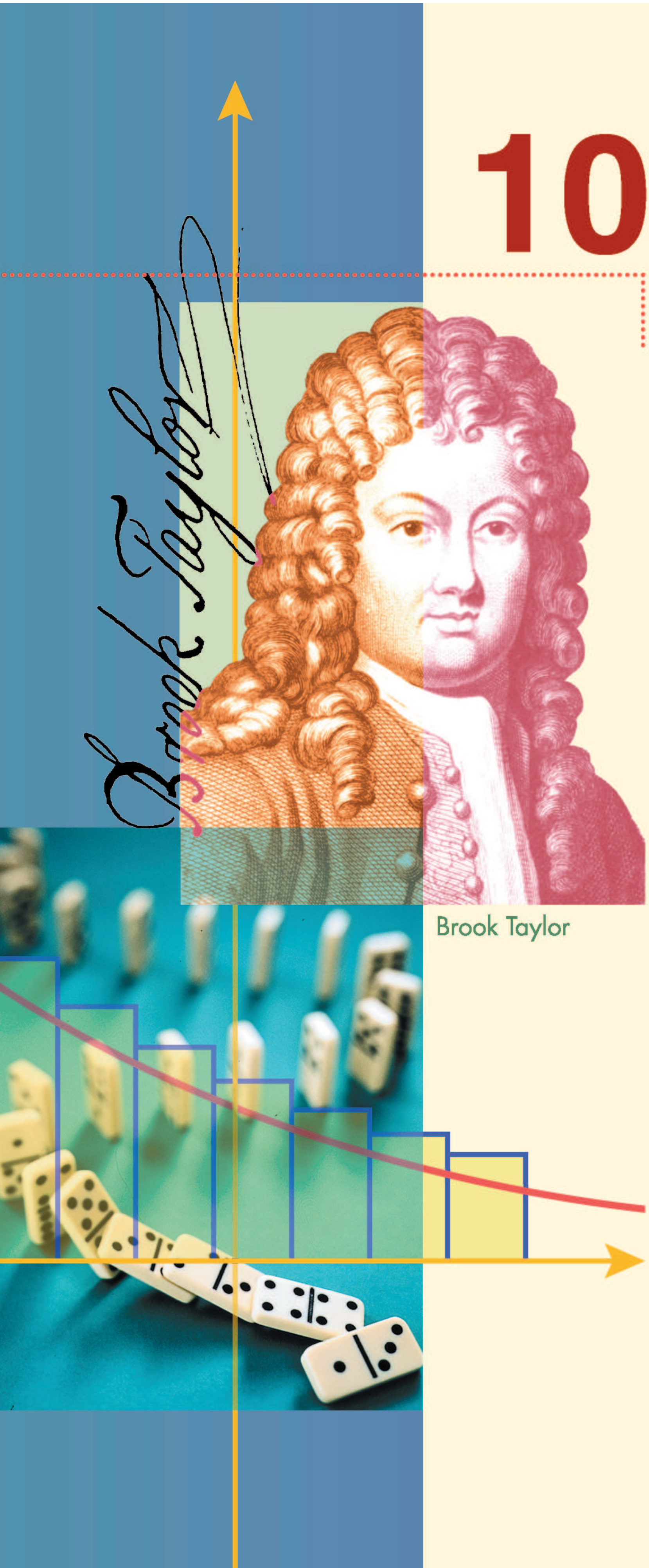


Brook Taylor

10

INFINITE SERIES

In this chapter we will be concerned with *infinite series*, which are sums that involve infinitely many terms. Infinite series play a fundamental role in both mathematics and science—they are used, for example, to approximate trigonometric functions and logarithms, to solve differential equations, to evaluate difficult integrals, to create new functions, and to construct mathematical models of physical laws. Since it is impossible to add up infinitely many numbers directly, one goal will be to define exactly what we mean by the sum of an infinite series. However, unlike finite sums, it turns out that not all infinite series actually have a sum, so we will need to develop tools for determining which infinite series have sums and which do not. Once the basic ideas have been developed we will begin to apply our work; we will show how infinite series are used to evaluate such quantities as $\sin 17^\circ$ and $\ln 5$, how they are used to create functions, and finally, how they are used to model physical laws.

10.1 MACLAURIN AND TAYLOR POLYNOMIAL APPROXIMATIONS

In Chapter 3 we used a tangent line to the graph of a function to obtain a linear approximation to the function near the point of tangency. In this section we will see how to improve such local approximations by using polynomials. We conclude the section by obtaining a bound on the error in these approximations. We have placed this section here for those who want an early discussion of Maclaurin and Taylor polynomials. If desired, this section can be delayed and used as a prelude to Section 10.8.

LOCAL QUADRATIC APPROXIMATIONS

Recall from Formula (1) in Section 3.8 that the local linear approximation of a function f at x_0 is

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) \tag{1}$$

In this formula, the approximating function

$$p(x) = f(x_0) + f'(x_0)(x - x_0)$$

is a first-degree polynomial satisfying $p(x_0) = f(x_0)$ and $p'(x_0) = f'(x_0)$ (verify). Thus, the local linear approximation of f at x_0 has the property that its value and the values of its first derivative match those of f at x_0.

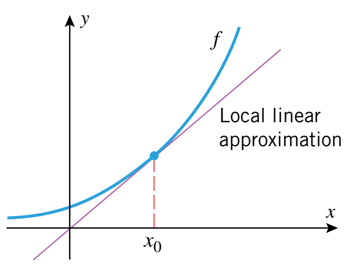


Figure 10.1.1

If the graph of a function f has a pronounced "bend" at x_0, then we can expect that the accuracy of the local linear approximation of f at x_0 will decrease rapidly as we progress away from x_0 (Figure 10.1.1). One way to deal with this problem is to approximate the function f at x_0 by a polynomial p of degree 2 with the property that the value of p and the values of its first two derivatives match those of f at x_0. This ensures that the graphs of f and p not only have the same tangent line at x_0, but they also bend in the same direction at x_0 (both concave up or concave down). As a result, we can expect that the graph of p will remain close to the graph of f over a larger interval around x_0 than the graph of the local linear approximation. The polynomial p is called the ***local quadratic approximation of f at $x = x_0$***.

To illustrate this idea, let us try to find a formula for the local quadratic approximation of a function f at $x = 0$. This approximation has the form

$$f(x) \approx c_0 + c_1 x + c_2 x^2 \tag{2}$$

where c_0, c_1, and c_2 must be chosen so that the values of

$$p(x) = c_0 + c_1 x + c_2 x^2$$

and its first two derivatives match those of f at 0. Thus, we want

$$p(0) = f(0), \quad p'(0) = f'(0), \quad p''(0) = f''(0) \tag{3}$$

But the values of $p(0)$, $p'(0)$, and $p''(0)$ are as follows:

$$\begin{aligned} p(x) &= c_0 + c_1 x + c_2 x^2 \qquad & p(0) &= c_0 \\ p'(x) &= c_1 + 2c_2 x & p'(0) &= c_1 \\ p''(x) &= 2c_2 & p''(0) &= 2c_2 \end{aligned}$$

Thus, it follows from (3) that

$$c_0 = f(0), \quad c_1 = f'(0), \quad c_2 = \frac{f''(0)}{2}$$

and substituting these in (2) yields the following formula for the local quadratic approximation of f at $x = 0$:

$$f(x) \approx f(0) + f'(0)x + \frac{f''(0)}{2}x^2 \tag{4}$$

REMARK. Observe that with $x_0 = 0$, Formula (1) becomes

$$f(x) \approx f(0) + f'(0)x \tag{5}$$

and hence the linear part of the local quadratic approximation of f at 0 is the local linear approximation of f at 0.

Example 1 Find the local linear and quadratic approximations of e^x at $x = 0$, and graph e^x and the two approximations together.

Solution. If we let $f(x) = e^x$, then $f'(x) = f''(x) = e^x$; and hence

$$f(0) = f'(0) = f''(0) = e^0 = 1$$

Thus, from (4) the local quadratic approximation of e^x at $x = 0$ is

$$e^x \approx 1 + x + \frac{x^2}{2}$$

and the local linear approximation (which is the linear part of the local quadratic approximation) is

$$e^x \approx 1 + x$$

The graphs of e^x and the two approximations are shown in Figure 10.1.2. As expected, the local quadratic approximation is more accurate than the local linear approximation near $x = 0$. ◀

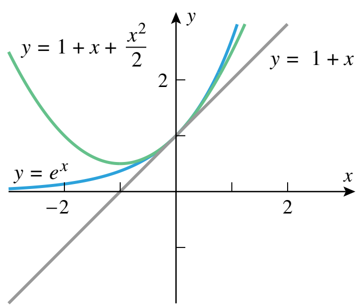


Figure 10.1.2

MACLAURIN POLYNOMIALS

It is natural to ask whether one can improve on the accuracy of a local quadratic approximation by using a polynomial of degree 3. Specifically, one might look for a polynomial of degree 3 with the property that its value and the values of its first three derivatives match those of f at a point; and if this provides an improvement in accuracy, why not go on to polynomials of even higher degree? Thus, we are led to consider the following general problem.

> **10.1.1** PROBLEM. Given a function f that can be differentiated n times at $x = x_0$, find a polynomial p of degree n with the property that the value of p and the values of its first n derivatives match those of f at x_0.

We will begin by solving this problem in the case where $x_0 = 0$. Thus, we want a polynomial

$$p(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n \tag{6}$$

such that

$$f(0) = p(0), \quad f'(0) = p'(0), \quad f''(0) = p''(0), \ldots, \quad f^{(n)}(0) = p^{(n)}(0) \tag{7}$$

But

$$\begin{aligned}
p(x) &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n \\
p'(x) &= c_1 + 2c_2 x + 3c_3 x^2 + \cdots + nc_n x^{n-1} \\
p''(x) &= 2c_2 + 3 \cdot 2c_3 x + \cdots + n(n-1)c_n x^{n-2} \\
p'''(x) &= 3 \cdot 2c_3 + \cdots + n(n-1)(n-2)c_n x^{n-3} \\
&\vdots \\
p^{(n)}(x) &= n(n-1)(n-2)\cdots(1)c_n
\end{aligned}$$

Thus, to satisfy (7) we must have*

$$\begin{aligned} f(0) &= p(0) &&= c_0 \\ f'(0) &= p'(0) &&= c_1 \\ f''(0) &= p''(0) &&= 2c_2 = 2!c_2 \\ f'''(0) &= p'''(0) &&= 3\cdot 2c_3 = 3!c_3 \\ &\vdots \\ f^{(n)}(0) &= p^{(n)}(0) &&= n(n-1)(n-2)\cdots(1)c_n = n!c_n \end{aligned}$$

which yields the following values for the coefficients of $p(x)$:

$$c_0 = f(0), \quad c_1 = f'(0), \quad c_2 = \frac{f''(0)}{2!}, \quad c_3 = \frac{f'''(0)}{3!}, \ldots, \quad c_n = \frac{f^{(n)}(0)}{n!}$$

The polynomial that results by using these coefficients in (6) is called the *nth Maclaurin*† *polynomial for* f.

10.1.2 DEFINITION. If f can be differentiated n times at 0, then we define the ***nth Maclaurin polynomial for f*** to be

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n \tag{8}$$

This polynomial has the property that its value and the values of its first n derivatives match the values of f and its first n derivatives at $x = 0$.

REMARK. Observe that $p_1(x)$ is the local linear approximation of f at 0 and $p_2(x)$ is the local quadratic approximation of f at $x = 0$.

Example 2 Find the Maclaurin polynomials p_0, p_1, p_2, p_3, and p_n for e^x.

Solution. Let $f(x) = e^x$. Thus,

$$f'(x) = f''(x) = f'''(x) = \cdots = f^{(n)}(x) = e^x$$

and

$$f(0) = f'(0) = f''(0) = f'''(0) = \cdots = f^{(n)}(0) = e^0 = 1$$

---

*Recall that if n is a positive integer, then the symbol $n!$ (read "n factorial") denotes the product of the first n positive integers; that is,

$$n! = 1\cdot 2\cdot 3\cdots n \quad \text{or equivalently,} \quad n! = n(n-1)(n-2)\cdots 1$$

Moreover, it is agreed by convention that $0! = 1$.

†COLIN MACLAURIN (1698–1746). Scottish mathematician. Maclaurin's father, a minister, died when the boy was only six months old, and his mother when he was nine years old. He was then raised by an uncle who was also a minister. Maclaurin entered Glasgow University as a divinity student, but transferred to mathematics after one year. He received his Master's degree at age 17 and, in spite of his youth, began teaching at Marischal College in Aberdeen, Scotland. He met Isaac Newton during a visit to London in 1719 and from that time on became Newton's disciple. During that era, some of Newton's analytic methods were bitterly attacked by major mathematicians and much of Maclaurin's important mathematical work resulted from his efforts to defend Newton's ideas geometrically. Maclaurin's work, *A Treatise of Fluxions* (1742), was the first systematic formulation of Newton's methods. The treatise was so carefully done that it was a standard of mathematical rigor in calculus until the work of Cauchy in 1821. Maclaurin was also an outstanding experimentalist; he devised numerous ingenious mechanical devices, made important astronomical observations, performed actuarial computations for insurance societies, and helped to improve maps of the islands around Scotland.

Therefore,

$$p_0(x) = f(0) = 1$$

$$p_1(x) = f(0) + f'(0)x = 1 + x$$

$$p_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 = 1 + x + \frac{x^2}{2!} = 1 + x + \frac{1}{2}x^2$$

$$p_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$$

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

$$= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$ ◀

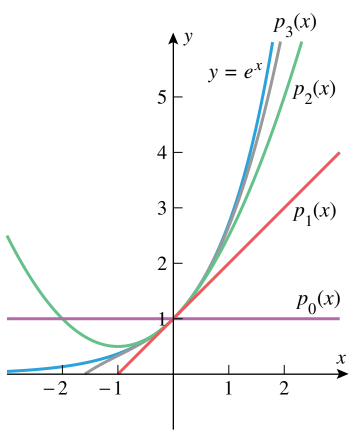


Figure 10.1.3

Figure 10.1.3 shows the graphs of e^x (in blue) and the graphs of the first four Maclaurin polynomials. Note that the graphs of $p_1(x)$, $p_2(x)$, and $p_3(x)$ are virtually indistinguishable from the graph of e^x near $x = 0$, so that these polynomials are good approximations of e^x for x near 0. However, the farther x is from 0, the poorer these approximations become. This is typical of the Maclaurin polynomials for a function $f(x)$; they provide good approximations of $f(x)$ near 0, but the accuracy diminishes as x progresses away from 0. However, it is usually the case that the higher the degree of the polynomial, the larger the interval on which it provides a specified accuracy. Accuracy issues will be investigated later.

TAYLOR POLYNOMIALS

Up to now we have focused on approximating a function f in the vicinity of $x = 0$. Now we will consider the more general case of approximating f in the vicinity of an arbitrary domain value x_0. The basic idea is the same as before; we want to find an nth-degree polynomial p with the property that its value and the values of its first n derivatives match those of f at x_0. However, rather than expressing $p(x)$ in powers of x, it will simplify the computations if we express it in powers of $x - x_0$; that is,

$$p(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + \cdots + c_n(x - x_0)^n \tag{9}$$

We will leave it as an exercise for you to imitate the computations used in the case where $x_0 = 0$ to show that

$$c_0 = f(x_0), \quad c_1 = f'(x_0), \quad c_2 = \frac{f''(x_0)}{2!}, \quad c_3 = \frac{f'''(x_0)}{3!}, \ldots, \quad c_n = \frac{f^{(n)}(x_0)}{n!}$$

Substituting these values in (9) we obtain a polynomial called the *nth Taylor* polynomial about $x = x_0$ for f.*

*BROOK TAYLOR (1685–1731). English mathematician. Taylor was born of well-to-do parents. Musicians and artists were entertained frequently in the Taylor home, which undoubtedly had a lasting influence on young Brook. In later years, Taylor published a definitive work on the mathematical theory of perspective and obtained major mathematical results about the vibrations of strings. There also exists an unpublished work, *On Musick*, that was intended to be part of a joint paper with Isaac Newton. Taylor's life was scarred with unhappiness, illness, and tragedy. Because his first wife was not rich enough to suit his father, the two men argued bitterly and parted ways. Subsequently, his wife died in childbirth. Then, after he remarried, his second wife also died in childbirth, though his daughter survived. Taylor's most productive period was from 1714 to 1719, during which time he wrote on a wide range of subjects—magnetism, capillary action, thermometers, perspective, and calculus. In his final years, Taylor devoted his writing efforts to religion and philosophy. According to Taylor, the results that bear his name were motivated by coffeehouse conversations about works of Newton on planetary motion and works of Halley ("Halley's comet") on roots of polynomials. Unfortunately, Taylor's writing style was so terse and hard to understand that he never received credit for many of his innovations.

10.1.3 DEFINITION. If f can be differentiated n times at x_0, then we define the ***nth Taylor polynomial for f about x = x₀*** to be

$$p_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \tag{10}$$

REMARK. Observe that the Maclaurin polynomials are special cases of the Taylor polynomials; that is, the nth-order Maclaurin polynomial is the nth-order Taylor polynomial about $x = 0$. Observe also that $p_1(x)$ is the local linear approximation of f at $x = x_0$ and $p_2(x)$ is the local quadratic approximation of f at $x = x_0$.

Example 3 Find the first four Taylor polynomials for $\ln x$ about $x = 2$.

Solution. Let $f(x) = \ln x$. Thus,

$$\begin{array}{ll} f(x) = \ln x & f(2) = \ln 2 \\ f'(x) = 1/x & f'(2) = 1/2 \\ f''(x) = -1/x^2 & f''(2) = -1/4 \\ f'''(x) = 2/x^3 & f'''(2) = 1/4 \end{array}$$

Substituting in (10) with $x_0 = 2$ yields

$$p_0(x) = f(2) = \ln 2$$

$$p_1(x) = f(2) + f'(2)(x - 2) = \ln 2 + \tfrac{1}{2}(x - 2)$$

$$p_2(x) = f(2) + f'(2)(x - 2) + \frac{f''(2)}{2!}(x - 2)^2 = \ln 2 + \tfrac{1}{2}(x - 2) - \tfrac{1}{8}(x - 2)^2$$

$$p_3(x) = f(2) + f'(2)(x - 2) + \frac{f''(2)}{2!}(x - 2)^2 + \frac{f'''(2)}{3!}(x - 2)^3$$

$$= \ln 2 + \tfrac{1}{2}(x - 2) - \tfrac{1}{8}(x - 2)^2 + \tfrac{1}{24}(x - 2)^3$$

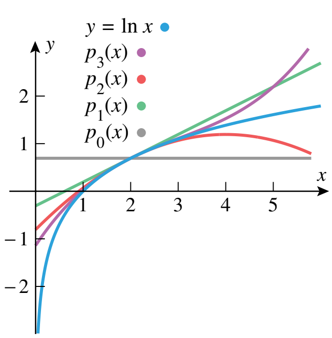


Figure 10.1.4

The graph of $\ln x$ (in blue) and its first four Taylor polynomials about $x = 2$ are shown in Figure 10.1.4. As expected, these polynomials produce their best approximations of $\ln x$ near 2. ◀

SIGMA NOTATION FOR TAYLOR AND MACLAURIN POLYNOMIALS

Frequently, we will want to express Formula (10) in sigma notation. To do this, we use the notation $f^{(k)}(x_0)$ to denote the kth derivative of f at $x = x_0$, and we make the convention that $f^{(0)}(x_0)$ denotes $f(x_0)$. This enables us to write

$$\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \tag{11}$$

In particular, we can write the nth-order Maclaurin polynomial for $f(x)$ as

$$\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n \tag{12}$$

Example 4 Find the nth Maclaurin polynomials for

(a) $\sin x$ (b) $\cos x$ (c) $\dfrac{1}{1 - x}$

Solution (a). In the Maclaurin polynomials for $\sin x$, only the odd powers of x appear explicitly. To see this, let $f(x) = \sin x$; thus,

$$\begin{aligned} f(x) &= \sin x & f(0) &= 0 \\ f'(x) &= \cos x & f'(0) &= 1 \\ f''(x) &= -\sin x & f''(0) &= 0 \\ f'''(x) &= -\cos x & f'''(0) &= -1 \end{aligned}$$

Since $f^{(4)}(x) = \sin x = f(x)$, the pattern 0, 1, 0, -1 will repeat as we evaluate successive derivatives at 0. Therefore, the successive Maclaurin polynomials for $\sin x$ are

$$\begin{aligned} p_0(x) &= 0 \\ p_1(x) &= 0 + x \\ p_2(x) &= 0 + x + 0 \\ p_3(x) &= 0 + x + 0 - \frac{x^3}{3!} \\ p_4(x) &= 0 + x + 0 - \frac{x^3}{3!} + 0 \\ p_5(x) &= 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} \\ p_6(x) &= 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + 0 \\ p_7(x) &= 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + 0 - \frac{x^7}{7!} \end{aligned}$$

Because of the zero terms, each even-order Maclaurin polynomial [after $p_0(x)$] is the same as the preceding odd-order Maclaurin polynomial. That is,

$$p_{2k+1}(x) = p_{2k+2}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^k \frac{x^{2k+1}}{(2k+1)!} \quad (k = 0, 1, 2, \ldots)$$

The graphs of $\sin x$, $p_1(x)$, $p_3(x)$, $p_5(x)$, and $p_7(x)$ are shown in Figure 10.1.5.

