward the entrance along the track line is \$0.06 per cubic meter.

Cost Analysis of Trenches

Assume that variations in elevation are negligible for short distances at right angles to the track, so that the cross sections of the dirt to be excavated always have the trapezoidal shape shown in Figure 2 (straight horizontal edges at the surface).

Exercise 1 Complete Table 1, and then use the table and Simpson's rule with $2n = 10$ to approximate the cost of a trench from town A to point M.

Table 1

DISTANCE x FROM TOWN A (m)	TERRAIN ELEVATION (m)	TRACK ELEVATION (m)	DEPTH OF CUT (m)	CROSS-SECTIONAL AREA $f(x)$ OF CUT (m^2)
0	100	100	0	0
2,000	105	101	4	56
4,000				
6,000				
8,000				
10,000				
12,000				
14,000				
16,000				
18,000				
20,000				

Exercise 2 As in Exercise 1, use Simpson's rule with $2n = 10$ to approximate the cost of constructing a trench from (a) point M to point N, and (b) point N to town B.

Exercise 3 Find the total cost of the project if a trench is used along the entire line from town A to town B.

Cost Analysis of a Tunnel

Exercise 4

(a) Find the volume of dirt that must be removed from the tunnel, and calculate the drilling and dirt-piling cost.

(b) Find an integral for the cost of moving all of the dirt inside the tunnel to the tunnel entrance. [*Suggestion:* Use Riemann sums.]

(c) Find the total cost of excavating the tunnel.

Exercise 5 Find the total cost of the project using a trench from town A to point M, a tunnel from point M to point N, and a trench from point N to town B. Compare the cost to that obtained in Exercise 3 and state which method is cheaper.

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