Solution (b). In the Maclaurin polynomials for $\cos x$, only the even powers of x appear explicitly; the computations are similar to those in part (a). The reader should be able to show that

$$\begin{aligned} p_0(x) &= p_1(x) = 1 \\ p_2(x) &= p_3(x) = 1 - \frac{x^2}{2!} \\ p_4(x) &= p_5(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \\ p_6(x) &= p_7(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \end{aligned}$$

In general, the Maclaurin polynomials for $\cos x$ are given by

$$p_{2k}(x) = p_{2k+1}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^k \frac{x^{2k}}{(2k)!} \quad (k = 0, 1, 2, \ldots)$$

The graphs of $\cos x$, $p_0(x)$, $p_2(x)$, $p_4(x)$, and $p_6(x)$ are shown in Figure 10.1.6.

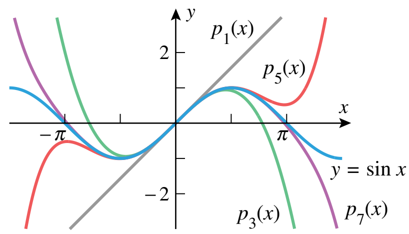


Figure 10.1.5

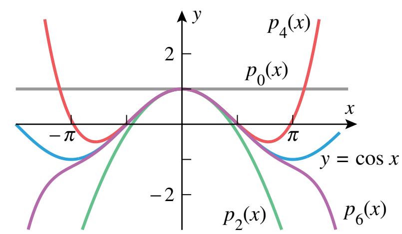


Figure 10.1.6

Solution (c). Let $f(x) = 1/(1-x)$. The values of f and its first k derivatives at $x = 0$ are as follows:

$$
\begin{aligned}
f(x) &= \frac{1}{1-x} & f(0) &= 1 = 0! \\
f'(x) &= \frac{1}{(1-x)^2} & f'(0) &= 1 = 1! \\
f''(x) &= \frac{2}{(1-x)^3} & f''(0) &= 2 = 2! \\
f'''(x) &= \frac{3 \cdot 2}{(1-x)^4} & f'''(0) &= 3! \\
f^{(4)}(x) &= \frac{4 \cdot 3 \cdot 2}{(1-x)^5} & f^{(4)}(0) &= 4! \\
&\vdots & &\vdots \\
f^{(k)}(x) &= \frac{k!}{(1-x)^{k+1}} & f^{(k)}(0) &= k!
\end{aligned}
$$

Thus, substituting $f^{(k)}(0) = k!$ into Formula (12) yields the nth Maclaurin polynomial for $1/(1-x)$:

$$p_n(x) = \sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n \quad (n = 0, 1, 2, \ldots)$$ ◀

Example 5 Find the nth Taylor polynomial for $1/x$ about $x = 1$.

Solution. Let $f(x) = 1/x$. The computations are similar to those in part (c) of Example 4. We leave it for you to show that

$$
\begin{aligned}
&f(1) = 1, \quad f'(1) = -1, \quad f''(1) = 2!, \quad f'''(1) = -3!, \\
&f^{(4)}(1) = 4!, \ldots, \quad f^{(k)}(1) = (-1)^k k!
\end{aligned}
$$

Thus, substituting $f^{(k)}(1) = (-1)^k k!$ into Formula (11) with $x_0 = 1$ yields the nth Taylor polynomial for $1/x$:

$$\sum_{k=0}^{n} (-1)^k (x-1)^k = 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots + (-1)^n (x-1)^n$$ ◀

FOR THE READER. CAS programs have commands for generating Taylor polynomials of any specified degree. If you have a CAS, read the documentation to determine how this is done, and then use the CAS to confirm the computations in the examples in this section.

THE *n*TH REMAINDER

The nth Taylor polynomial p_n for a function f about $x = x_0$ has been introduced as a tool to obtain good approximations to values of $f(x)$ for x near x_0. We now develop a method to forecast how good these approximations will be.

It is convenient to develop a notation for the error in using $p_n(x)$ to approximate $f(x)$, so we define $R_n(x)$ to be the difference between $f(x)$ and its nth Taylor polynomial. That is,

$$R_n(x) = f(x) - p_n(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \tag{13}$$

This can also be written as

$$f(x) = p_n(x) + R_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + R_n(x) \tag{14}$$

which is called ***Taylor's formula with remainder***.

Finding a bound for $R_n(x)$ gives an indication of the accuracy of the approximation $p_n(x) \approx f(x)$. The following theorem, which is proved in Appendix G, provides such a bound.

10.1.4 THEOREM (***The Remainder Estimation Theorem***). *If the function f can be differentiated $n+1$ times on an interval I containing the number x_0, and if M is an upper bound for $|f^{(n+1)}(x)|$ on I, that is, $|f^{(n+1)}(x)| \le M$ for all x in I, then*

$$|R_n(x)| \le \frac{M}{(n+1)!}|x - x_0|^{n+1} \tag{15}$$

for all x in I.

Example 6 Use an nth Maclaurin polynomial for e^x to approximate e to five decimal-place accuracy.

Solution. We note first that the exponential function e^x has derivatives of all orders for every real number x. From Example 2, the nth Maclaurin polynomial for e^x is

$$\sum_{k=0}^{n} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

from which we have

$$e = e^1 \approx \sum_{k=0}^{n} \frac{1^k}{k!} = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!}$$

Thus, our problem is to determine how many terms to include in a Maclaurin polynomial for e^x to achieve five decimal-place accuracy; that is, we want to choose n so that the absolute value of the nth remainder at $x = 1$ in the Maclaurin series satisfies

$$|R_n(x)| \le 0.000005$$

To determine n we apply the Remainder Estimation Theorem with $f(x) = e^x$, $x = 1$, $x_0 = 0$, and I being the interval $[0, 1]$. In this case it follows from Formula (15) that

$$|R_n(1)| \le \frac{M}{(n+1)!} \tag{16}$$

where M is an upper bound on the value of $f^{(n+1)}(x) = e^x$ for x in the interval $[0, 1]$. However, e^x is an increasing function, so its maximum value on the interval $[0, 1]$ occurs at $x = 1$; that is, $e^x \le e$ on this interval. Thus, we can take $M = e$ in (16) to obtain

$$|R_n(1)| \le \frac{e}{(n+1)!} \tag{17}$$

Unfortunately, this inequality is not very useful because it involves e, which is the very quantity we are trying to approximate. However, if we accept that $e < 3$, then we can replace (17) with the following less precise, but more easily applied, inequality:

$$|R_n(1)| \le \frac{3}{(n+1)!}$$

Thus, we can achieve five decimal-place accuracy by choosing n so that

$$\frac{3}{(n+1)!} \le 0.000005 \quad \text{or} \quad (n+1)! \ge 600{,}000$$

Since $9! = 362{,}880$ and $10! = 3{,}628{,}800$, the smallest value of n that meets this criterion is $n = 9$. Thus, to five decimal-place accuracy

$$e \approx 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!} + \frac{1}{9!} \approx 2.71828$$

As a check, a calculator's twelve-digit representation of e is $e \approx 2.71828182846$, which agrees with the preceding approximation when rounded to five decimal places. ◀

EXERCISE SET 10.1 Graphing Utility C CAS

1. In each part, find the local quadratic approximation of f at $x = x_0$, and use that approximation to find the local linear approximation of f at x_0.
   (a) $f(x) = e^{-x}$; $x_0 = 0$
   (b) $f(x) = \cos x$; $x_0 = 0$
   (c) $f(x) = \sin x$; $x_0 = \pi/2$
   (d) $f(x) = \sqrt{x}$; $x_0 = 1$

C 2. In each part, use a CAS to find the local quadratic approximation of f at $x = x_0$, and use that approximation to find the local linear approximation of f at $x = x_0$.
   (a) $f(x) = e^{\sin x}$; $x_0 = 0$
   (b) $f(x) = \sqrt{x}$; $x_0 = 9$
   (c) $f(x) = \sec^{-1} x$; $x_0 = 2$
   (d) $f(x) = \sin^{-1} x$; $x_0 = 0$

3. (a) Find the local quadratic approximation of $\sqrt{x}$ at $x_0 = 1$.
   (b) Use the result obtained in part (a) to approximate $\sqrt{1.1}$, and compare your approximation to that produced directly by your calculating utility. [See Example 1 of Section 3.8.]

4. (a) Find the local quadratic approximation of $\cos x$ at $x_0 = 0$.
   (b) Use the result obtained in part (a) to approximate $\cos 2^\circ$, and compare the approximation to that produced directly by your calculating utility.

5. Use an appropriate local quadratic approximation to approximate $\tan 61^\circ$, and compare the result to that produced directly by your calculating utility.

6. Use an appropriate local quadratic approximation to approximate $\sqrt{36.03}$, and compare the result to that produced directly by your calculating utility.

In Exercises 7–16, find the Maclaurin polynomials of orders $n = 0$, 1, 2, 3, and 4, and then find the nth Maclaurin polynomials for the function in sigma notation.

7. e^{-x}
8. e^{ax}
9. $\cos \pi x$
10. $\sin \pi x$
11. $\ln(1+x)$
12. $\dfrac{1}{1+x}$
13. $\cosh x$
14. $\sinh x$
15. $x \sin x$
16. xe^x

In Exercises 17–24, find the Taylor polynomials of orders $n = 0$, 1, 2, 3, and 4 about $x = x_0$, and then find the nth Taylor polynomials for the function in sigma notation.

17. e^x; $x_0 = 1$
18. e^{-x}; $x_0 = \ln 2$
19. $\dfrac{1}{x}$; $x_0 = -1$
20. $\dfrac{1}{x+2}$; $x_0 = 3$
21. $\sin \pi x$; $x_0 = \dfrac{1}{2}$
22. $\cos x$; $x_0 = \dfrac{\pi}{2}$
23. $\ln x$; $x_0 = 1$
24. $\ln x$; $x_0 = e$

25. (a) Find the third Maclaurin polynomial for
$$f(x) = 1 + 2x - x^2 + x^3$$
   (b) Find the third Taylor polynomial about $x = 1$ for
$$f(x) = 1 + 2(x-1) - (x-1)^2 + (x-1)^3$$

26. (a) Find the nth Maclaurin polynomial for
$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$$
   (b) Find the nth Taylor polynomial about $x = 1$ for
$$f(x) = c_0 + c_1(x-1) + c_2(x-1)^2 + \cdots + c_n(x-1)^n$$

In Exercises 27–30, find the first four distinct Taylor polynomials about $x = x_0$, and use a graphing utility to graph the given function and the Taylor polynomials on the same screen.

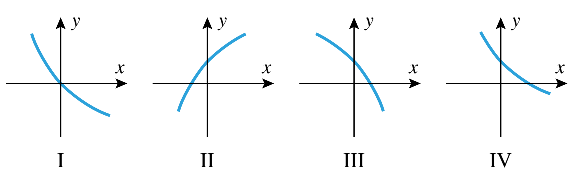
27. $f(x) = e^{-2x};\ x_0 = 0$ **28.** $f(x) = \sin x;\ x_0 = \pi/2$

29. $f(x) = \cos x;\ x_0 = \pi$ **30.** $\ln(x+1);\ x_0 = 0$

31. Use the method of Example 6 to approximate $\sqrt{e}$ to four decimal-place accuracy, and check your work by comparing your answer to that produced directly by your calculating utility. [*Suggestion:* Write $\sqrt{e}$ as $e^{0.5}$.]

32. Use the method of Example 6 to approximate $1/e$ to three decimal-place accuracy, and check your work by comparing your answer to that produced directly by your calculating utility.

33. Which of the functions graphed in the following figure is most likely to have $p(x) = 1 - x + 2x^2$ as its second-order Maclaurin polynomial? Explain your reasoning.



34. Suppose that the values of a function f and its first three derivatives at $x = 1$ are

$$f(1) = 2, \quad f'(1) = -3, \quad f''(1) = 0, \quad f'''(1) = 6$$

Find as many Taylor polynomials for f as you can about $x = 1$.

35. Show that the nth Taylor polynomial for $\sinh x$ about $x = \ln 4$ is

$$\sum_{k=0}^{n} \frac{16 - (-1)^k}{8k!}(x - \ln 4)^k$$

36. (a) The accompanying figure shows a sector of radius r and central angle 2α. Assuming that the angle α is small, use the local quadratic approximation of $\cos\alpha$ at $\alpha = 0$ to show that $x \approx r\alpha^2/2$.

(b) Assuming that the Earth is a sphere of radius 4000 mi, use the result in part (a) to approximate the maximum amount by which a 100-mi arc along the equator will diverge from its chord.

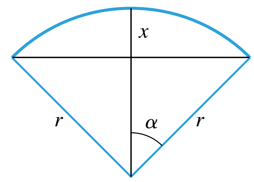


Figure Ex-36

37. Let $p_1(x)$ and $p_2(x)$ be the local linear and local quadratic approximations of $f(x) = e^{\sin x}$ at $x = 0$.

(a) Use a graphing utility to generate the graphs of $f(x)$, $p_1(x)$, and $p_2(x)$ on the same screen for $-1 \le x \le 1$.

(b) Construct a table of values of $f(x)$, $p_1(x)$, and $p_2(x)$ for $x = -1.00, -0.75, -0.50, -0.25, 0, 0.25, 0.50, 0.75, 1.00$. Round the values to three decimal places.

(c) Generate the graph of $|f(x) - p_1(x)|$, and use the graph to determine an interval on which $p_1(x)$ approximates $f(x)$ with an error of at most ± 0.01. [*Suggestion:* Review the discussion relating to Figure 3.8.5.]

(d) Generate the graph of $|f(x) - p_2(x)|$, and use the graph to determine an interval on which $p_2(x)$ approximates $f(x)$ with an error of at most ± 0.01.

38. (a) Find an interval $[0, b]$ over which e^x can be approximated by $1 + x + (x^2/2!)$ to three decimal-place accuracy throughout the interval.

(b) Check your answer in part (a) by graphing

$$\left| e^x - \left(1 + x + \frac{x^2}{2!}\right) \right|$$

over the interval you obtained.

39. (a) Use the Remainder Estimation Theorem to find an interval containing $x = 0$ over which $\sin x$ can be approximated by $x - (x^3/3!)$ to three decimal-place accuracy throughout the interval.

(b) Check your answer in part (a) by graphing

$$\left| \sin x - \left(x - \frac{x^3}{3!}\right) \right|$$

over the interval you obtained.

10.2 SEQUENCES

In everyday language, the term "sequence" means a succession of things in a definite order—chronological order, size order, or logical order, for example. In mathematics, the term "sequence" is commonly used to denote a succession of numbers whose order is determined by a rule or a function. In this section, we will develop some of the basic ideas concerning sequences of numbers.

DEFINITION OF A SEQUENCE

Stated informally, an ***infinite sequence***, or more simply a ***sequence***, is an unending succession of numbers, called ***terms***. It is understood that the terms have a definite order; that is, there is a first term a_1, a second term a_2, a third term a_3, a fourth term a_4, and so forth. Such a sequence would typically be written as

$$a_1, a_2, a_3, a_4, \ldots$$

where the dots are used to indicate that the sequence continues indefinitely. Some specific examples are

$$1, 2, 3, 4, \ldots, \qquad 1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots,$$
$$2, 4, 6, 8, \ldots, \qquad 1, -1, 1, -1, \ldots$$

Each of these sequences has a definite pattern that makes it easy to generate additional terms if we assume that those terms follow the same pattern as the displayed terms. However, such patterns can be deceiving, so it is better to have a rule or formula for generating the terms. One way of doing this is to look for a function that relates each term in the sequence to its term number. For example, in the sequence

$$2, 4, 6, 8, \ldots$$

each term is twice the term number; that is, the nth term in the sequence is given by the formula $2n$. We denote this by writing the sequence as

$$2, 4, 6, 8, \ldots, 2n, \ldots$$

We call the function $f(n) = 2n$ the *general term* of this sequence. Now, if we want to know a specific term in the sequence, we need only substitute its term number in the formula for the general term. For example, the 37th term in the sequence is $2 \cdot 37 = 74$.

Example 1 In each part, find the general term of the sequence.

(a) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$ (b) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

(c) $\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \ldots$ (d) $1, 3, 5, 7, \ldots$

Solution (a). In Table 10.2.1, the four known terms have been placed below their term numbers, from which we see that the numerator is the same as the term number and the denominator is one greater than the term number. This suggests that the nth term has numerator n and denominator $n + 1$, as indicated in the table. Thus, the sequence can be expressed as

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots$$

Solution (b). In Table 10.2.2, the denominators of the four known terms have been expressed as powers of 2 and the first four terms have been placed below their term numbers, from which we see that the exponent in the denominator is the same as the term number. This suggests that the denominator of the nth term is 2^n, as indicated in the table. Thus, the sequence can be expressed as

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots, \frac{1}{2^n}, \ldots$$

Table 10.2.1

TERM NUMBER	1	2	3	4	⋯	n	⋯
TERM	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	⋯	$\frac{n}{n+1}$	⋯

Table 10.2.2

TERM NUMBER	1	2	3	4	⋯	n	⋯
TERM	$\frac{1}{2}$	$\frac{1}{2^2}$	$\frac{1}{2^3}$	$\frac{1}{2^4}$	⋯	$\frac{1}{2^n}$	⋯

Solution (c). This sequence is identical to that in part (a), except for the alternating signs. Thus, the nth term in the sequence can be obtained by multiplying the nth term in part (a) by $(-1)^{n+1}$. This factor produces the correct alternating signs, since its successive values, starting with $n = 1$, are $1, -1, 1, -1, \ldots$. Thus, the sequence can be written as

$$\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \ldots, (-1)^{n+1}\frac{n}{n+1}, \ldots$$

Solution (d). In Table 10.2.3, the four known terms have been placed below their term numbers, from which we see that each term is one less than twice its term number. This suggests that the nth term in the sequence is $2n - 1$, as indicated in the table. Thus, the sequence can be expressed as

$$1, 3, 5, 7, \ldots, 2n - 1, \ldots$$ ◀

Table 10.2.3

TERM NUMBER	1	2	3	4	⋯	n	⋯
TERM	1	3	5	7	⋯	$2n-1$	⋯

FOR THE READER. Consider the sequence whose general term is

$$f(n) = \tfrac{1}{3}(3 - 5n + 6n^2 - n^3)$$

Calculate the first three terms, and make a conjecture about the fourth term. Check your conjecture by calculating the fourth term. What message does this convey?

When the general term of a sequence

$$a_1, a_2, a_3, \ldots, a_n, \ldots \tag{1}$$

is known, there is no need to write out the initial terms, and it is common to write only the general term enclosed in braces. Thus, (1) might be written as

$$\{a_n\}_{n=1}^{+\infty}$$

For example, here are the four sequences in Example 1 expressed in brace notation.

SEQUENCE	BRACE NOTATION
$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots$	$\left\{\frac{n}{n+1}\right\}_{n=1}^{+\infty}$
$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots, \frac{1}{2^n}, \ldots$	$\left\{\frac{1}{2^n}\right\}_{n=1}^{+\infty}$
$\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \ldots, (-1)^{n+1}\frac{n}{n+1}, \ldots$	$\left\{(-1)^{n+1}\frac{n}{n+1}\right\}_{n=1}^{+\infty}$
$1, 3, 5, 7, \ldots, 2n-1, \ldots$	$\{2n-1\}_{n=1}^{+\infty}$

The letter n in (1) is called the ***index*** for the sequence. It is not essential to use n for the index; any letter not reserved for another purpose can be used. For example, we might view the general term of the sequence $a_1, a_2, a_3, \ldots$ to be the kth term, in which case we would denote this sequence as $\{a_k\}_{k=1}^{+\infty}$. Moreover, it is not essential to start the index at 1; sometimes it is more convenient to start it at 0 (or some other integer). For example, consider the sequence

$$1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots$$

One way to write this sequence is

$$\left\{\frac{1}{2^{n-1}}\right\}_{n=1}^{+\infty}$$

However, the general term will be simpler if we think of the initial term in the sequence as the zeroth term, in which case we can write the sequence as

$$\left\{\frac{1}{2^{n}}\right\}_{n=0}^{+\infty}$$

REMARK. In general discussions that involve sequences in which the specific terms and the starting point for the index are not important, it is common to write $\{a_n\}$ rather than $\{a_n\}_{n=1}^{+\infty}$ or $\{a_n\}_{n=0}^{+\infty}$. Moreover, we can distinguish between different sequences by using different letters for their general terms; thus, $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ denote three different sequences.

We began this section by describing a sequence as an unending succession of numbers. Although this conveys the general idea, it is not a satisfactory mathematical definition because it relies on the term "succession," which is itself an undefined term. To motivate a precise definition, consider the sequence

$$2, 4, 6, 8, \ldots, 2n, \ldots$$

If we denote the general term by $f(n) = 2n$, then we can write this sequence as

$$f(1), f(2), f(3), \ldots, f(n), \ldots$$

which is a "list" of values of the function

$$f(n) = 2n, \quad n = 1, 2, 3, \ldots$$

whose domain is the set of positive integers. This suggests the following definition.

10.2.1 DEFINITION. A ***sequence*** is a function whose domain is a set of integers. Specifically, we will regard the expression $\{a_n\}_{n=1}^{+\infty}$ to be an alternative notation for the function $f(n) = a_n$, $n = 1, 2, 3, \ldots.$

GRAPHS OF SEQUENCES

Since sequences are functions, it makes sense to talk about the graph of a sequence. For example, the graph of the sequence $\{1/n\}_{n=1}^{+\infty}$ is the graph of the equation

$$y = \frac{1}{n}, \quad n = 1, 2, 3, \ldots$$

Because the right side of this equation is defined only for positive integer values of n, the graph consists of a succession of isolated points (Figure 10.2.1*a*). This is in distinction to the graph of

$$y = \frac{1}{x}, \quad x \geq 1$$

which is a continuous curve (Figure 10.2.1*b*).

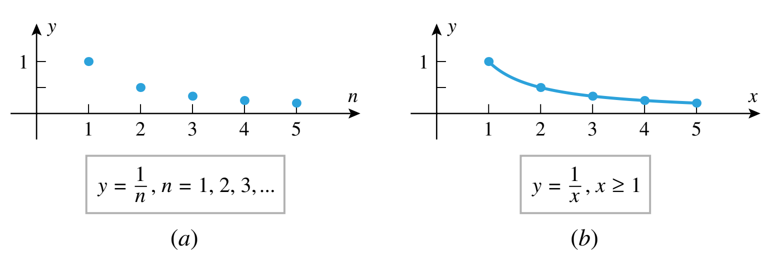


Figure 10.2.1

LIMIT OF A SEQUENCE

Since sequences are functions, we can inquire about their limits. However, because a sequence $\{a_n\}$ is only defined for integer values of n, the only limit that makes sense is the limit of a_n as $n \to +\infty$. In Figure 10.2.2 we have shown the graphs of four sequences, each of which behaves differently as $n \to +\infty$:

- The terms in the sequence $\{n+1\}$ increase without bound.
- The terms in the sequence $\{(-1)^{n+1}\}$ oscillate between -1 and 1.
- The terms in the sequence $\{n/(n+1)\}$ increase toward a "limiting value" of 1.
- The terms in the sequence $\left\{1 + \left(-\frac{1}{2}\right)^n\right\}$ also tend toward a "limiting value" of 1, but do so in an oscillatory fashion.

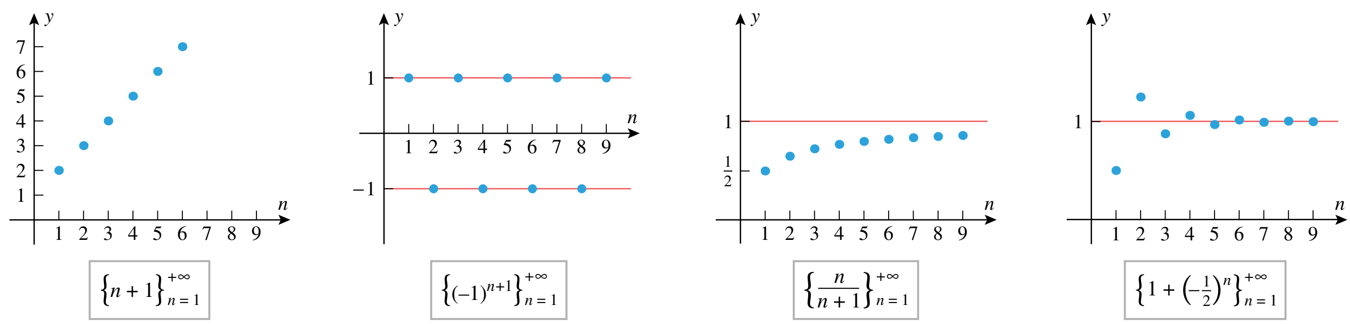


Figure 10.2.2

Informally speaking, the limit of a sequence $\{a_n\}$ is intended to describe how a_n behaves as $n \to +\infty$. To be more specific, we will say that *a sequence* $\{a_n\}$ *approaches a limit* L *if the terms in the sequence eventually become arbitrarily close to* L. Geometrically, this means that for any positive number ϵ there is a point in the sequence after which all terms lie between the lines $y = L - \epsilon$ and $y = L + \epsilon$ (Figure 10.2.3).

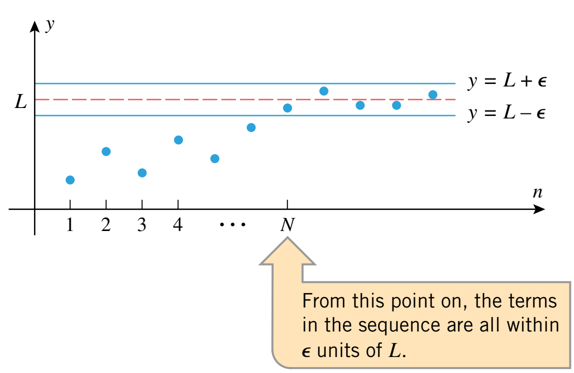


Figure 10.2.3

The following definition makes these ideas precise.

10.2.2 DEFINITION. A sequence $\{a_n\}$ is said to ***converge*** to the ***limit*** L if given any $\epsilon > 0$, there is a positive integer N such that $|a_n - L| < \epsilon$ for $n \geq N$. In this case we write

$$\lim_{n \to +\infty} a_n = L$$

A sequence that does not converge to some finite limit is said to ***diverge***.

Example 2 The first two sequences in Figure 10.2.2 diverge, and the second two converge to 1; that is,

$$\lim_{n\to+\infty} \frac{n}{n+1} = 1 \quad \text{and} \quad \lim_{n\to+\infty} \left(1 + \left(-\tfrac{1}{2}\right)^n\right) = 1$$ ◀

FOR THE READER. How would you define

$$\lim_{n\to+\infty} a_n = +\infty \quad \text{and} \quad \lim_{n\to+\infty} a_n = -\infty?$$

The following theorem, which we state without proof, shows that the familiar properties of limits apply to sequences. This theorem ensures that the algebraic techniques used to find limits of the form $\lim_{x\to+\infty}$ can also be used for limits of the form $\lim_{n\to+\infty}$.

10.2.3 THEOREM. *Suppose that the sequences $\{a_n\}$ and $\{b_n\}$ converge to limits L_1 and L_2, respectively, and c is a constant. Then*

(*a*) $\lim_{n\to+\infty} c = c$

(*b*) $\lim_{n\to+\infty} ca_n = c \lim_{n\to+\infty} a_n = cL_1$

(*c*) $\lim_{n\to+\infty} (a_n + b_n) = \lim_{n\to+\infty} a_n + \lim_{n\to+\infty} b_n = L_1 + L_2$

(*d*) $\lim_{n\to+\infty} (a_n - b_n) = \lim_{n\to+\infty} a_n - \lim_{n\to+\infty} b_n = L_1 - L_2$

(*e*) $\lim_{n\to+\infty} (a_n b_n) = \lim_{n\to+\infty} a_n \cdot \lim_{n\to+\infty} b_n = L_1 L_2$

(*f*) $\lim_{n\to+\infty} \left(\dfrac{a_n}{b_n}\right) = \dfrac{\lim_{n\to+\infty} a_n}{\lim_{n\to+\infty} b_n} = \dfrac{L_1}{L_2}$ (*if* $L_2 \neq 0$)

Example 3 In each part, determine whether the sequence converges or diverges. If it converges, find the limit.

(a) $\left\{\dfrac{n}{2n+1}\right\}_{n=1}^{+\infty}$ (b) $\left\{(-1)^{n+1}\dfrac{n}{2n+1}\right\}_{n=1}^{+\infty}$

(c) $\left\{(-1)^{n+1}\dfrac{1}{n}\right\}_{n=1}^{+\infty}$ (d) $\{8 - 2n\}_{n=1}^{+\infty}$

Solution (a). Dividing numerator and denominator by n yields

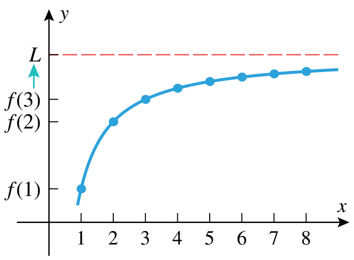
$$\lim_{n\to+\infty} \frac{n}{2n+1} = \lim_{n\to+\infty} \frac{1}{2 + 1/n} = \frac{\lim_{n\to+\infty} 1}{\lim_{n\to+\infty} (2 + 1/n)} = \frac{\lim_{n\to+\infty} 1}{\lim_{n\to+\infty} 2 + \lim_{n\to+\infty} 1/n}$$

$$= \frac{1}{2+0} = \frac{1}{2}$$

Thus, the sequence converges to $\frac{1}{2}$.

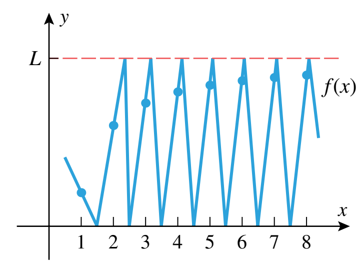
Solution (b). This sequence is the same as that in part (a), except for the factor of $(-1)^{n+1}$, which oscillates between $+1$ and -1. Thus, the terms in this sequence oscillate between positive and negative values, with the odd-numbered terms being identical to those in part (a) and the even-numbered terms being the negatives of those in part (a). Since the sequence in part (a) has a limit of $\frac{1}{2}$, it follows that the odd-numbered terms in this sequence approach $\frac{1}{2}$, and the even-numbered terms approach $-\frac{1}{2}$. Therefore, this sequence has no limit—it diverges.

Solution (c). Since $\lim_{n\to+\infty} 1/n = 0$, the product $(-1)^{n+1}(1/n)$ oscillates between positive and negative values, with the odd-numbered terms approaching 0 through positive



If $f(x) \to L$ as $x \to +\infty$, then $f(n) \to L$ as $n \to +\infty$.

(*a*)



$f(n) \to L$ as $n \to +\infty$, but $f(x)$ diverges by oscillation as $x \to +\infty$.

(*b*)

Figure 10.2.4

values and the even-numbered terms approaching 0 through negative values. Thus,

$$\lim_{n\to+\infty} (-1)^{n+1}\frac{1}{n} = 0$$

so the sequence converges to 0.

Solution (d). $\lim_{n\to+\infty} (8-2n) = -\infty$, so the sequence $\{8-2n\}_{n=1}^{+\infty}$ diverges. ◀

If the general term of a sequence is $f(n)$, and if we replace n by x, where x can vary over the entire interval $[1, +\infty)$, then the values of $f(n)$ can be viewed as "sample values" of $f(x)$ taken at the positive integers. Thus, if $f(x) \to L$ as $x \to +\infty$, then it must also be true that $f(n) \to L$ as $n \to +\infty$ (Figure 10.2.4*a*). However, the converse is not true; that is, one cannot infer that $f(x) \to L$ as $x \to +\infty$ from the fact that $f(n) \to L$ as $n \to +\infty$ (Figure 10.2.4*b*).

Example 4 In each part, determine whether the sequence converges, and if so, find its limit.

(a) $1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots, \frac{1}{2^n}, \ldots$ (b) $1, 2, 2^2, 2^3, \ldots, 2^n, \ldots$

Solution. Replacing n by x in the first sequence produces the power function $(1/2)^x$, and replacing n by x in the second sequence produces the power function 2^x. Now recall that if $0 < b < 1$, then $b^x \to 0$ as $x \to +\infty$, and if $b > 1$, then $b^x \to +\infty$ as $x \to +\infty$ (Figure 7.2.1). Thus,

$$\lim_{n\to+\infty} \frac{1}{2^n} = 0 \quad \text{and} \quad \lim_{n\to+\infty} 2^n = +\infty$$ ◀

Example 5 Find the limit of the sequence $\left\{\frac{n}{e^n}\right\}_{n=1}^{+\infty}$.

Solution. The expression n/e^n is an indeterminate form of type ∞/∞ as $n \to +\infty$, so L'Hôpital's rule is indicated. However, we cannot apply this rule directly to n/e^n because the functions n and e^n have been defined here only at the positive integers, and hence are not differentiable functions. To circumvent this problem, we extend the domains of these functions to all real numbers, here implied by replacing n by x, and apply L'Hôpital's rule to the limit of the quotient x/e^x. This yields

$$\lim_{x\to+\infty} \frac{x}{e^x} = \lim_{x\to+\infty} \frac{1}{e^x} = 0$$

from which we can conclude that

$$\lim_{n\to+\infty} \frac{n}{e^n} = 0$$ ◀

Example 6 Show that $\lim_{n\to+\infty} \sqrt[n]{n} = 1$.

Solution.

$$\lim_{n\to+\infty} \sqrt[n]{n} = \lim_{n\to+\infty} n^{1/n} = \lim_{n\to+\infty} e^{(1/n)\ln n} = e^0 = 1$$ ◀

By L'Hôpital's rule applied to $(1/x)\ln x$

Sometimes the even-numbered and odd-numbered terms of a sequence behave sufficiently differently that it is desirable to investigate their convergence separately. The following theorem, whose proof is omitted, is helpful for that purpose.

> **10.2.4** THEOREM. *A sequence converges to a limit L if and only if the sequences of even-numbered terms and odd-numbered terms both converge to L.*

Example 7 The sequence

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{2^2}, \frac{1}{3^2}, \frac{1}{2^3}, \frac{1}{3^3}, \ldots$$

converges to 0, since the even-numbered terms and the odd-numbered terms both converge to 0, and the sequence

$$1, \tfrac{1}{2}, 1, \tfrac{1}{3}, 1, \tfrac{1}{4}, \ldots$$

diverges, since the odd-numbered terms converge to 1 and the even-numbered terms converge to 0. ◀

THE SQUEEZING THEOREM FOR SEQUENCES

The following theorem, which we state without proof, is an adaptation of the Squeezing Theorem (2.6.2) to sequences. This theorem will be useful for finding limits of sequences that cannot be obtained directly.

> **10.2.5 THEOREM (*The Squeezing Theorem for Sequences*).** *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences such that*
>
> $$a_n \le b_n \le c_n \quad (\textit{for all values of } n \textit{ beyond some index } N)$$
>
> *If the sequences $\{a_n\}$ and $\{c_n\}$ have a common limit L as $n \to +\infty$, then $\{b_n\}$ also has the limit L as $n \to +\infty$.*

Table 10.2.4

n	$\dfrac{n!}{n^n}$
1	1.0000000000
2	0.5000000000
3	0.2222222222
4	0.0937500000
5	0.0384000000
6	0.0154320988
7	0.0061198990
8	0.0024032593
9	0.0009366567
10	0.0003628800
11	0.0001399059
12	0.0000537232

Example 8 Use numerical evidence to make a conjecture about the limit of the sequence*

$$\left\{\frac{n!}{n^n}\right\}_{n=1}^{+\infty}$$

and then confirm that your conjecture is correct.

Solution. Table 10.2.4, which was obtained with a calculating utility, suggests that the limit of the sequence may be 0. To confirm this we need to examine the limit of

$$a_n = \frac{n!}{n^n}$$

as $n \to +\infty$. Although this is an indeterminate form of type ∞/∞, L'Hôpital's rule is not helpful because we have no definition of $x!$ for values of x that are not integers. However, let us write out some of the initial terms and the general term in the sequence:

$$a_1 = 1, \quad a_2 = \frac{1 \cdot 2}{2 \cdot 2}, \quad a_3 = \frac{1 \cdot 2 \cdot 3}{3 \cdot 3 \cdot 3}, \ldots, \quad a_n = \frac{1 \cdot 2 \cdot 3 \cdots n}{n \cdot n \cdot n \cdots n}, \ldots$$

We can rewrite the general term as

$$a_n = \frac{1}{n}\left(\frac{2 \cdot 3 \cdots n}{n \cdot n \cdots n}\right)$$

from which it is evident that

$$0 \le a_n \le \frac{1}{n}$$

However, the two outside expressions have a limit of 0 as $n \to +\infty$; thus, the Squeezing Theorem for Sequences implies that $a_n \to 0$ as $n \to +\infty$, which confirms our conjecture. ◀

The following theorem is often useful for finding the limit of a sequence with both positive and negative terms—it states that if the sequence $\{|a_n|\}$ that is obtained by taking the absolute value of each term in the sequence $\{a_n\}$ converges to 0, then $\{a_n\}$ also converges to 0.

*The symbol $n!$ (read "n factorial") is defined on page 642.

10.2.6 THEOREM. *If* $\lim_{n\to+\infty} |a_n| = 0$, *then* $\lim_{n\to+\infty} a_n = 0$.

Proof. Depending on the sign of a_n, either $a_n = |a_n|$ or $a_n = -|a_n|$. Thus, in all cases we have

$$-|a_n| \le a_n \le |a_n|$$

However, the limit of the two outside terms is 0, and hence the limit of a_n is 0 by the Squeezing Theorem for Sequences. ■

Example 9 Consider the sequence

$$1, -\frac{1}{2}, \frac{1}{2^2}, -\frac{1}{2^3}, \ldots, (-1)^n\frac{1}{2^n}, \ldots$$

If we take the absolute value of each term, we obtain the sequence

$$1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots, \frac{1}{2^n}, \ldots$$

which, as shown in Example 4, converges to 0. Thus, from Theorem 10.2.6 we have

$$\lim_{n\to+\infty}\left[(-1)^n\frac{1}{2^n}\right] = 0$$ ◀

SEQUENCES DEFINED RECURSIVELY

Some sequences do not arise from a formula for the general term, but rather from a formula or set of formulas that specify how to generate each term in the sequence from terms that precede it; such sequences are said to be defined ***recursively***, and the defining formulas are called ***recursion formulas***. A good example is the mechanic's rule for approximating square roots. In Exercise 19 of Section 4.7 you were asked to show that

$$x_1 = 1, \quad x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \tag{2}$$

describes the sequence produced by Newton's Method to approximate $\sqrt{a}$ as a root of the function $f(x) = x^2 - a$. Table 10.2.5 shows the first five terms in an application of the mechanic's rule to approximate $\sqrt{2}$.

Table 10.2.5

n	$x_1 = 1, \quad x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$	DECIMAL APPROXIMATION
	$x_1 = 1$ (Starting value)	1.00000000000
1	$x_2 = \frac{1}{2}\left[1 + \frac{2}{1}\right] = \frac{3}{2}$	1.50000000000
2	$x_3 = \frac{1}{2}\left[\frac{3}{2} + \frac{2}{3/2}\right] = \frac{17}{12}$	1.41666666667
3	$x_4 = \frac{1}{2}\left[\frac{17}{12} + \frac{2}{17/12}\right] = \frac{577}{408}$	1.41421568627
4	$x_5 = \frac{1}{2}\left[\frac{577}{408} + \frac{2}{577/408}\right] = \frac{665{,}857}{470{,}832}$	1.41421356237
5	$x_6 = \frac{1}{2}\left[\frac{665{,}857}{470{,}832} + \frac{2}{665{,}857/470{,}832}\right] = \frac{886{,}731{,}088{,}897}{627{,}013{,}566{,}048}$	1.41421356237

It would take us too far afield to investigate the convergence of sequences defined recursively, but we will conclude this section with a useful technique that can sometimes be used to compute limits of such sequences.

Example 10 Assuming that the sequence in Table 10.2.5 converges, show that the limit is $\sqrt{2}$.

Solution. Assume that $x_n \to L$, where L is to be determined. Since $n+1 \to +\infty$ as $n \to +\infty$, it is also true that $x_{n+1} \to L$ as $n \to +\infty$. Thus, if we take the limit of the expression

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$$

as $n \to +\infty$, we obtain

$$L = \frac{1}{2}\left(L + \frac{2}{L}\right)$$

which can be rewritten as $L^2 = 2$. The negative solution of this equation is extraneous because $x_n > 0$ for all n, so $L = \sqrt{2}$. ◀

EXERCISE SET 10.2 Graphing Utility C CAS

1. In each part, find a formula for the general term of the sequence, starting with $n = 1$.

(a) $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$ (b) $1, -\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \ldots$

(c) $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \ldots$ (d) $\frac{1}{\sqrt{\pi}}, \frac{4}{\sqrt[3]{\pi}}, \frac{9}{\sqrt[4]{\pi}}, \frac{16}{\sqrt[5]{\pi}}, \ldots$

2. In each part, find two formulas for the general term of the sequence, one starting with $n = 1$ and the other with $n = 0$.
(a) $1, -r, r^2, -r^3, \ldots$ (b) $r, -r^2, r^3, -r^4, \ldots$

3. (a) Write out the first four terms of the sequence $\{1 + (-1)^n\}$, starting with $n = 0$.
(b) Write out the first four terms of the sequence $\{\cos n\pi\}$, starting with $n = 0$.
(c) Use the results in parts (a) and (b) to express the general term of the sequence $4, 0, 4, 0, \ldots$ in two different ways, starting with $n = 0$.

4. In each part, find a formula for the general term using factorials and starting with $n = 1$.
(a) $1 \cdot 2, 1 \cdot 2 \cdot 3 \cdot 4, 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6, 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8, \ldots$
(b) $1, 1 \cdot 2 \cdot 3, 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5, 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7, \ldots$

In Exercises 5–22, write out the first five terms of the sequence, determine whether the sequence converges, and if so find its limit.

5. $\left\{\frac{n}{n+2}\right\}_{n=1}^{+\infty}$ **6.** $\left\{\frac{n^2}{2n+1}\right\}_{n=1}^{+\infty}$ **7.** $\{2\}_{n=1}^{+\infty}$

8. $\left\{\ln\left(\frac{1}{n}\right)\right\}_{n=1}^{+\infty}$ **9.** $\left\{\frac{\ln n}{n}\right\}_{n=1}^{+\infty}$ **10.** $\left\{n \sin \frac{\pi}{n}\right\}_{n=1}^{+\infty}$

11. $\{1 + (-1)^n\}_{n=1}^{+\infty}$ **12.** $\left\{\frac{(-1)^{n+1}}{n^2}\right\}_{n=1}^{+\infty}$

13. $\left\{(-1)^n \frac{2n^3}{n^3+1}\right\}_{n=1}^{+\infty}$ **14.** $\left\{\frac{n}{2^n}\right\}_{n=1}^{+\infty}$

15. $\left\{\frac{(n+1)(n+2)}{2n^2}\right\}_{n=1}^{+\infty}$ **16.** $\left\{\frac{\pi^n}{4^n}\right\}_{n=1}^{+\infty}$

17. $\left\{\cos \frac{3}{n}\right\}_{n=1}^{+\infty}$ **18.** $\left\{\cos \frac{\pi n}{2}\right\}_{n=1}^{+\infty}$

19. $\{n^2 e^{-n}\}_{n=1}^{+\infty}$ **20.** $\{\sqrt{n^2+3n} - n\}_{n=1}^{+\infty}$

21. $\left\{\left(\frac{n+3}{n+1}\right)^n\right\}_{n=1}^{+\infty}$ **22.** $\left\{\left(1 - \frac{2}{n}\right)^n\right\}_{n=1}^{+\infty}$

In Exercises 23–30, find the general term of the sequence, starting with $n = 1$, determine whether the sequence converges, and if so find its limit.

23. $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \ldots$ **24.** $0, \frac{1}{2^2}, \frac{2}{3^2}, \frac{3}{4^2}, \ldots$

25. $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots$ **26.** $-1, 2, -3, 4, -5, \ldots$

27. $\left(1 - \frac{1}{2}\right), \left(\frac{1}{2} - \frac{1}{3}\right), \left(\frac{1}{3} - \frac{1}{4}\right), \left(\frac{1}{4} - \frac{1}{5}\right), \ldots$

28. $3, \frac{3}{2}, \frac{3}{2^2}, \frac{3}{2^3}, \ldots$

29. $(\sqrt{2} - \sqrt{3}), (\sqrt{3} - \sqrt{4}), (\sqrt{4} - \sqrt{5}), \ldots$

30. $\frac{1}{3^5}, -\frac{1}{3^6}, \frac{1}{3^7}, -\frac{1}{3^8}, \ldots$

31. (a) Starting with $n = 1$, write out the first six terms of the sequence $\{a_n\}$, where

$$a_n = \begin{cases} 1, & \text{if } n \text{ is odd} \\ n, & \text{if } n \text{ is even} \end{cases}$$

(b) Starting with $n = 1$, and considering the even and odd terms separately, find a formula for the general term of the sequence

$$1, \frac{1}{2^2}, 3, \frac{1}{2^4}, 5, \frac{1}{2^6}, \ldots$$

(c) Starting with $n = 1$, and considering the even and odd terms separately, find a formula for the general term of the sequence

$$1, \frac{1}{3}, \frac{1}{3}, \frac{1}{5}, \frac{1}{5}, \frac{1}{7}, \frac{1}{7}, \frac{1}{9}, \frac{1}{9}, \ldots$$

(d) Determine whether the sequences in parts (a), (b), and (c) converge. For those that do, find the limit.

32. For what positive values of b does the sequence $b, 0, b^2, 0, b^3, 0, b^4, \ldots$ converge? Justify your answer.

C **33.** (a) Use numerical evidence to make a conjecture about the limit of the sequence $\{\sqrt[n]{n^3}\}_{n=2}^{+\infty}$.
(b) Use a CAS to confirm your conjecture.

C **34.** (a) Use numerical evidence to make a conjecture about the limit of the sequence $\{\sqrt[n]{3^n + n^3}\}_{n=2}^{+\infty}$.
(b) Use a CAS to confirm your conjecture.

35. Assuming that the sequence given in Formula (2) of this section converges, use the method of Example 10 to show that the limit of this sequence is $\sqrt{a}$.

36. Consider the sequence

$$a_1 = \sqrt{6}$$
$$a_2 = \sqrt{6 + \sqrt{6}}$$
$$a_3 = \sqrt{6 + \sqrt{6 + \sqrt{6}}}$$
$$a_4 = \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6}}}}$$
$$\vdots$$

(a) Find a recursion formula for a_{n+1}.
(b) Assuming that the sequence converges, use the method of Example 10 to find the limit.

37. Consider the sequence $\{a_n\}_{n=1}^{+\infty}$, where

$$a_n = \frac{1}{n^2} + \frac{2}{n^2} + \cdots + \frac{n}{n^2}$$

(a) Find a_1, a_2, a_3, and a_4.
(b) Use numerical evidence to make a conjecture about the limit of the sequence.
(c) Confirm your conjecture by expressing a_n in closed form and calculating the limit.

38. Follow the directions in Exercise 37 with

$$a_n = \frac{1^2}{n^3} + \frac{2^2}{n^3} + \cdots + \frac{n^2}{n^3}$$

In Exercises 39 and 40, use numerical evidence to make a conjecture about the limit of the sequence, and then use the Squeezing Theorem for Sequences (Theorem 10.2.5) to confirm that your conjecture is correct.

39. $\lim_{n \to +\infty} \dfrac{\sin^2 n}{n}$ **40.** $\lim_{n \to +\infty} \left(\dfrac{1+n}{2n}\right)^n$

41. (a) A bored student enters the number 0.5 in a calculator display and then repeatedly computes the square of the number in the display. Taking $a_0 = 0.5$, find a formula for the general term of the sequence $\{a_n\}$ of numbers that appear in the display.
(b) Try this with a calculator and make a conjecture about the limit of a_n.
(c) Confirm your conjecture by finding the limit of a_n.
(d) For what values of a_0 will this procedure produce a convergent sequence?

42. Let

$$f(x) = \begin{cases} 2x, & 0 \le x < 0.5 \\ 2x - 1, & 0.5 \le x < 1 \end{cases}$$

Does the sequence $f(0.2), f(f(0.2)), f(f(f(0.2))), \ldots$ converge? Justify your reasoning.

43. (a) Use a graphing utility to generate the graph of the equation $y = (2^x + 3^x)^{1/x}$, and then use the graph to make a conjecture about the limit of the sequence

$$\{(2^n + 3^n)^{1/n}\}_{n=1}^{+\infty}$$

(b) Confirm your conjecture by calculating the limit.

44. Consider the sequence $\{a_n\}_{n=1}^{+\infty}$ whose nth term is

$$a_n = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{1 + (k/n)}$$

Show that $\lim_{n \to +\infty} a_n = \ln 2$ by interpreting a_n as the Riemann sum of a definite integral.

45. Let a_n be the average value of $f(x) = 1/x$ over the interval $[1, n]$. Determine whether the sequence $\{a_n\}$ converges, and if so find its limit.

46. The sequence whose terms are 1, 1, 2, 3, 5, 8, 13, 21, ... is called the ***Fibonacci sequence*** in honor of Leonardo ("Fibonacci") da Pisa (c. 1170–1250). This sequence has the property that after starting with two 1's, each term is the sum of the preceding two.
(a) Denoting the sequence by $\{a_n\}$ and starting with $a_1 = 1$ and $a_2 = 1$, show that

$$\frac{a_{n+2}}{a_{n+1}} = 1 + \frac{a_n}{a_{n+1}} \quad \text{if } n \ge 1$$

(b) Give a reasonable informal argument to show that if the sequence $\{a_{n+1}/a_n\}$ converges to some limit L, then the sequence $\{a_{n+2}/a_{n+1}\}$ must also converge to L.
(c) Assuming that the sequence $\{a_{n+1}/a_n\}$ converges, show that its limit is $(1 + \sqrt{5})/2$.

47. If we accept the fact that the sequence $\{1/n\}_{n=1}^{+\infty}$ converges to the limit $L = 0$, then according to Definition 10.2.2, for every $\epsilon > 0$, there exists a positive integer N such that $|a_n - L| = |(1/n) - 0| < \epsilon$ when $n \geq N$. In each part, find the smallest possible value of N for the given value of ϵ.
(a) $\epsilon = 0.5$ (b) $\epsilon = 0.1$ (c) $\epsilon = 0.001$

48. If we accept the fact that the sequence
$$\left\{\frac{n}{n+1}\right\}_{n=1}^{+\infty}$$
converges to the limit $L = 1$, then according to Definition 10.2.2, for every $\epsilon > 0$ there exists an integer N such that
$$|a_n - L| = \left|\frac{n}{n+1} - 1\right| < \epsilon$$
when $n \geq N$. In each part, find the smallest value of N for the given value of ϵ.
(a) $\epsilon = 0.25$ (b) $\epsilon = 0.1$ (c) $\epsilon = 0.001$

49. Use Definition 10.2.2 to prove that
(a) the sequence $\{1/n\}_{n=1}^{+\infty}$ converges to 0
(b) the sequence $\left\{\frac{n}{n+1}\right\}_{n=1}^{+\infty}$ converges to 1.

50. Find $\lim_{n\to+\infty} r^n$, where r is a real number. [*Hint:* Consider the cases $|r| < 1$, $|r| > 1$, $r = 1$, and $r = -1$ separately.]

10.3 MONOTONE SEQUENCES

There are many situations in which it is important to know whether a sequence converges, but the value of the limit is not relevant to the problem at hand. In this section we will study several techniques that can be used to determine whether a sequence converges.

TERMINOLOGY

We begin with some terminology.

10.3.1 DEFINITION. A sequence $\{a_n\}_{n=1}^{+\infty}$ is called
strictly increasing if $a_1 < a_2 < a_3 < \cdots < a_n < \cdots$
increasing if $a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n \leq \cdots$
strictly decreasing if $a_1 > a_2 > a_3 > \cdots > a_n > \cdots$
decreasing if $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq \cdots$

In words, a sequence is strictly increasing if each term is larger than its predecessor, increasing if each term is the same as or larger than its predecessor, strictly decreasing if each term is smaller than its predecessor, and decreasing if each term is the same as or smaller than its predecessor. It follows that every strictly increasing sequence is increasing (but not conversely), and every strictly decreasing sequence is decreasing (but not conversely). A sequence that is either strictly increasing or strictly decreasing is called ***strictly monotone***, and a sequence that is either increasing or decreasing is called ***monotone***.

Example 1

SEQUENCE	DESCRIPTION
$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$	Strictly increasing
$1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$	Strictly decreasing
$1, 1, 2, 2, 3, 3, \ldots$	Increasing; not strictly increasing
$1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \ldots$	Decreasing; not strictly decreasing
$1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots, (-1)^{n+1}\frac{1}{n}, \ldots$	Neither increasing nor decreasing

The first and second sequences are strictly monotone, and the third and fourth sequences are monotone but not strictly monotone. The fifth sequence is not monotone. ◀

• FOR THE READER. Can a sequence be both increasing and decreasing? Explain.

TESTING FOR MONOTONICITY

In order for a sequence to be strictly increasing, *all* pairs of successive terms, a_n and a_{n+1}, must satisfy $a_n < a_{n+1}$ or, equivalently, $a_{n+1} - a_n > 0$. More generally, monotone sequences can be classified as follows:

DIFFERENCE BETWEEN SUCCESSIVE TERMS	CLASSIFICATION
$a_{n+1} - a_n > 0$	Strictly increasing
$a_{n+1} - a_n < 0$	Strictly decreasing
$a_{n+1} - a_n \geq 0$	Increasing
$a_{n+1} - a_n \leq 0$	Decreasing

Frequently, one can *guess* whether a sequence is monotone or strictly monotone by writing out some of the initial terms. However, to be certain that the guess is correct, one must give a precise mathematical argument. The following example illustrates one method for doing this.

Example 2 Show that

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$$

is a strictly increasing sequence.

Solution. The pattern of the initial terms suggests that the sequence is strictly increasing. To prove that this is so, let

$$a_n = \frac{n}{n+1}$$

We can obtain a_{n+1} by replacing n by $n+1$ in this formula. This yields

$$a_{n+1} = \frac{n+1}{(n+1)+1} = \frac{n+1}{n+2}$$

Thus, for $n \geq 1$

$$a_{n+1} - a_n = \frac{n+1}{n+2} - \frac{n}{n+1} = \frac{n^2+2n+1-n^2-2n}{(n+1)(n+2)} = \frac{1}{(n+1)(n+2)} > 0$$

which proves that the sequence is strictly increasing. ◀

If a_n and a_{n+1} are any successive terms in a strictly increasing sequence, then $a_n < a_{n+1}$. If the terms in the sequence are all positive, then we can divide both sides of this inequality by a_n to obtain $1 < a_{n+1}/a_n$ or, equivalently, $a_{n+1}/a_n > 1$. More generally, monotone sequences with *positive* terms can be classified as follows:

RATIO OF SUCCESSIVE TERMS	CONCLUSION
$a_{n+1}/a_n > 1$	Strictly increasing
$a_{n+1}/a_n < 1$	Strictly decreasing
$a_{n+1}/a_n \geq 1$	Increasing
$a_{n+1}/a_n \leq 1$	Decreasing

Example 3 Show that the sequence in Example 2 is strictly increasing by examining the ratio of successive terms.

Solution. As shown in the solution of Example 2,

$$a_n = \frac{n}{n+1} \quad \text{and} \quad a_{n+1} = \frac{n+1}{n+2}$$

Thus,

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)/(n+2)}{n/(n+1)} = \frac{n+1}{n+2} \cdot \frac{n+1}{n} = \frac{n^2+2n+1}{n^2+2n} \tag{1}$$

Since the numerator in (1) exceeds the denominator, it follows that $a_{n+1}/a_n > 1$ for $n \geq 1$. This proves that the sequence is strictly increasing. ◀

The following example illustrates still a third technique for determining whether a sequence is strictly monotone.

Example 4 In Examples 2 and 3 we proved that the sequence

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$$

is strictly increasing by considering the difference and ratio of successive terms. Alternatively, we can proceed as follows. Let

$$f(x) = \frac{x}{x+1}$$

so that the nth term in the given sequence is $a_n = f(n)$. The function f is increasing for $x \geq 1$ since

$$f'(x) = \frac{(x+1)(1) - x(1)}{(x+1)^2} = \frac{1}{(x+1)^2} > 0$$

Thus,

$$a_n = f(n) < f(n+1) = a_{n+1}$$

which proves that the given sequence is strictly increasing. ◀

In general, if $f(n) = a_n$ is the nth term of a sequence, and if f is differentiable for $x \geq 1$, then we have the following results:

DERIVATIVE OF f FOR $x \geq 1$	CONCLUSION FOR THE SEQUENCE WITH $a_n = f(n)$
$f'(x) > 0$	Strictly increasing
$f'(x) < 0$	Strictly decreasing
$f'(x) \geq 0$	Increasing
$f'(x) \leq 0$	Decreasing

PROPERTIES THAT HOLD EVENTUALLY

Sometimes a sequence will behave erratically at first and then settle down into a definite pattern. For example, the sequence

$$9, -8, -17, 12, 1, 2, 3, 4, \ldots \tag{2}$$

is strictly increasing from the fifth term on, but the sequence as a whole cannot be classified as strictly increasing because of the erratic behavior of the first four terms. To describe such sequences, we introduce the following terminology.

10.3.2 DEFINITION. If discarding finitely many terms from the beginning of a sequence produces a sequence with a certain property, then the original sequence is said to have that property ***eventually***.

For example, although we cannot say that sequence (2) is strictly increasing, we can say that it is eventually strictly increasing.

Example 5 Show that the sequence $\left\{\dfrac{10^n}{n!}\right\}_{n=1}^{+\infty}$ is eventually strictly decreasing.

Solution. We have

$$a_n = \frac{10^n}{n!} \quad \text{and} \quad a_{n+1} = \frac{10^{n+1}}{(n+1)!}$$

so

$$\frac{a_{n+1}}{a_n} = \frac{10^{n+1}/(n+1)!}{10^n/n!} = \frac{10^{n+1}n!}{10^n(n+1)!} = 10\frac{n!}{(n+1)n!} = \frac{10}{n+1} \tag{3}$$

From (3), $a_{n+1}/a_n < 1$ for all $n \geq 10$, so the sequence is eventually strictly decreasing. ◀

AN INTUITIVE VIEW OF CONVERGENCE

Informally stated, the convergence or divergence of a sequence does not depend on the behavior of its *initial terms*, but rather on how the terms behave *eventually*. For example, the sequence

$$3, -9, -13, 17, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$

eventually behaves like the sequence

$$1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$$

and hence has a limit of 0.

CONVERGENCE OF MONOTONE SEQUENCES

The following two theorems, whose proofs are discussed at the end of this section, show that a monotone sequence either converges or becomes infinite—divergence by oscillation cannot occur.

10.3.3 THEOREM. *If a sequence $\{a_n\}$ is eventually increasing, then there are two possibilities:*

(*a*) *There is a constant M, called an **upper bound** for the sequence, such that $a_n \leq M$ for all n, in which case the sequence converges to a limit L satisfying $L \leq M$.*

(*b*) *No upper bound exists, in which case $\lim_{n\to+\infty} a_n = +\infty$.*

10.3.4 THEOREM. *If a sequence $\{a_n\}$ is eventually decreasing, then there are two possibilities:*

(*a*) *There is a constant M, called a **lower bound** for the sequence, such that $a_n \geq M$ for all n, in which case the sequence converges to a limit L satisfying $L \geq M$.*

(*b*) *No lower bound exists, in which case $\lim_{n\to+\infty} a_n = -\infty$.*

Note that these results do not give a method for obtaining limits; they tell us only whether a limit exists.

Example 6 Show that the sequence $\left\{\dfrac{10^n}{n!}\right\}_{n=1}^{+\infty}$ converges and find its limit.

Solution. We showed in Example 5 that the sequence is eventually strictly decreasing. Since all terms in the sequence are positive, it is bounded below by $M = 0$, and hence Theorem 10.3.4 guarantees that it converges to a nonnegative limit L. However, the limit is not evident directly from the formula $10^n/n!$ for the nth term, so we will need some ingenuity to obtain it.

Recall from Formula (3) of Example 5 that successive terms in the given sequence are related by the recursion formula

$$a_{n+1} = \frac{10}{n+1}a_n \tag{4}$$

where $a_n = 10^n/n!$. We will take the limit as $n \to +\infty$ of both sides of (4) and use the fact that

$$\lim_{n\to+\infty} a_{n+1} = \lim_{n\to+\infty} a_n = L$$

We obtain

$$L = \lim_{n\to+\infty} a_{n+1} = \lim_{n\to+\infty}\left(\frac{10}{n+1}a_n\right) = \lim_{n\to+\infty}\frac{10}{n+1}\lim_{n\to+\infty} a_n = 0 \cdot L = 0$$

so that

$$L = \lim_{n\to+\infty}\frac{10^n}{n!} = 0$$

◀

REMARK. In the exercises we will show that the technique illustrated in this example can be adapted to obtain the limit

$$\lim_{n\to+\infty}\frac{x^n}{n!} = 0 \tag{5}$$

for any real value of x (Exercise 26). This result will be useful in our later work.

THE COMPLETENESS AXIOM

In this text we have accepted the familiar properties of real numbers without proof, and indeed, we have not even attempted to define the term *real number*. Although this is sufficient for many purposes, it was recognized by the late nineteenth century that the study of limits and functions in calculus requires a precise axiomatic formulation of the real numbers analogous to the axiomatic development of Euclidean geometry. Although we will not attempt to pursue this development, we will need to discuss one of the axioms about real numbers in order to prove Theorems 10.3.3 and 10.3.4. But first we will introduce some terminology.

If S is a nonempty set of real numbers, then we call u an ***upper bound*** for S if u is greater than or equal to every number in S, and we call ℓ a ***lower bound*** for S if ℓ is smaller than or equal to every number in S. For example, if S is the set of numbers in the interval $(1, 3)$, then $u = 4$, 10, and 100 are upper bounds for S and $\ell = -10$, 0, and $\frac{1}{2}$ are lower bounds for S. Observe also that $u = 3$ is the smallest of all upper bounds and $\ell = 1$ is the largest of all lower bounds. The existence of a smallest upper bound and a greatest lower bound for S is not accidental; it is a consequence of the following axiom.

10.3.5 AXIOM (***The Completeness Axiom***). *If a nonempty set S of real numbers has an upper bound, then it has a smallest upper bound (called the **least upper bound**), and if a nonempty set S of real numbers has a lower bound, then it has a largest lower bound (called the **greatest lower bound**).*

Proof of Theorem 10.3.3.

(*a*) We will prove the result for increasing sequences, and leave it for the reader to adapt the argument to sequences that are eventually increasing. Assume there exists a number M such that $a_n \leq M$ for $n = 1, 2, \ldots$. Then M is an upper bound for the set of terms in the sequence. By the Completeness Axiom there is a least upper bound for the terms, call it L. Now let ϵ be any positive number. Since L is the least upper bound for the terms, $L - \epsilon$ is not an upper bound for the terms, which means that there is at least one term a_N such that

$$a_N > L - \epsilon$$

Moreover, since $\{a_n\}$ is an increasing sequence, we must have

$$a_n \geq a_N > L - \epsilon \tag{6}$$

when $n \geq N$. But a_n cannot exceed L since L is an upper bound for the terms. This observation together with (6) tells us that $L \geq a_n > L - \epsilon$ for $n \geq N$, so all terms from the Nth on are within ϵ units of L. This is exactly the requirement to have

$$\lim_{n \to +\infty} a_n = L$$

Finally, $L \leq M$ since M is an upper bound for the terms and L is the least upper bound. This proves part (*a*).

(*b*) If there is no number M such that $a_n \leq M$ for $n = 1, 2, \ldots$, then no matter how large we choose M, there is a term a_N such that

$$a_N > M$$

and, since the sequence is increasing,

$$a_n \geq a_N > M$$

when $n \geq N$. Thus, the terms in the sequence become arbitrarily large as n increases. That is,

$$\lim_{n \to +\infty} a_n = +\infty$$ ■

The proof of Theorem 10.3.4 will be omitted since it is similar to that of 10.3.3.

EXERCISE SET 10.3

In Exercises 1–6, use $a_{n+1} - a_n$ to show that the given sequence $\{a_n\}$ is strictly increasing or strictly decreasing.

1. $\left\{\frac{1}{n}\right\}_{n=1}^{+\infty}$ **2.** $\left\{1 - \frac{1}{n}\right\}_{n=1}^{+\infty}$ **3.** $\left\{\frac{n}{2n+1}\right\}_{n=1}^{+\infty}$

4. $\left\{\frac{n}{4n-1}\right\}_{n=1}^{+\infty}$ **5.** $\{n - 2^n\}_{n=1}^{+\infty}$ **6.** $\{n - n^2\}_{n=1}^{+\infty}$

In Exercises 7–12, use a_{n+1}/a_n to show that the given sequence $\{a_n\}$ is strictly increasing or strictly decreasing.

7. $\left\{\frac{n}{2n+1}\right\}_{n=1}^{+\infty}$ **8.** $\left\{\frac{2^n}{1+2^n}\right\}_{n=1}^{+\infty}$ **9.** $\{ne^{-n}\}_{n=1}^{+\infty}$

10. $\left\{\frac{10^n}{(2n)!}\right\}_{n=1}^{+\infty}$ **11.** $\left\{\frac{n^n}{n!}\right\}_{n=1}^{+\infty}$ **12.** $\left\{\frac{5^n}{2^{(n^2)}}\right\}_{n=1}^{+\infty}$

In Exercises 13–18, use differentiation to show that the sequence is strictly increasing or strictly decreasing.

13. $\left\{\frac{n}{2n+1}\right\}_{n=1}^{+\infty}$ **14.** $\left\{3 - \frac{1}{n}\right\}_{n=1}^{+\infty}$

15. $\left\{\frac{1}{n + \ln n}\right\}_{n=1}^{+\infty}$ **16.** $\{ne^{-2n}\}_{n=1}^{+\infty}$

17. $\left\{\frac{\ln(n+2)}{n+2}\right\}_{n=1}^{+\infty}$ **18.** $\{\tan^{-1} n\}_{n=1}^{+\infty}$

In Exercises 19–24, use any method to show that the given sequence is eventually strictly increasing or eventually strictly decreasing.

19. $\{2n^2 - 7n\}_{n=1}^{+\infty}$ **20.** $\{n^3 - 4n^2\}_{n=1}^{+\infty}$

21. $\left\{\dfrac{n}{n^2+10}\right\}_{n=1}^{+\infty}$

22. $\left\{n+\dfrac{17}{n}\right\}_{n=1}^{+\infty}$

23. $\left\{\dfrac{n!}{3^n}\right\}_{n=1}^{+\infty}$

24. $\{n^5 e^{-n}\}_{n=1}^{+\infty}$

25. (a) Suppose that $\{a_n\}$ is a monotone sequence such that $1 \le a_n \le 2$. Must the sequence converge? If so, what can you say about the limit?
(b) Suppose that $\{a_n\}$ is a monotone sequence such that $a_n \le 2$. Must the sequence converge? If so, what can you say about the limit?

26. The goal in this exercise is to prove Formula (5) in this section. The case where $x = 0$ is obvious, so we will focus on the case where $x \neq 0$.
(a) Let $a_n = |x|^n/n!$. Show that

$$a_{n+1} = \frac{|x|}{n+1}a_n$$

(b) Show that the sequence $\{a_n\}$ is eventually strictly decreasing.
(c) Show that the sequence $\{a_n\}$ converges.
(d) Use the results in parts (a) and (c) to show that $a_n \to 0$ as $n \to +\infty$.
(e) Obtain Formula (5) from the result in part (d).

27. Let $\{a_n\}$ be the sequence defined recursively by $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + a_n}$ for $n \ge 1$.
(a) List the first three terms of the sequence.
(b) Show that $a_n < 2$ for $n \ge 1$.
(c) Show that $a_{n+1}^2 - a_n^2 = (2 - a_n)(1 + a_n)$ for $n \ge 1$.
(d) Use the results in parts (b) and (c) to show that $\{a_n\}$ is a strictly increasing sequence. [*Hint:* If x and y are positive real numbers such that $x^2 - y^2 > 0$, then it follows by factoring that $x - y > 0$.]
(e) Show that $\{a_n\}$ converges and find its limit L.

28. Let $\{a_n\}$ be the sequence defined recursively by $a_1 = 1$ and $a_{n+1} = \frac{1}{2}[a_n + (3/a_n)]$ for $n \ge 1$.
(a) Show that $a_n \ge \sqrt{3}$ for $n \ge 2$. [*Hint:* What is the minimum value of $\frac{1}{2}[x + (3/x)]$ for $x > 0$?]
(b) Show that $\{a_n\}$ is eventually decreasing. [*Hint:* Examine $a_{n+1} - a_n$ or a_{n+1}/a_n and use the result in part (a).]
(c) Show that $\{a_n\}$ converges and find its limit L.

29. (a) Compare appropriate areas in the accompanying figure to deduce the following inequalities for $n \ge 2$:

$$\int_1^n \ln x\,dx < \ln n! < \int_1^{n+1} \ln x\,dx$$

(b) Use the result in part (a) to show that

$$\frac{n^n}{e^{n-1}} < n! < \frac{(n+1)^{n+1}}{e^n}, \quad n > 1$$

(c) Use the Squeezing Theorem for Sequences (Theorem 10.2.5) and the result in part (b) to show that

$$\lim_{n\to+\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}$$

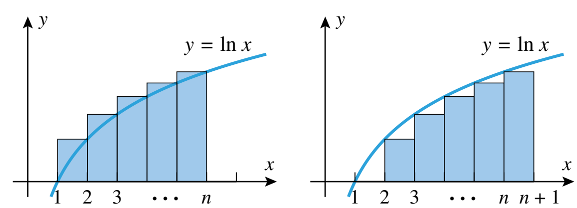


Figure Ex-29

30. Use the left inequality in Exercise 29(b) to show that

$$\lim_{n\to+\infty} \sqrt[n]{n!} = +\infty$$

10.4 INFINITE SERIES

The purpose of this section is to discuss sums that contain infinitely many terms. The most familiar examples of such sums occur in the decimal representations of real numbers. For example, when we write $\frac{1}{3}$ in the decimal form $\frac{1}{3} = 0.3333\ldots$, we mean

$$\frac{1}{3} = 0.3 + 0.03 + 0.003 + 0.0003 + \cdots$$

which suggests that the decimal representation of $\frac{1}{3}$ can be viewed as a sum of infinitely many real numbers.

SUMS OF INFINITE SERIES

Our first objective is to define what is meant by the "sum" of infinitely many real numbers. We begin with some terminology.

10.4.1 DEFINITION. An ***infinite series*** is an expression that can be written in the form

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + u_3 + \cdots + u_k + \cdots$$

The numbers $u_1, u_2, u_3, \ldots$ are called the ***terms*** of the series.

Since it is impossible to add infinitely many numbers together directly, sums of infinite series are defined and computed by an indirect limiting process. To motivate the basic idea, consider the decimal

$$0.3333\ldots \tag{1}$$

This can be viewed as the infinite series

$$0.3 + 0.03 + 0.003 + 0.0003 + \cdots$$

or, equivalently,

$$\frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \cdots \tag{2}$$

Since (1) is the decimal expansion of $\frac{1}{3}$, any reasonable definition for the sum of an infinite series should yield $\frac{1}{3}$ for the sum of (2). To obtain such a definition, consider the following sequence of (finite) sums:

$$s_1 = \frac{3}{10} = 0.3$$

$$s_2 = \frac{3}{10} + \frac{3}{10^2} = 0.33$$

$$s_3 = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} = 0.333$$

$$s_4 = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} = 0.3333$$

$$\vdots$$

The sequence of numbers $s_1, s_2, s_3, s_4, \ldots$ can be viewed as a succession of approximations to the "sum" of the infinite series, which we want to be $\frac{1}{3}$. As we progress through the sequence, more and more terms of the infinite series are used, and the approximations get better and better, suggesting that the desired sum of $\frac{1}{3}$ might be the *limit* of this sequence of approximations. To see that this is so, we must calculate the limit of the general term in the sequence of approximations, namely

$$s_n = \frac{3}{10} + \frac{3}{10^2} + \cdots + \frac{3}{10^n} \tag{3}$$

The problem of calculating

$$\lim_{n\to+\infty} s_n = \lim_{n\to+\infty} \left(\frac{3}{10} + \frac{3}{10^2} + \cdots + \frac{3}{10^n}\right)$$

is complicated by the fact that both the last term and the number of terms in the sum change with n. It is best to rewrite such limits in a closed form in which the number of terms does not vary, if possible. (See the discussion of closed form and open form following Example 3 in Section 5.4.) To do this, we multiply both sides of (3) by $\frac{1}{10}$ to obtain

$$\frac{1}{10}s_n = \frac{3}{10^2} + \frac{3}{10^3} + \cdots + \frac{3}{10^n} + \frac{3}{10^{n+1}} \tag{4}$$

and then subtract (4) from (3) to obtain

$$s_n - \frac{1}{10}s_n = \frac{3}{10} - \frac{3}{10^{n+1}}$$

$$\frac{9}{10}s_n = \frac{3}{10}\left(1 - \frac{1}{10^n}\right)$$

$$s_n = \frac{1}{3}\left(1 - \frac{1}{10^n}\right)$$

Since $1/10^n \to 0$ as $n \to +\infty$, it follows that

$$\lim_{n\to+\infty} s_n = \lim_{n\to+\infty} \frac{1}{3}\left(1 - \frac{1}{10^n}\right) = \frac{1}{3}$$

which we denote by writing

$$\frac{1}{3} = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots + \frac{3}{10^n} + \cdots$$

Motivated by the preceding example, we are now ready to define the general concept of the "sum" of an infinite series

$$u_1 + u_2 + u_3 + \cdots + u_k + \cdots$$

We begin with some terminology: Let s_n denote the sum of the initial terms of the series, up to and including the term with index n. Thus,

$$s_1 = u_1$$
$$s_2 = u_1 + u_2$$
$$s_3 = u_1 + u_2 + u_3$$
$$\vdots$$
$$s_n = u_1 + u_2 + u_3 + \cdots + u_n = \sum_{k=1}^{n} u_k$$

The number s_n is called the ***nth partial sum*** of the series and the sequence $\{s_n\}_{n=1}^{+\infty}$ is called the ***sequence of partial sums***.

WARNING. In everyday language the words "sequence" and "series" are often used interchangeably. However, this is not so in mathematics—mathematically, a sequence is a *succession* and a series is a *sum*. It is essential that you keep this distinction in mind.

As n increases, the partial sum $s_n = u_1 + u_2 + \cdots + u_n$ includes more and more terms of the series. Thus, if s_n tends toward a limit as $n \to +\infty$, it is reasonable to view this limit as the sum of *all* the terms in the series. This suggests the following definition.

10.4.2 DEFINITION. Let $\{s_n\}$ be the sequence of partial sums of the series

$$u_1 + u_2 + u_3 + \cdots + u_k + \cdots$$

If the sequence $\{s_n\}$ converges to a limit S, then the series is said to ***converge*** to S, and S is called the ***sum*** of the series. We denote this by writing

$$S = \sum_{k=1}^{\infty} u_k$$

If the sequence of partial sums diverges, then the series is said to ***diverge***. A divergent series has no sum.

REMARK. Sometimes it will be desirable to start the summation index in an infinite series at $k = 0$ rather than $k = 1$, in which case we will view u_0 as the zeroth term and $s_0 = u_0$ as the zeroth partial sum. It can be proved that changing the starting value for the index has no effect on the convergence or divergence of an infinite series.

Example 1 Determine whether the series

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

converges or diverges. If it converges, find the sum.

Solution. It is tempting to conclude that the sum of the series is zero by arguing that the positive and negative terms cancel one another. However, this is *not correct*; the problem is that algebraic operations that hold for finite sums do not carry over to infinite series in all cases. Later, we will discuss conditions under which familiar algebraic operations can be applied to infinite series, but for this example we turn directly to Definition 10.4.2. The partial sums are

$$\begin{aligned} s_1 &= 1 \\ s_2 &= 1 - 1 = 0 \\ s_3 &= 1 - 1 + 1 = 1 \\ s_4 &= 1 - 1 + 1 - 1 = 0 \end{aligned}$$

and so forth. Thus, the sequence of partial sums is

$$1, 0, 1, 0, 1, 0, \ldots$$

Since this is a divergent sequence, the given series diverges and consequently has no sum. ◀

GEOMETRIC SERIES

In many important geometric series, each term is obtained by multiplying the preceding term by some fixed constant. Thus, if the initial term of the series is a and each term is obtained by multiplying the preceding term by r, then the series has the form

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \cdots + ar^k + \cdots \qquad (a \neq 0)$$

Such series are called ***geometric series***, and the number r is called the ***ratio*** for the series. Here are some examples:

$$1 + 2 + 4 + 8 + \cdots + 2^k + \cdots \qquad \boxed{a = 1, r = 2}$$

$$\frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots + \frac{3}{10^k} + \cdots \qquad \boxed{a = \tfrac{3}{10}, r = \tfrac{1}{10}}$$

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots + (-1)^{k+1}\frac{1}{2^k} + \cdots \qquad \boxed{a = \tfrac{1}{2}, r = -\tfrac{1}{2}}$$

$$1 + 1 + 1 + \cdots + 1 + \cdots \qquad \boxed{a = 1, r = 1}$$

$$1 - 1 + 1 - 1 + \cdots + (-1)^{k+1} + \cdots \qquad \boxed{a = 1, r = -1}$$

$$1 + x + x^2 + x^3 + \cdots + x^k + \cdots \qquad \boxed{a = 1, r = x}$$

The following theorem is the fundamental result on convergence of geometric series.

> **10.4.3 THEOREM.** *A geometric series*
>
> $$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + \cdots + ar^k + \cdots \qquad (a \neq 0)$$
>
> *converges if* $|r| < 1$ *and diverges if* $|r| \geq 1$. *If the series converges, then the sum is*
>
> $$\sum_{k=0}^{\infty} ar^k = \frac{a}{1 - r}$$

Proof. Let us treat the case $|r| = 1$ first. If $r = 1$, then the series is

$$a + a + a + a + \cdots$$

so the nth partial sum is $s_n = (n + 1)a$ and $\lim_{n\to+\infty} s_n = \lim_{n\to+\infty}(n + 1)a = \pm\infty$ (the sign depending on whether a is positive or negative). This proves divergence. If $r = -1$, the series is

$$a - a + a - a + \cdots$$

so the sequence of partial sums is

$$a, 0, a, 0, a, 0, \ldots$$

which diverges.

Now let us consider the case where $|r| \neq 1$. The nth partial sum of the series is

$$s_n = a + ar + ar^2 + \cdots + ar^n \tag{5}$$

Multiplying both sides of (5) by r yields

$$rs_n = ar + ar^2 + \cdots + ar^n + ar^{n+1} \tag{6}$$

and subtracting (6) from (5) gives

$$s_n - rs_n = a - ar^{n+1}$$

or

$$(1 - r)s_n = a - ar^{n+1} \tag{7}$$

Since $r \neq 1$ in the case we are considering, this can be rewritten as

$$s_n = \frac{a - ar^{n+1}}{1 - r} = \frac{a}{1 - r} - \frac{ar^{n+1}}{1 - r} \tag{8}$$

If $|r| < 1$, then $\lim_{n\to+\infty} r^{n+1} = 0$ (can you see why?), so $\{s_n\}$ converges. From (8)

$$\lim_{n\to+\infty} s_n = \frac{a}{1 - r}$$

If $|r| > 1$, then either $r > 1$ or $r < -1$. In the case $r > 1$, $\lim_{n\to+\infty} r^{n+1} = +\infty$, and in the case $r < -1$, r^{n+1} oscillates between positive and negative values that grow in magnitude, so $\{s_n\}$ diverges in both cases. ∎

Example 2 The series

$$\sum_{k=0}^{\infty} \frac{5}{4^k} = 5 + \frac{5}{4} + \frac{5}{4^2} + \cdots + \frac{5}{4^k} + \cdots$$

is a geometric series with $a = 5$ and $r = \frac{1}{4}$. Since $|r| = \frac{1}{4} < 1$, the series converges and the sum is

$$\frac{a}{1 - r} = \frac{5}{1 - \frac{1}{4}} = \frac{20}{3}$$ ◀

Example 3 Find the rational number represented by the repeating decimal

$$0.784784784\ldots$$

Solution. We can write

$$0.784784784\ldots = 0.784 + 0.000784 + 0.000000784 + \cdots$$

so the given decimal is the sum of a geometric series with $a = 0.784$ and $r = 0.001$. Thus,

$$0.784784784\ldots = \frac{a}{1 - r} = \frac{0.784}{1 - 0.001} = \frac{0.784}{0.999} = \frac{784}{999}$$ ◀

Example 4 In each part, determine whether the series converges, and if so find its sum.

(a) $\sum_{k=1}^{\infty} 3^{2k} 5^{1-k}$ (b) $\sum_{k=0}^{\infty} x^k$

Solution (a). This is a geometric series in a concealed form, since we can rewrite it as

$$\sum_{k=1}^{\infty} 3^{2k}5^{1-k} = \sum_{k=1}^{\infty} \frac{9^k}{5^{k-1}} = \sum_{k=1}^{\infty} 9\left(\frac{9}{5}\right)^{k-1}$$

Since $r = \frac{9}{5} > 1$, the series diverges.

Solution (b). The expanded form of the series is

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots + x^k + \cdots$$

The series is a geometric series with $a = 1$ and $r = x$, so it converges if $|x| < 1$ and diverges otherwise. When the series converges its sum is

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$ ◀

TELESCOPING SUMS

Example 5 Determine whether the series

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots$$

converges or diverges. If it converges, find the sum.

Solution. The nth partial sum of the series is

$$s_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)}$$

To calculate $\lim_{n\to+\infty} s_n$ we will rewrite s_n in closed form. This can be accomplished by using the method of partial fractions to obtain (verify)

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

from which we obtain the telescoping sum

$$\begin{aligned} s_n &= \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 + \left(-\frac{1}{2} + \frac{1}{2}\right) + \left(-\frac{1}{3} + \frac{1}{3}\right) + \cdots + \left(-\frac{1}{n} + \frac{1}{n}\right) - \frac{1}{n+1} \\ &= 1 - \frac{1}{n+1} \end{aligned}$$

so

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \lim_{n\to+\infty} s_n = \lim_{n\to+\infty}\left(1 - \frac{1}{n+1}\right) = 1$$ ◀

FOR THE READER. If you have a CAS, read the documentation to determine how to find sums of infinite series; then use the CAS to check the results in Example 5.

HARMONIC SERIES

One of the most important of all diverging series is the ***harmonic series***,

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

which arises in connection with the overtones produced by a vibrating musical string. It is not immediately evident that this series diverges. However, the divergence will become apparent when we examine the partial sums in detail. Because the terms in the series are all positive, the partial sums

$$s_1 = 1, \quad s_2 = 1 + \tfrac{1}{2}, \quad s_3 = 1 + \tfrac{1}{2} + \tfrac{1}{3}, \quad s_4 = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4}, \ldots$$

form a strictly increasing sequence

$$s_1 < s_2 < s_3 < \cdots < s_n < \cdots$$

Thus, by Theorem 10.3.3 we can prove divergence by demonstrating that there is no constant M that is greater than or equal to *every* partial sum. To this end, we will consider some selected partial sums, namely $s_2, s_4, s_8, s_{16}, s_{32}, \ldots$. Note that the subscripts are successive powers of 2, so that these are the partial sums of the form s_{2^n}. These partial sums satisfy the inequalities

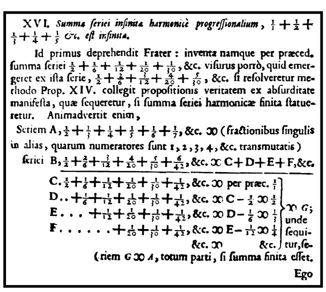
$$\begin{aligned}
s_2 &= 1 + \tfrac{1}{2} > \tfrac{1}{2} + \tfrac{1}{2} = \tfrac{2}{2} \\
s_4 &= s_2 + \tfrac{1}{3} + \tfrac{1}{4} > s_2 + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) = s_2 + \tfrac{1}{2} > \tfrac{3}{2} \\
s_8 &= s_4 + \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8} > s_4 + \left(\tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8}\right) = s_4 + \tfrac{1}{2} > \tfrac{4}{2} \\
s_{16} &= s_8 + \tfrac{1}{9} + \tfrac{1}{10} + \tfrac{1}{11} + \tfrac{1}{12} + \tfrac{1}{13} + \tfrac{1}{14} + \tfrac{1}{15} + \tfrac{1}{16} \\
&\quad > s_8 + \left(\tfrac{1}{16} + \tfrac{1}{16} + \tfrac{1}{16} + \tfrac{1}{16} + \tfrac{1}{16} + \tfrac{1}{16} + \tfrac{1}{16} + \tfrac{1}{16}\right) = s_8 + \tfrac{1}{2} > \tfrac{5}{2} \\
&\ \vdots \\
s_{2^n} &> \frac{n+1}{2}
\end{aligned}$$

If M is any constant, we can find a positive integer n such that $(n+1)/2 > M$. But for this n

$$s_{2^n} > \frac{n+1}{2} > M$$

so that no constant M is greater than or equal to *every* partial sum of the harmonic series. This proves divergence.

This divergence proof, which predates the discovery of calculus, is due to a French bishop and teacher, Nicole Oresme (1323–1382). This series eventually attracted the interest of Johann and Jakob Bernoulli (p. 94 and led them to begin thinking about the general concept of convergence, which was a new idea at that time.


XVI. *Summa seriei infinitæ harmonicè progressionalium*, $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$ *&c. est infinita.*

Id primus deprehendit Frater: inventa namque per præced. summa seriei $\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}$, &c. visurus porrò, quid emergeret ex ista serie, $\frac{1}{2}+\frac{2}{6}+\frac{3}{12}+\frac{4}{20}+\frac{5}{30}$, &c. si resolveretur methodo Prop. XIV. collegit propositionis veritatem ex absurditate manifesta, quæ sequeretur, si summa seriei harmonicæ finita statueretur. Animadvertit enim,

Seriem A, $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}$, &c. ∞ (fractionibus singulis in alias, quarum numeratores sunt 1, 2, 3, 4, &c. transmutatis)

seriei B, $\frac{1}{2}+\frac{2}{6}+\frac{3}{12}+\frac{4}{20}+\frac{5}{30}+\frac{6}{42}$, &c. ∞ C + D + E + F, &c.

C. $\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}+\frac{1}{42}$, &c. ∞ per præc. $\frac{1}{1}$
D. . $+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}+\frac{1}{42}$, &c. ∞ C − $\frac{1}{2}$ ∞ $\frac{1}{2}$
E . . . $+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}+\frac{1}{42}$, &c. ∞ D − $\frac{1}{6}$ ∞ $\frac{1}{3}$
F . . . . . . $+\frac{1}{20}+\frac{1}{30}+\frac{1}{42}$, &c. ∞ E − $\frac{1}{12}$ ∞ $\frac{1}{4}$
&c. ∞ &c.
} ∞ G; unde sequitur, seriem G ∞ A, totum parti, si summa finita esset.

Ego

This is a proof of the divergence of the harmonic series, as it appeared in an appendix of Jakob Bernoulli's posthumous publication, *Ars Conjectandi*, which appeared in 1713.

EXERCISE SET 10.4 [C] CAS

1. In each part, find exact values for the first four partial sums, find a closed form for the nth partial sum, and determine whether the series converges by calculating the limit of the nth partial sum. If the series converges, then state its sum.
   (a) $2 + \dfrac{2}{5} + \dfrac{2}{5^2} + \cdots + \dfrac{2}{5^{k-1}} + \cdots$
   (b) $\dfrac{1}{4} + \dfrac{2}{4} + \dfrac{2^2}{4} + \cdots + \dfrac{2^{k-1}}{4} + \cdots$
   (c) $\dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \dfrac{1}{4 \cdot 5} + \cdots + \dfrac{1}{(k+1)(k+2)} + \cdots$

2. In each part, find exact values for the first four partial sums, find a closed form for the nth partial sum, and determine whether the series converges by calculating the limit of the nth partial sum. If the series converges, then state its sum.
   (a) $\displaystyle\sum_{k=1}^{\infty} \left(\frac{1}{4}\right)^k$ (b) $\displaystyle\sum_{k=1}^{\infty} 4^{k-1}$ (c) $\displaystyle\sum_{k=1}^{\infty} \left(\frac{1}{k+3} - \frac{1}{k+4}\right)$

In Exercises 3–14, determine whether the series converges, and if so, find its sum.

3. $\sum_{k=1}^{\infty}\left(-\frac{3}{4}\right)^{k-1}$

4. $\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^{k+2}$

5. $\sum_{k=1}^{\infty}(-1)^{k-1}\frac{7}{6^{k-1}}$

6. $\sum_{k=1}^{\infty}\left(-\frac{3}{2}\right)^{k+1}$

7. $\sum_{k=1}^{\infty}\frac{1}{(k+2)(k+3)}$

8. $\sum_{k=1}^{\infty}\left(\frac{1}{2^k}-\frac{1}{2^{k+1}}\right)$

9. $\sum_{k=1}^{\infty}\frac{1}{9k^2+3k-2}$

10. $\sum_{k=2}^{\infty}\frac{1}{k^2-1}$

11. $\sum_{k=3}^{\infty}\frac{1}{k-2}$

12. $\sum_{k=5}^{\infty}\left(\frac{e}{\pi}\right)^{k-1}$

13. $\sum_{k=1}^{\infty}\frac{4^{k+2}}{7^{k-1}}$

14. $\sum_{k=1}^{\infty}5^{3k}7^{1-k}$

In Exercises 15–20, express the given repeating decimal as a fraction.

15. 0.4444 . . .

16. 0.9999 . . .

17. 5.373737 . . .

18. 0.159159159 . . .

19. 0.782178217821 . . .

20. 0.451141414 . . .

21. A ball is dropped from a height of 10 m. Each time it strikes the ground it bounces vertically to a height that is $\frac{3}{4}$ of the preceding height. Find the total distance the ball will travel if it is assumed to bounce infinitely often.

22. The accompanying figure shows an "infinite staircase" constructed from cubes. Find the total volume of the staircase, given that the largest cube has a side of length 1 and each successive cube has a side whose length is half that of the preceding cube.

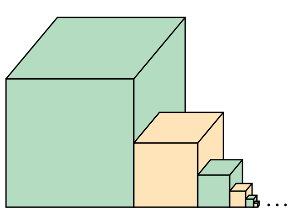


Figure Ex-22

23. In each part, find a closed form for the nth partial sum of the series, and determine whether the series converges. If so, find its sum.

(a) $\ln\frac{1}{2}+\ln\frac{2}{3}+\ln\frac{3}{4}+\cdots+\ln\frac{n}{n+1}+\cdots$

(b) $\ln\left(1-\frac{1}{4}\right)+\ln\left(1-\frac{1}{9}\right)+\ln\left(1-\frac{1}{16}\right)+\cdots$

$$+\ln\left(1-\frac{1}{(k+1)^2}\right)+\cdots$$

24. Use geometric series to show that

(a) $\sum_{k=0}^{\infty}(-1)^k x^k=\frac{1}{1+x}$ if $-1<x<1$

(b) $\sum_{k=0}^{\infty}(x-3)^k=\frac{1}{4-x}$ if $2<x<4$

(c) $\sum_{k=0}^{\infty}(-1)^k x^{2k}=\frac{1}{1+x^2}$ if $-1<x<1$.

25. In each part, find all values of x for which the series converges, and find the sum of the series for those values of x.

(a) $x-x^3+x^5-x^7+x^9-\cdots$

(b) $\frac{1}{x^2}+\frac{2}{x^3}+\frac{4}{x^4}+\frac{8}{x^5}+\frac{16}{x^6}+\cdots$

(c) $e^{-x}+e^{-2x}+e^{-3x}+e^{-4x}+e^{-5x}+\cdots$

26. Show: $\sum_{k=1}^{\infty}\frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k^2+k}}=1$.

27. Show: $\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+2}\right)=\frac{3}{2}$.

28. Show: $\frac{1}{1\cdot 3}+\frac{1}{2\cdot 4}+\frac{1}{3\cdot 5}+\cdots=\frac{3}{4}$.

29. Show: $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots=\frac{1}{2}$.

30. Show that for all real values of x

$$\sin x-\frac{1}{2}\sin^2 x+\frac{1}{4}\sin^3 x-\frac{1}{8}\sin^4 x+\cdots=\frac{2\sin x}{2+\sin x}$$

31. Let a_1 be any real number, and let $\{a_n\}$ be the sequence defined recursively by

$$a_{n+1}=\tfrac{1}{2}(a_n+1)$$

Make a conjecture about the limit of the sequence, and confirm your conjecture by expressing a_n in terms of a_1 and taking the limit.

32. Recall that a *terminating decimal* is a decimal whose digits are all 0 from some point on ($0.5 = 0.50000\ldots$, for example). Show that a decimal of the form $0.a_1a_2\ldots a_n9999\ldots$, where $a_n \neq 9$, can be expressed as a terminating decimal.

33. The great Swiss mathematician Leonhard Euler (biography on p. 11) sometimes reached incorrect conclusions in his pioneering work on infinite series. For example, Euler deduced that

$$\tfrac{1}{2}=1-1+1-1+\cdots$$

and

$$-1=1+2+4+8+\cdots$$

by substituting $x=-1$ and $x=2$ in the formula

$$\frac{1}{1-x}=1+x+x^2+x^3+\cdots$$

What was the problem with his reasoning?

34. As shown in the accompanying figure, suppose that lines L_1 and L_2 form an angle θ, $0<\theta<\pi/2$, at their point of intersection P. A point P_0 is chosen that is on L_1 and a units from P. Starting from P_0 a zig-zag path is constructed by successively going back and forth between L_1 and L_2

along a perpendicular from one line to the other. Find the following sums in terms of θ.

(a) $P_0P_1 + P_1P_2 + P_2P_3 + \cdots$

(b) $P_0P_1 + P_2P_3 + P_4P_5 + \cdots$

(c) $P_1P_2 + P_3P_4 + P_5P_6 + \cdots$

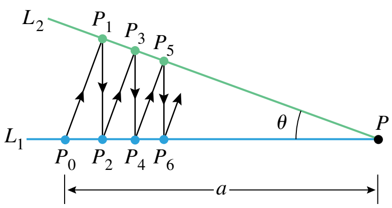


Figure Ex-34

35. As shown in the accompanying figure, suppose that an angle θ is bisected using a straightedge and compass to produce ray R_1, then the angle between R_1 and the initial side is bisected to produce ray R_2. Thereafter, rays R_3, R_4, $R_5, \ldots$ are constructed in succession by bisecting the angle between the preceding two rays. Show that the sequence of angles that these rays make with the initial side has a limit of $\theta/3$. [This problem is based on *Trisection of an Angle in an Infinite Number of Steps* by Eric Kincannon, which appeared in *The College Mathematics Journal*, Vol. 21, No. 5, November 1990.]

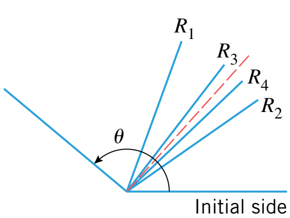


Figure Ex-35

36. In his *Treatise on the Configurations of Qualities and Motions* (written in the 1350s), the French Bishop of Lisieux, Nicole Oresme, used a geometric method to find the sum of the series

$$\sum_{k=1}^{\infty} \frac{k}{2^k} = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \cdots$$

In part (*a*) of the accompanying figure, each term in the series is represented by the area of a rectangle, and in part (*b*) the configuration in part (*a*) has been divided into rectangles with areas $A_1, A_2, A_3, \ldots$. Find the sum $A_1 + A_2 + A_3 + \cdots$.

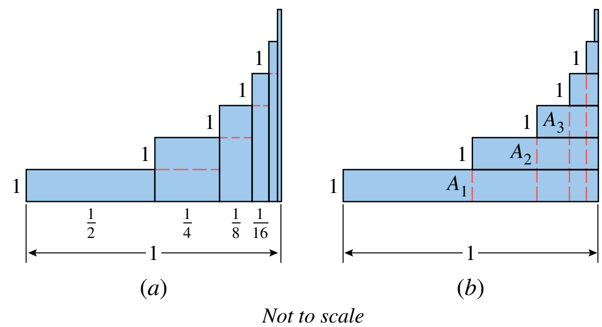


Figure Ex-36

C **37.** (a) See if your CAS can find the sum of the series

$$\sum_{k=1}^{\infty} \frac{6^k}{(3^{k+1} - 2^{k+1})(3^k - 2^k)}$$

(b) Find A and B such that

$$\frac{6^k}{(3^{k+1} - 2^{k+1})(3^k - 2^k)} = \frac{2^k A}{3^k - 2^k} + \frac{2^k B}{3^{k+1} - 2^{k+1}}$$

(c) Use the result in part (b) to find a closed form for the nth partial sum, and then find the sum of the series.

[This exercise is adapted from a problem that appeared in the Forty-Fifth Annual William Lowell Putnam Competition.]

C **38.** In each part, use a CAS to find the sum of the series if it converges, and then confirm the result by hand calculation.

(a) $\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} 2^k 3^{2-k}$ (b) $\displaystyle\sum_{k=1}^{\infty} \frac{3^{3k}}{5^{k-1}}$ (c) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{4k^2 - 1}$

10.5 CONVERGENCE TESTS

In the last section we showed how to find the sum of a series by finding a closed form for the nth partial sum and taking its limit. However, it is relatively rare that one can find a closed form for the nth partial sum of a series, so alternative methods are needed for finding the sum of a series. One possibility is to prove that the series converges, and then to approximate the sum by a partial sum with sufficiently many terms to achieve the desired degree of accuracy. In this section we will develop various tests that can be used to determine whether a given series converges or diverges.

THE DIVERGENCE TEST

In stating general results about convergence or divergence of series, it is convenient to use the notation $\sum u_k$ as a generic template for a series, thus avoiding the issue of whether the sum begins with $k = 0$ or $k = 1$ or some other value. Indeed, we will see shortly that the starting index value is irrelevant to the issue of convergence. The kth term in an infinite series $\sum u_k$ is called the ***general term*** of the series. The following theorem establishes a relationship between the limit of the general term and the convergence properties of a series.

> **10.5.1 THEOREM (*The Divergence Test*).**
> (*a*) *If* $\lim_{k\to+\infty} u_k \neq 0$, *then the series* $\sum u_k$ *diverges.*
> (*b*) *If* $\lim_{k\to+\infty} u_k = 0$, *then the series* $\sum u_k$ *may either converge or diverge.*

Proof (a). To prove this result, it suffices to show that if the series converges, then $\lim_{k\to+\infty} u_k = 0$ (why?). We will prove this alternative form of (*a*).

Let us assume that the series converges. The general term u_k can be written as

$$u_k = s_k - s_{k-1} \tag{1}$$

where s_k is the sum of the terms through u_k and s_{k-1} is the sum of the terms through u_{k-1}. If S denotes the sum of the series, then $\lim_{k\to+\infty} s_k = S$, and since $(k-1)\to+\infty$ as $k\to+\infty$, we also have $\lim_{k\to+\infty} s_{k-1} = S$. Thus, from (1)

$$\lim_{k\to+\infty} u_k = \lim_{k\to+\infty} (s_k - s_{k-1}) = S - S = 0$$

Proof (b). To prove this result, it suffices to produce both a convergent series and a divergent series for which $\lim_{k\to+\infty} u_k = 0$. The following series both have this property:

$$\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^k} + \cdots \quad \text{and} \quad 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} + \cdots$$

The first is a convergent geometric series and the second is the divergent harmonic series. ∎

The alternative form of part (*a*) given in the preceding proof is sufficiently important that we state it separately for future reference.

> **10.5.2 THEOREM.** *If the series* $\sum u_k$ *converges, then* $\lim_{k\to+\infty} u_k = 0$.

Example 1 The series

$$\sum_{k=1}^{\infty} \frac{k}{k+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{k}{k+1} + \cdots$$

diverges since

$$\lim_{k\to+\infty} \frac{k}{k+1} = \lim_{k\to+\infty} \frac{1}{1+1/k} = 1 \neq 0$$ ◀

WARNING. The converse of Theorem 10.5.2 is false. To prove that a series converges it does not suffice to show that $\lim_{k\to+\infty} u_k = 0$, since this property may hold for divergent as well as convergent series, as we saw in the proof of part (*b*) of Theorem 10.5.1.

ALGEBRAIC PROPERTIES OF INFINITE SERIES

For brevity, the proof of the following result is omitted.

10.5.3 THEOREM.

(a) *If* $\sum u_k$ *and* $\sum v_k$ *are convergent series, then* $\sum(u_k + v_k)$ *and* $\sum(u_k - v_k)$ *are convergent series and the sums of these series are related by*

$$\sum_{k=1}^{\infty}(u_k + v_k) = \sum_{k=1}^{\infty} u_k + \sum_{k=1}^{\infty} v_k$$

$$\sum_{k=1}^{\infty}(u_k - v_k) = \sum_{k=1}^{\infty} u_k - \sum_{k=1}^{\infty} v_k$$

(b) *If c is a nonzero constant, then the series* $\sum u_k$ *and* $\sum cu_k$ *both converge or both diverge. In the case of convergence, the sums are related by*

$$\sum_{k=1}^{\infty} cu_k = c\sum_{k=1}^{\infty} u_k$$

(c) *Convergence or divergence is unaffected by deleting a finite number of terms from a series; in particular, for any positive integer* K, *the series*

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + u_3 + \cdots$$

$$\sum_{k=K}^{\infty} u_k = u_K + u_{K+1} + u_{K+2} + \cdots$$

both converge or both diverge.

REMARK. Do not read too much into part (*c*) of this theorem. Although the convergence is not affected when a finite number of terms is deleted from the beginning of a convergent series, the *sum* of a convergent series is changed by the removal of these terms.

Example 2 Find the sum of the series

$$\sum_{k=1}^{\infty}\left(\frac{3}{4^k} - \frac{2}{5^{k-1}}\right)$$

Solution. The series

$$\sum_{k=1}^{\infty}\frac{3}{4^k} = \frac{3}{4} + \frac{3}{4^2} + \frac{3}{4^3} + \cdots$$

is a convergent geometric series $\left(a = \frac{3}{4}, r = \frac{1}{4}\right)$, and the series

$$\sum_{k=1}^{\infty}\frac{2}{5^{k-1}} = 2 + \frac{2}{5} + \frac{2}{5^2} + \frac{2}{5^3} + \cdots$$

is also a convergent geometric series $\left(a = 2, r = \frac{1}{5}\right)$. Thus, from Theorems 10.5.3(*a*) and 10.4.3 the given series converges and

$$\sum_{k=1}^{\infty}\left(\frac{3}{4^k} - \frac{2}{5^{k-1}}\right) = \sum_{k=1}^{\infty}\frac{3}{4^k} - \sum_{k=1}^{\infty}\frac{2}{5^{k-1}} = \frac{\frac{3}{4}}{1-\frac{1}{4}} - \frac{2}{1-\frac{1}{5}} = -\frac{3}{2}$$ ◀

Example 3 Determine whether the following series converge or diverge.

(a) $\displaystyle\sum_{k=1}^{\infty}\frac{5}{k} = 5 + \frac{5}{2} + \frac{5}{3} + \cdots + \frac{5}{k} + \cdots$ (b) $\displaystyle\sum_{k=10}^{\infty}\frac{1}{k} = \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \cdots$

Solution. The first series is a constant times the divergent harmonic series, and hence diverges by part (*b*) of Theorem 10.5.3. The second series results by deleting the first nine terms from the divergent harmonic series, and hence diverges by part (*c*) of Theorem 10.5.3. ◀

THE INTEGRAL TEST

The expressions

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \quad \text{and} \quad \int_1^{+\infty} \frac{1}{x^2}\,dx$$

are related in that the integrand in the improper integral results when the index k in the general term of the series is replaced by x and the limits of summation in the series are replaced by the corresponding limits of integration. The following theorem shows that there is a relationship between the convergence of the series and the integral.

10.5.4 THEOREM (*The Integral Test*). *Let $\sum u_k$ be a series with positive terms, and let $f(x)$ be the function that results when k is replaced by x in the general term of the series. If f is decreasing and continuous on the interval $[a, +\infty)$, then*

$$\sum_{k=1}^{\infty} u_k \quad \textit{and} \quad \int_a^{+\infty} f(x)\,dx$$

both converge or both diverge.

Example 4 Use the integral test to determine whether the following series converge or diverge.

$$\text{(a)} \ \sum_{k=1}^{\infty} \frac{1}{k} \qquad \text{(b)} \ \sum_{k=1}^{\infty} \frac{1}{k^2}$$

Solution (a). We already know that this is the divergent harmonic series, so the integral test will simply provide another way of establishing the divergence. If we replace k by x in the general term $1/k$, we obtain the function $f(x) = 1/x$, which is decreasing and continuous for $x \geq 1$ (as required to apply the integral test with $a = 1$). Since

$$\int_1^{+\infty} \frac{1}{x}\,dx = \lim_{\ell \to +\infty} \int_1^{\ell} \frac{1}{x}\,dx = \lim_{\ell \to +\infty} [\ln \ell - \ln 1] = +\infty$$

the integral diverges and consequently so does the series.

Solution (b). If we replace k by x in the general term $1/k^2$, we obtain the function $f(x) = 1/x^2$, which is decreasing and continuous for $x \geq 1$. Since

$$\int_1^{+\infty} \frac{1}{x^2}\,dx = \lim_{\ell \to +\infty} \int_1^{\ell} \frac{dx}{x^2} = \lim_{\ell \to +\infty} \left[-\frac{1}{x}\right]_1^{\ell} = \lim_{\ell \to +\infty} \left[1 - \frac{1}{\ell}\right] = 1$$

the integral converges and consequently the series converges by the integral test with $a = 1$. ◀

REMARK. In part (b) of the last example, do *not* erroneously conclude that the sum of the series is 1 because the value of the corresponding integral is 1. It can be proved that the sum of the series is actually $\pi^2/6$ and, indeed, the sum of the first two terms alone exceeds 1.

***p*-SERIES**

The series in Example 4 are special cases of a class of series called ***p-series*** or ***hyperharmonic series***. A p-series is an infinite series of the form

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{k^p} + \cdots$$

where $p > 0$. Examples of p-series are

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} + \cdots \qquad p = 1$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} + \cdots \qquad p = 2$$

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} + \cdots \qquad p = \tfrac{1}{2}$$

The following theorem tells when a p-series converges.

10.5.5 THEOREM (*Convergence of p-Series*).

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{k^p} + \cdots$$

converges if $p > 1$ and diverges if $0 < p \le 1$.

Proof. To establish this result when $p \neq 1$, we will use the integral test.

$$\int_1^{+\infty} \frac{1}{x^p}\, dx = \lim_{\ell \to +\infty} \int_1^{\ell} x^{-p}\, dx = \lim_{\ell \to +\infty} \frac{x^{1-p}}{1-p}\bigg]_1^{\ell} = \lim_{\ell \to +\infty} \left[\frac{\ell^{1-p}}{1-p} - \frac{1}{1-p}\right]$$

If $p > 1$, then $1 - p < 0$, so $\ell^{1-p} \to 0$ as $\ell \to +\infty$. Thus, the integral converges [its value is $-1/(1-p)$] and consequently the series also converges. For $0 < p < 1$, it follows that $1 - p > 0$ and $\ell^{1-p} \to +\infty$ as $\ell \to +\infty$, so the integral and the series diverge. The case $p = 1$ is the harmonic series, which was previously shown to diverge. ∎

Example 5

$$1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \cdots + \frac{1}{\sqrt[3]{k}} + \cdots$$

diverges since it is a p-series with $p = \frac{1}{3} < 1$. ◀

PROOF OF THE INTEGRAL TEST

Before we can prove the integral test, we need a basic result about convergence of series with *nonnegative* terms. If $u_1 + u_2 + u_3 + \cdots + u_k + \cdots$ is such a series, then its sequence of partial sums is increasing, that is,

$$s_1 \le s_2 \le s_3 \le \cdots \le s_n \le \cdots$$

Thus, from Theorem 10.3.3 the sequence of partial sums converges to a limit S if and only if it has some upper bound M, in which case $S \le M$. If no upper bound exists, then the sequence of partial sums diverges. Since convergence of the sequence of partial sums corresponds to convergence of the series, we have the following theorem.

10.5.6 THEOREM. *If $\sum u_k$ is a series with nonnegative terms, and if there is a constant M such that*

$$s_n = u_1 + u_2 + \cdots + u_n \le M$$

for every n, then the series converges and the sum S satisfies $S \le M$. If no such M exists, then the series diverges.

In words, this theorem implies that *a series with nonnegative terms converges if and only if its sequence of partial sums is bounded above*.

Proof of Theorem 10.5.4. We need only show that the series converges when the integral converges and that the series diverges when the integral diverges. For simplicity, we will limit the proof to the case where $a = 1$. Assume that $f(x)$ satisfies the hypotheses of the theorem for $x \geq 1$. Since

$$f(1) = u_1,\ f(2) = u_2, \ldots, f(n) = u_n, \ldots$$

the values of $u_1, u_2, \ldots, u_n, \ldots$ can be interpreted as the areas of the rectangles shown in Figure 10.5.1.

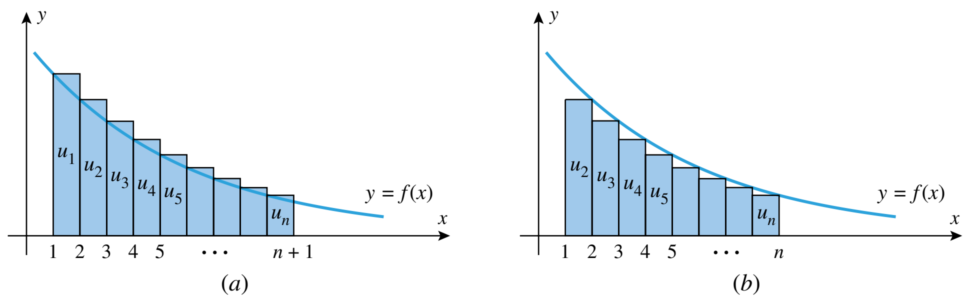


Figure 10.5.1

The following inequalities result by comparing the areas under the curve $y = f(x)$ to the areas of the rectangles in Figure 10.5.1 for $n > 1$:

$$\int_1^{n+1} f(x)\,dx < u_1 + u_2 + \cdots + u_n = s_n \qquad \text{Figure 10.5.1}a$$

$$s_n - u_1 = u_2 + u_3 + \cdots + u_n < \int_1^n f(x)\,dx \qquad \text{Figure 10.5.1}b$$

These inequalities can be combined as

$$\int_1^{n+1} f(x)\,dx < s_n < u_1 + \int_1^n f(x)\,dx \tag{2}$$

If the integral $\int_1^{\infty} f(x)\,dx$ converges to a finite value L, then from the right-hand inequality in (2)

$$s_n < u_1 + \int_1^n f(x)\,dx < u_1 + \int_1^{\infty} f(x)\,dx = u_1 + L$$

Thus, each partial sum is less than the finite constant $u_1 + L$, and the series converges by Theorem 10.5.6. On the other hand, if the integral $\int_1^{\infty} f(x)\,dx$ diverges, then

$$\lim_{n\to+\infty} \int_1^{n+1} f(x)\,dx = +\infty$$

so that from the left-hand inequality in (2), $\lim_{n\to+\infty} s_n = +\infty$. This implies that the series also diverges. ■

EXERCISE SET 10.5 Graphing Utility C CAS

1. In each part, use Theorem 10.5.3 to find the sum of the series.

(a) $\left(\frac{1}{2} + \frac{1}{4}\right) + \left(\frac{1}{2^2} + \frac{1}{4^2}\right) + \cdots + \left(\frac{1}{2^k} + \frac{1}{4^k}\right) + \cdots$

(b) $\sum_{k=1}^{\infty} \left(\frac{1}{5^k} - \frac{1}{k(k+1)}\right)$

2. In each part, use Theorem 10.5.3 to find the sum of the series.

(a) $\sum_{k=2}^{\infty} \left[\frac{1}{k^2 - 1} - \frac{7}{10^{k-1}}\right]$ (b) $\sum_{k=1}^{\infty} \left[7^{-k}3^{k+1} - \frac{2^{k+1}}{5^k}\right]$

In Exercises 3 and 4, various p-series are given. In each case, find p and determine whether the series converges.

3. (a) $\sum_{k=1}^{\infty} \frac{1}{k^3}$ (b) $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ (c) $\sum_{k=1}^{\infty} k^{-1}$ (d) $\sum_{k=1}^{\infty} k^{-2/3}$

4. (a) $\sum_{k=1}^{\infty} k^{-4/3}$ (b) $\sum_{k=1}^{\infty} \frac{1}{\sqrt[4]{k}}$ (c) $\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{k^5}}$ (d) $\sum_{k=1}^{\infty} \frac{1}{k^{\pi}}$

In Exercises 5 and 6, apply the divergence test, and state what it tells you about the series.

5. (a) $\sum_{k=1}^{\infty} \frac{k^2+k+3}{2k^2+1}$ (b) $\sum_{k=1}^{\infty} \left(1+\frac{1}{k}\right)^k$

(c) $\sum_{k=1}^{\infty} \cos k\pi$ (d) $\sum_{k=1}^{\infty} \frac{1}{k!}$

6. (a) $\sum_{k=1}^{\infty} \frac{k}{e^k}$ (b) $\sum_{k=1}^{\infty} \ln k$ (c) $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ (d) $\sum_{k=1}^{\infty} \frac{\sqrt{k}}{\sqrt{k}+3}$

In Exercises 7 and 8, confirm that the integral test is applicable, and use it to determine whether the series converges.

7. (a) $\sum_{k=1}^{\infty} \frac{1}{5k+2}$ (b) $\sum_{k=1}^{\infty} \frac{1}{1+9k^2}$

8. (a) $\sum_{k=1}^{\infty} \frac{k}{1+k^2}$ (b) $\sum_{k=1}^{\infty} \frac{1}{(4+2k)^{3/2}}$

In Exercises 9–24, use any method to determine whether the series converges.

9. $\sum_{k=1}^{\infty} \frac{1}{k+6}$ **10.** $\sum_{k=1}^{\infty} \frac{3}{5k}$ **11.** $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k+5}}$

12. $\sum_{k=1}^{\infty} \frac{1}{\sqrt[k]{e}}$ **13.** $\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{2k-1}}$ **14.** $\sum_{k=3}^{\infty} \frac{\ln k}{k}$

15. $\sum_{k=1}^{\infty} \frac{k}{\ln(k+1)}$ **16.** $\sum_{k=1}^{\infty} ke^{-k^2}$ **17.** $\sum_{k=1}^{\infty} \left(1+\frac{1}{k}\right)^{-k}$

18. $\sum_{k=1}^{\infty} \frac{k^2+1}{k^2+3}$ **19.** $\sum_{k=1}^{\infty} \frac{\tan^{-1} k}{1+k^2}$ **20.** $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k^2+1}}$

21. $\sum_{k=1}^{\infty} k^2 \sin^2\left(\frac{1}{k}\right)$ **22.** $\sum_{k=1}^{\infty} k^2 e^{-k^3}$

23. $\sum_{k=5}^{\infty} 7k^{-1.01}$ **24.** $\sum_{k=1}^{\infty} \operatorname{sech}^2 k$

In Exercises 25 and 26, use the integral test to investigate the relationship between the value of p and the convergence of the series.

25. $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^p}$ **26.** $\sum_{k=3}^{\infty} \frac{1}{k(\ln k)[\ln(\ln k)]^p}$

C **27.** Use a CAS to confirm that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}$$

and then use these results in each part to find the sum of the series.

(a) $\sum_{k=1}^{\infty} \frac{3k^2-1}{k^4}$ (b) $\sum_{k=3}^{\infty} \frac{1}{k^2}$ (c) $\sum_{k=2}^{\infty} \frac{1}{(k-1)^4}$

28. Suppose that the series $\sum u_k$ converges and the series $\sum v_k$ diverges.

(a) Show that the series $\sum(u_k+v_k)$ and $\sum(u_k-v_k)$ both diverge. [*Hint:* Assume that each series converges and use Theorem 10.5.3 to obtain a contradiction.]

(b) Find examples to show that if $\sum u_k$ and $\sum v_k$ both diverge, then the series $\sum(u_k+v_k)$ and $\sum(u_k-v_k)$ may either converge or diverge.

29. In each part, use the results in Exercise 28, if needed, to determine whether the series diverges.

(a) $\sum_{k=1}^{\infty} \left[\left(\frac{2}{3}\right)^{k-1} + \frac{1}{k}\right]$ (b) $\sum_{k=1}^{\infty} \left[\frac{1}{3k+2} - \frac{1}{k^{3/2}}\right]$

(c) $\sum_{k=2}^{\infty} \left[\frac{1}{k(\ln k)^2} - \frac{1}{k^2}\right]$

Exercise 30 will show how a partial sum can be used to obtain upper and lower bounds on the sum of the series when the hypotheses of the integral test are satisfied. This result will be needed in Exercises 31–35.

30. (a) Let $\sum_{k=1}^{\infty} u_k$ be a convergent series with positive terms, let $f(x)$ be the function that results when k is replaced by x in the general term of the series, and suppose that f satisfies the hypotheses of the integral test for $x \ge n$ (Theorem 10.5.4). Use an area argument and the accompanying figure (see page 681) to show that

$$\int_{n+1}^{+\infty} f(x)\,dx < \sum_{k=n+1}^{\infty} u_k < \int_{n}^{+\infty} f(x)\,dx$$

(b) Show that if S is the sum of the series $\sum_{k=1}^{\infty} u_k$ and s_n is the nth partial sum, then

$$s_n + \int_{n+1}^{+\infty} f(x)\,dx < S < s_n + \int_{n}^{+\infty} f(x)\,dx$$

31. (a) It was stated in Exercise 27 that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

Show that if s_n is the nth partial sum of this series, then

$$s_n + \frac{1}{n+1} < \frac{\pi^2}{6} < s_n + \frac{1}{n}$$

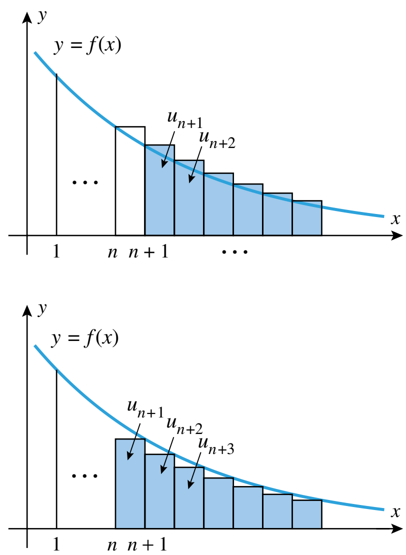


Figure Ex-30

(b) Calculate s_3 exactly, and then use the result in part (a) to show that

$$\frac{29}{18} < \frac{\pi^2}{6} < \frac{61}{36}$$

(c) Use a calculating utility to confirm that the inequalities in part (b) are correct.

(d) Find upper and lower bounds on the error that results if the sum of the series is approximated by the 10th partial sum.

33. In each part, find upper and lower bounds on the error that results if the sum of the series is approximated by the 10th partial sum.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(2k+1)^2}$ (b) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2+1}$ (c) $\displaystyle\sum_{k=1}^{\infty} \frac{k}{e^k}$

34. Our objective in this problem is to approximate the sum of the series $\sum_{k=1}^{\infty} 1/k^3$ to two decimal-place accuracy.

(a) Show that if S is the sum of the series and s_n is the nth partial sum, then

$$s_n + \frac{1}{2(n+1)^2} < S < s_n + \frac{1}{2n^2}$$

(b) For two decimal-place accuracy, the error must be less than 0.005 (see Table 2.5.1 on p. 154). We can achieve this by finding an interval of length 0.01 (or less) that contains S and approximating S by the midpoint of that interval. Find the smallest value of n such that the interval containing S in part (a) has a length of 0.01 or less.

(c) Approximate S to two decimal-place accuracy.

35. (a) Use the method of Exercise 33 to approximate the sum of the series $\sum_{k=1}^{\infty} 1/k^4$ to two decimal-place accuracy.

(b) It was stated in Exercise 27 that the sum of this series is $\pi^4/90$. Use a calculating utility to confirm that your answer in part (a) is accurate to two decimal places.

36. We showed in Section 10.4 that the harmonic series $\sum_{k=1}^{\infty} 1/k$ diverges. Our objective in this problem is to demonstrate that although the partial sums of this series approach $+\infty$, they increase extremely slowly.

(a) Use inequality (2) to show that for $n \geq 2$

$$\ln(n+1) < s_n < 1 + \ln n$$

(b) Use the inequalities in part (a) to find upper and lower bounds on the sum of the first million terms in the series.

(c) Show that the sum of the first billion terms in the series is less than 22.

(d) Find a value of n so that the sum of the first n terms is greater than 100.

37. Investigate the relationship between the value of a and the convergence of the series $\sum_{k=1}^{\infty} k^{-\ln a}$.

38. Use a graphing utility to confirm that the integral test applies to the series $\sum_{k=1}^{\infty} k^2 e^{-k}$, and then determine whether the series converges.

C **39.** (a) Show that the integral test applies to the series $\sum_{k=1}^{\infty} 1/(k^3+1)$.

(b) Use a CAS and the integral test to confirm that the series converges.

(c) Construct a table of partial sums for $n = 10, 20, 30, \ldots, 100$, showing at least six decimal places.

(d) Based on your table, make a conjecture about the sum of the series to three decimal-place accuracy.

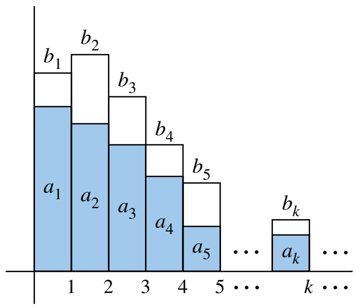
(e) Use part (b) of Exercise 30 to check your conjecture.

10.6 THE COMPARISON, RATIO, AND ROOT TESTS

In this section we will develop some more basic convergence tests for series with nonnegative terms. Later, we will use some of these tests to study the convergence of Taylor series.

THE COMPARISON TEST

We will begin with a test that is useful in its own right and is also the building block for other important convergence tests. The underlying idea of this test is to use the known convergence or divergence of a series to deduce the convergence or divergence of another series.



For each rectangle, b_k is the entire area and a_k is the area of the blue portion.

Figure 10.6.1

10.6.1 THEOREM (*The Comparison Test*). *Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be series with non-negative terms and suppose that*

$$a_1 \le b_1,\ a_2 \le b_2,\ a_3 \le b_3, \ldots, a_k \le b_k, \ldots$$

(a) *If the "bigger series" Σb_k converges, then the "smaller series" Σa_k also converges.*

(b) *If the "smaller series" Σa_k diverges, then the "bigger series" Σb_k also diverges.*

We have left the proof of this theorem for the exercises; however, it is easy to visualize why the theorem is true by interpreting the terms in the series as areas of rectangles (Figure 10.6.1). The comparison test states that if the total area $\sum b_k$ is finite, then the total area $\sum a_k$ must also be finite; and if the total area $\sum a_k$ is infinite, then the total area $\sum b_k$ must also be infinite.

REMARK. As one would expect, it is not essential in Theorem 10.6.1 that the condition $a_k \le b_k$ hold for all k, as stated; the conclusions of the theorem remain true if this condition is eventually true.

USING THE COMPARISON TEST

There are two steps required for using the comparison test to determine whether a series $\sum u_k$ with positive terms converges:

- Guess at whether the series $\sum u_k$ converges or diverges.
- Find a series that proves the guess to be correct. That is, if the guess is divergence, we must find a divergent series whose terms are "smaller" than the corresponding terms of $\sum u_k$, and if the guess is convergence, we must find a convergent series whose terms are "bigger" than the corresponding terms of $\sum u_k$.

In most cases, the series $\sum u_k$ being considered will have its general term u_k expressed as a fraction. To help with the guessing process in the first step, we have formulated two principles that are based on the form of the denominator for u_k. These principles sometimes *suggest* whether a series is likely to converge or diverge. We have called these "informal principles" because they are not intended as formal theorems. In fact, we will not guarantee that they *always* work. However, they work often enough to be useful.

10.6.2 INFORMAL PRINCIPLE. *Constant summands in the denominator of u_k can usually be deleted without affecting the convergence or divergence of the series.*

10.6.3 INFORMAL PRINCIPLE. *If a polynomial in k appears as a factor in the numerator or denominator of u_k, all but the leading term in the polynomial can usually be discarded without affecting the convergence or divergence of the series.*

Example 1 Use the comparison test to determine whether the following series converge or diverge.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k} - \frac{1}{2}}$ (b) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2k^2 + k}$

Solution (a). According to Principle 10.6.2, we should be able to drop the constant in the denominator without affecting the convergence or divergence. Thus, the given series is likely to behave like

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \qquad (1)$$

which is a divergent p-series $\left(p = \frac{1}{2}\right)$. Thus, we will guess that the given series diverges and try to prove this by finding a divergent series that is "smaller" than the given series. However, series (1) does the trick since

$$\frac{1}{\sqrt{k} - \frac{1}{2}} > \frac{1}{\sqrt{k}} \quad \text{for } k = 1, 2, \ldots$$

Thus, we have proved that the given series diverges.

Solution (b). According to Principle 10.6.3, we should be able to discard all but the leading term in the polynomial without affecting the convergence or divergence. Thus, the given series is likely to behave like

$$\sum_{k=1}^{\infty} \frac{1}{2k^2} = \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{k^2} \tag{2}$$

which converges since it is a constant times a convergent p-series ($p = 2$). Thus, we will guess that the given series converges and try to prove this by finding a convergent series that is "bigger" than the given series. However, series (2) does the trick since

$$\frac{1}{2k^2 + k} < \frac{1}{2k^2} \quad \text{for } k = 1, 2, \ldots$$

Thus, we have proved that the given series converges. ◀

THE LIMIT COMPARISON TEST

In the last example, Principles 10.6.2 and 10.6.3 provided the guess about convergence or divergence as well as the series needed to apply the comparison test. Unfortunately, it is not always so straightforward to find the series required for comparison, so we will now consider an alternative to the comparison test that is usually easier to apply. The proof is given in Appendix G.

10.6.4 THEOREM (*The Limit Comparison Test*). *Let $\sum a_k$ and $\sum b_k$ be series with positive terms and suppose that*

$$\rho = \lim_{k\to+\infty} \frac{a_k}{b_k}$$

If ρ is finite and $\rho > 0$, then the series both converge or both diverge.

The cases where $\rho = 0$ or $\rho = +\infty$ are discussed in the exercises (Exercise 54).

Example 2 Use the limit comparison test to determine whether the following series converge or diverge.

(a) $\displaystyle\sum_{k=2}^{\infty} \frac{1}{\sqrt{k} - 1}$ (b) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2k^2 + k}$ (c) $\displaystyle\sum_{k=1}^{\infty} \frac{3k^3 - 2k^2 + 4}{k^7 - k^3 + 2}$

Solution (a). As in Example 1, Principle 10.6.2 suggests that the series is likely to behave like the divergent p-series (1). To prove that the given series diverges, we will apply the limit comparison test with

$$a_k = \frac{1}{\sqrt{k} - 1} \quad \text{and} \quad b_k = \frac{1}{\sqrt{k}}$$

We obtain

$$\rho = \lim_{k\to+\infty} \frac{a_k}{b_k} = \lim_{k\to+\infty} \frac{\sqrt{k}}{\sqrt{k} - 1} = \lim_{k\to+\infty} \frac{1}{1 - \frac{1}{\sqrt{k}}} = 1$$

Since ρ is finite and positive, it follows from Theorem 10.6.4 that the given series diverges.

Solution (b). As in Example 1, Principle 10.6.3 suggests that the series is likely to behave like the convergent series (2). To prove that the given series converges, we will apply the limit comparison test with

$$a_k = \frac{1}{2k^2 + k} \quad \text{and} \quad b_k = \frac{1}{2k^2}$$

We obtain

$$\rho = \lim_{k \to +\infty} \frac{a_k}{b_k} = \lim_{k \to +\infty} \frac{2k^2}{2k^2 + k} = \lim_{k \to +\infty} \frac{2}{2 + \frac{1}{k}} = 1$$

Since ρ is finite and positive, it follows from Theorem 10.6.4 that the given series converges, which agrees with the conclusion reached in Example 1 using the comparison test.

Solution (c). From Principle 10.6.3, the series is likely to behave like

$$\sum_{k=1}^{\infty} \frac{3k^3}{k^7} = \sum_{k=1}^{\infty} \frac{3}{k^4} \tag{3}$$

which converges since it is a constant times a convergent p-series. Thus, the given series is likely to converge. To prove this, we will apply the limit comparison test to series (3) and the given series. We obtain

$$\rho = \lim_{k \to +\infty} \frac{\dfrac{3k^3 - 2k^2 + 4}{k^7 - k^3 + 2}}{\dfrac{3}{k^4}} = \lim_{k \to +\infty} \frac{3k^7 - 2k^6 + 4k^4}{3k^7 - 3k^3 + 6} = 1$$

Since ρ is finite and nonzero, it follows from Theorem 10.6.4 that the given series converges, since (3) converges. ◀

THE RATIO TEST

The comparison test and the limit comparison test hinge on first making a guess about convergence and then finding an appropriate series for comparison, both of which can be difficult tasks in cases where Principles 10.6.2 and 10.6.3 cannot be applied. In such cases the next test can often be used, since it works exclusively with the terms of the given series—it requires neither an initial guess about convergence nor the discovery of a series for comparison. Its proof is given in Appendix G.

10.6.5 THEOREM (*The Ratio Test*). *Let $\sum u_k$ be a series with positive terms and suppose that*

$$\rho = \lim_{k \to +\infty} \frac{u_{k+1}}{u_k}$$

(*a*) *If $\rho < 1$, the series converges.*

(*b*) *If $\rho > 1$ or $\rho = +\infty$, the series diverges.*

(*c*) *If $\rho = 1$, the series may converge or diverge, so that another test must be tried.*

Example 3 Use the ratio test to determine whether the following series converge or diverge.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k!}$ (b) $\displaystyle\sum_{k=1}^{\infty} \frac{k}{2^k}$ (c) $\displaystyle\sum_{k=1}^{\infty} \frac{k^k}{k!}$ (d) $\displaystyle\sum_{k=3}^{\infty} \frac{(2k)!}{4^k}$ (e) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2k - 1}$

Solution (a). The series converges, since

$$\rho = \lim_{k \to +\infty} \frac{u_{k+1}}{u_k} = \lim_{k \to +\infty} \frac{1/(k+1)!}{1/k!} = \lim_{k \to +\infty} \frac{k!}{(k+1)!} = \lim_{k \to +\infty} \frac{1}{k+1} = 0 < 1$$

Solution (b). The series converges, since

$$\rho = \lim_{k\to+\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to+\infty} \frac{k+1}{2^{k+1}} \cdot \frac{2^k}{k} = \frac{1}{2} \lim_{k\to+\infty} \frac{k+1}{k} = \frac{1}{2} < 1$$

Solution (c). The series diverges, since

$$\rho = \lim_{k\to+\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to+\infty} \frac{(k+1)^{k+1}}{(k+1)!} \cdot \frac{k!}{k^k} = \lim_{k\to+\infty} \frac{(k+1)^k}{k^k} = \lim_{k\to+\infty} \left(1+\frac{1}{k}\right)^k = e > 1$$

See Theorem 7.5.6(*b*)

Solution (d). The series diverges, since

$$\rho = \lim_{k\to+\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to+\infty} \frac{[2(k+1)]!}{4^{k+1}} \cdot \frac{4^k}{(2k)!} = \lim_{k\to+\infty} \left(\frac{(2k+2)!}{(2k)!} \cdot \frac{1}{4}\right)$$

$$= \frac{1}{4} \lim_{k\to+\infty} (2k+2)(2k+1) = +\infty$$

Solution (e). The ratio test is of no help since

$$\rho = \lim_{k\to+\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to+\infty} \frac{1}{2(k+1)-1} \cdot \frac{2k-1}{1} = \lim_{k\to+\infty} \frac{2k-1}{2k+1} = 1$$

However, the integral test proves that the series diverges since

$$\int_1^{+\infty} \frac{dx}{2x-1} = \lim_{\ell\to+\infty} \int_1^{\ell} \frac{dx}{2x-1} = \lim_{\ell\to+\infty} \frac{1}{2} \ln(2x-1)\Big]_1^{\ell} = +\infty$$

Both the comparison test and the limit comparison test would also have worked here (verify). ◀

THE ROOT TEST

In cases where it is difficult or inconvenient to find the limit required for the ratio test, the next test is sometimes useful. Since its proof is similar to the proof of the ratio test, we will omit it.

> **10.6.6 THEOREM (*The Root Test*).** *Let $\sum u_k$ be a series with positive terms and suppose that*
>
> $$\rho = \lim_{k\to+\infty} \sqrt[k]{u_k} = \lim_{k\to+\infty} (u_k)^{1/k}$$
>
> (a) *If $\rho < 1$, the series converges.*
> (b) *If $\rho > 1$ or $\rho = +\infty$, the series diverges.*
> (c) *If $\rho = 1$, the series may converge or diverge, so that another test must be tried.*

Example 4 Use the root test to determine whether the following series converge or diverge.

(a) $\displaystyle\sum_{k=2}^{\infty} \left(\frac{4k-5}{2k+1}\right)^k$ (b) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(\ln(k+1))^k}$

Solution (a). The series diverges, since

$$\rho = \lim_{k\to+\infty} (u_k)^{1/k} = \lim_{k\to+\infty} \frac{4k-5}{2k+1} = 2 > 1$$

Solution (b). The series converges, since

$$\rho = \lim_{k\to+\infty} (u_k)^{1/k} = \lim_{k\to+\infty} \frac{1}{\ln(k+1)} = 0 < 1$$ ◀

Exercise Set 10.6 C CAS

In Exercises 1 and 2, make a guess about the convergence or divergence of the series, and confirm your guess using the comparison test.

1. (a) $\sum_{k=1}^{\infty} \frac{1}{5k^2 - k}$ (b) $\sum_{k=1}^{\infty} \frac{3}{k - \frac{1}{4}}$

2. (a) $\sum_{k=2}^{\infty} \frac{k+1}{k^2 - k}$ (b) $\sum_{k=1}^{\infty} \frac{2}{k^4 + k}$

3. In each part, use the comparison test to show that the series converges.
(a) $\sum_{k=1}^{\infty} \frac{1}{3^k + 5}$ (b) $\sum_{k=1}^{\infty} \frac{5 \sin^2 k}{k!}$

4. In each part, use the comparison test to show that the series diverges.
(a) $\sum_{k=1}^{\infty} \frac{\ln k}{k}$ (b) $\sum_{k=1}^{\infty} \frac{k}{k^{3/2} - \frac{1}{2}}$

In Exercises 5–10, use the limit comparison test to determine whether the series converges.

5. $\sum_{k=1}^{\infty} \frac{4k^2 - 2k + 6}{8k^7 + k - 8}$ **6.** $\sum_{k=1}^{\infty} \frac{1}{9k + 6}$

7. $\sum_{k=1}^{\infty} \frac{5}{3^k + 1}$ **8.** $\sum_{k=1}^{\infty} \frac{k(k+3)}{(k+1)(k+2)(k+5)}$

9. $\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{8k^2 - 3k}}$ **10.** $\sum_{k=1}^{\infty} \frac{1}{(2k+3)^{17}}$

In Exercises 11–16, use the ratio test to determine whether the series converges. If the test is inconclusive, then say so.

11. $\sum_{k=1}^{\infty} \frac{3^k}{k!}$ **12.** $\sum_{k=1}^{\infty} \frac{4^k}{k^2}$ **13.** $\sum_{k=1}^{\infty} \frac{1}{5k}$

14. $\sum_{k=1}^{\infty} k\left(\frac{1}{2}\right)^k$ **15.** $\sum_{k=1}^{\infty} \frac{k!}{k^3}$ **16.** $\sum_{k=1}^{\infty} \frac{k}{k^2 + 1}$

In Exercises 17–20, use the root test to determine whether the series converges. If the test is inconclusive, then say so.

17. $\sum_{k=1}^{\infty} \left(\frac{3k+2}{2k-1}\right)^k$ **18.** $\sum_{k=1}^{\infty} \left(\frac{k}{100}\right)^k$

19. $\sum_{k=1}^{\infty} \frac{k}{5^k}$ **20.** $\sum_{k=1}^{\infty} (1 - e^{-k})^k$

In Exercises 21–44, use any method to determine whether the series converges.

21. $\sum_{k=0}^{\infty} \frac{7^k}{k!}$ **22.** $\sum_{k=1}^{\infty} \frac{1}{2k+1}$ **23.** $\sum_{k=1}^{\infty} \frac{k^2}{5^k}$

24. $\sum_{k=1}^{\infty} \frac{k!10^k}{3^k}$ **25.** $\sum_{k=1}^{\infty} k^{50} e^{-k}$ **26.** $\sum_{k=1}^{\infty} \frac{k^2}{k^3 + 1}$

27. $\sum_{k=1}^{\infty} \frac{\sqrt{k}}{k^3 + 1}$ **28.** $\sum_{k=1}^{\infty} \frac{4}{2 + 3^k k}$

29. $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k(k+1)}}$ **30.** $\sum_{k=1}^{\infty} \frac{2 + (-1)^k}{5^k}$

31. $\sum_{k=1}^{\infty} \frac{2 + \sqrt{k}}{(k+1)^3 - 1}$ **32.** $\sum_{k=1}^{\infty} \frac{4 + |\cos k|}{k^3}$

33. $\sum_{k=1}^{\infty} \frac{1}{1 + \sqrt{k}}$ **34.** $\sum_{k=1}^{\infty} \frac{k!}{k^k}$ **35.** $\sum_{k=1}^{\infty} \frac{\ln k}{e^k}$

36. $\sum_{k=1}^{\infty} \frac{k!}{e^{k^2}}$ **37.** $\sum_{k=0}^{\infty} \frac{(k+4)!}{4!k!4^k}$ **38.** $\sum_{k=1}^{\infty} \left(\frac{k}{k+1}\right)^{k^2}$

39. $\sum_{k=1}^{\infty} \frac{1}{4 + 2^{-k}}$ **40.** $\sum_{k=1}^{\infty} \frac{\sqrt{k}\ln k}{k^3 + 1}$ **41.** $\sum_{k=1}^{\infty} \frac{\tan^{-1} k}{k^2}$

42. $\sum_{k=1}^{\infty} \frac{5^k + k}{k! + 3}$ **43.** $\sum_{k=0}^{\infty} \frac{(k!)^2}{(2k)!}$ **44.** $\sum_{k=1}^{\infty} \frac{(k!)^2 2^k}{(2k+2)!}$

In Exercises 45 and 46, find the general term of the series, and use the ratio test to show that the series converges.

45. $1 + \frac{1 \cdot 2}{1 \cdot 3} + \frac{1 \cdot 2 \cdot 3}{1 \cdot 3 \cdot 5} + \frac{1 \cdot 2 \cdot 3 \cdot 4}{1 \cdot 3 \cdot 5 \cdot 7} + \cdots$

46. $1 + \frac{1 \cdot 3}{3!} + \frac{1 \cdot 3 \cdot 5}{5!} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{7!} + \cdots$

In Exercises 47 and 48, use a CAS to investigate the convergence of the series.

C **47.** $\sum_{k=1}^{\infty} \frac{\ln k}{3^k}$ C **48.** $\sum_{k=1}^{\infty} \frac{[\pi(k+1)]^k}{k^{k+1}}$

49. (a) Make a conjecture about the convergence of the series $\sum_{k=1}^{\infty} \sin(\pi/k)$ by considering the local linear approximation of $\sin x$ near $x = 0$.
(b) Try to confirm your conjecture using the limit comparison test.

50. (a) Make a conjecture about the convergence of the series
$$\sum_{k=1}^{\infty} \left[1 - \cos\left(\frac{1}{k}\right)\right]$$
by considering the local quadratic approximation of $\cos x$ near $x = 0$.
(b) Try to confirm your conjecture using the limit comparison test.

51. Show that $\ln x < \sqrt{x}$ if $x > 0$, and use this result to investigate the convergence of
(a) $\sum_{k=1}^{\infty} \frac{\ln k}{k^2}$ (b) $\sum_{k=2}^{\infty} \frac{1}{(\ln k)^2}$

52. For which positive values of α does the series $\sum_{k=1}^{\infty}(\alpha^k/k^\alpha)$ converge?

53. Use Theorem 10.5.6 to prove the comparison test (Theorem 10.6.1).

54. Let $\sum a_k$ and $\sum b_k$ be series with positive terms. Prove:
(a) If $\lim_{k\to+\infty}(a_k/b_k) = 0$ and $\sum b_k$ converges, then $\sum a_k$ converges.
(b) If $\lim_{k\to+\infty}(a_k/b_k) = +\infty$ and $\sum b_k$ diverges, then $\sum a_k$ diverges.

10.7 ALTERNATING SERIES; CONDITIONAL CONVERGENCE

Up to now we have focused exclusively on series with nonnegative terms. In this section we will discuss series that contain both positive and negative terms.

ALTERNATING SERIES

Series whose terms alternate between positive and negative, called ***alternating series***, are of special importance. Some examples are

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

$$\sum_{k=1}^{\infty}(-1)^{k}\frac{1}{k} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots$$

In general, an alternating series has one of the following two forms:

$$\sum_{k=1}^{\infty}(-1)^{k+1}a_k = a_1 - a_2 + a_3 - a_4 + \cdots \tag{1}$$

$$\sum_{k=1}^{\infty}(-1)^{k}a_k = -a_1 + a_2 - a_3 + a_4 - \cdots \tag{2}$$

where the a_k's are assumed to be positive in both cases.

The following theorem is the key result on convergence of alternating series.

10.7.1 THEOREM (*Alternating Series Test*). *An alternating series of either form* (1) *or form* (2) *converges if the following two conditions are satisfied:*

(*a*) $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_k \geq \cdots$

(*b*) $\lim_{k\to+\infty} a_k = 0$

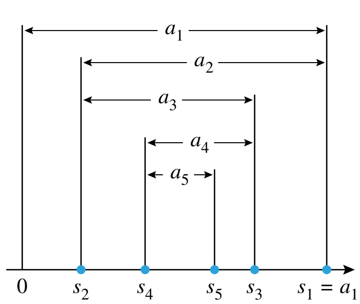


Figure 10.7.1

Proof. We will consider only alternating series of form (1). The idea of the proof is to show that if conditions (a) and (b) hold, then the sequences of even-numbered and odd-numbered partial sums converge to a common limit S. It will then follow from Theorem 10.2.4 that the entire sequence of partial sums converges to S.

Figure 10.7.1 shows how successive partial sums satisfying conditions (a) and (b) appear when plotted on a horizontal axis. The even-numbered partial sums

$$s_2, s_4, s_6, s_8, \ldots, s_{2n}, \ldots$$

form an increasing sequence bounded above by a_1, and the odd-numbered partial sums

$$s_1, s_3, s_5, \ldots, s_{2n-1}, \ldots$$

form a decreasing sequence bounded below by 0. Thus, by Theorems 10.3.3 and 10.3.4, the even-numbered partial sums converge to some limit S_E and the odd-numbered partial sums converge to some limit S_O. To complete the proof we must show that $S_E = S_O$. But the $(2n)$-th term in the series is $-a_{2n}$, so that $s_{2n} - s_{2n-1} = -a_{2n}$, which can be written as

$$s_{2n-1} = s_{2n} + a_{2n}$$

However, $2n \to +\infty$ and $2n - 1 \to +\infty$ as $n \to +\infty$, so that

$$S_O = \lim_{n\to+\infty} s_{2n-1} = \lim_{n\to+\infty} (s_{2n} + a_{2n}) = S_E + 0 = S_E$$

which completes the proof. ∎

REMARK. As might be expected, it is not essential for condition (*a*) in the alternating series test to hold for all terms; an alternating series will converge if condition (*b*) is true and condition (*a*) holds eventually.

Example 1 Use the alternating series test to show that the following series converge.

$$\text{(a)} \sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k} \qquad \text{(b)} \sum_{k=1}^{\infty}(-1)^{k+1}\frac{k+3}{k(k+1)}$$

Solution (a). The two conditions in the alternating series test are satisfied since

$$a_k = \frac{1}{k} > \frac{1}{k+1} = a_{k+1} \quad \text{and} \quad \lim_{k\to+\infty} a_k = \lim_{k\to+\infty} \frac{1}{k} = 0$$

Solution (b). The two conditions in the alternating series test are satisfied since

$$\frac{a_{k+1}}{a_k} = \frac{k+4}{(k+1)(k+2)} \cdot \frac{k(k+1)}{k+3} = \frac{k^2+4k}{k^2+5k+6} = \frac{k^2+4k}{(k^2+4k)+(k+6)} < 1$$

so

$$a_k > a_{k+1}$$

and

$$\lim_{k\to+\infty} a_k = \lim_{k\to+\infty} \frac{k+3}{k(k+1)} = \lim_{k\to+\infty} \frac{\dfrac{1}{k} + \dfrac{3}{k^2}}{1 + \dfrac{1}{k}} = 0$$ ◀

REMARK. The series in part (a) of the last example is called the ***alternating harmonic series***. Observe that this series converges, whereas the harmonic series diverges.

REMARK. If an alternating series violates condition (*b*) of the alternating series test, then the series must diverge by the divergence test (Theorem 10.5.1). However, if condition (*b*) is satisfied, but condition (*a*) is not, the series can either converge or diverge.*

APPROXIMATING SUMS OF ALTERNATING SERIES

The following theorem is concerned with the error that results when the sum of an alternating series is approximated by a partial sum.

10.7.2 THEOREM. *If an alternating series satisfies the hypotheses of the alternating series test, and if S is the sum of the series, then:*

(*a*) *S lies between any two successive partial sums; that is, either*

$$s_n < S < s_{n+1} \quad \text{or} \quad s_{n+1} < S < s_n \tag{3}$$

depending on which partial sum is larger.

(*b*) *If S is approximated by s_n, then the absolute error $|S - s_n|$ satisfies*

$$|S - s_n| < a_{n+1} \tag{4}$$

Moreover, the sign of the error $S - s_n$ is the same as that of the coefficient of a_{n+1}.

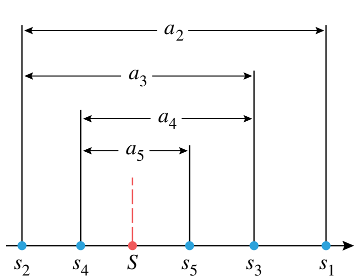


Figure 10.7.2

Proof. We will prove the theorem for series of form (1). Referring to Figure 10.7.2 and keeping in mind our observation in the proof of Theorem 10.7.1 that the odd-numbered

*The interested reader will find some nice examples in an article by R. Lariviere, "On a Convergence Test for Alternating Series," *Mathematics Magazine*, Vol. 29, 1956, p. 88.

partial sums form a decreasing sequence converging to S and the even-numbered partial sums form an increasing sequence converging to S, we see that successive partial sums oscillate from one side of S to the other in smaller and smaller steps with the odd-numbered partial sums being larger than S and the even-numbered partial sums being smaller than S. Thus, depending on whether n is even or odd, we have

$$s_n < S < s_{n+1} \quad \text{or} \quad s_{n+1} < S < s_n$$

which proves (3). Moreover, in either case we have

$$|S - s_n| < |s_{n+1} - s_n| \tag{5}$$

But $s_{n+1} - s_n = \pm a_{n+1}$ (the sign depending on whether n is even or odd). Thus, it follows from (5) that $|S - s_n| < a_{n+1}$, which proves (4). Finally, since the odd-numbered partial sums are larger than S and the even-numbered partial sums are smaller than S, it follows that $S - s_n$ has the same sign as the coefficient of a_{n+1} (verify). ■

REMARK. In words, inequality (4) states that for a series satisfying the hypotheses of the alternating series test, the magnitude of the error that results from approximating S by s_n is less than that of the first term that is *not* included in the partial sum.

Example 2 Later in this chapter we will show that the sum of the alternating harmonic series is

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{k+1}\frac{1}{k} + \cdots$$

(a) Accepting this to be so, find an upper bound on the magnitude of the error that results if ln 2 is approximated by the sum of the first eight terms in the series.

(b) Find a partial sum that approximates ln 2 to one decimal-place accuracy (the nearest tenth).

Solution (a). It follows from (4) that

$$|\ln 2 - s_8| < a_9 = \frac{1}{9} < 0.12 \tag{6}$$

As a check, let us compute s_8 exactly. We obtain

$$s_8 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} = \frac{533}{840}$$

Thus, with the help of a calculator

$$|\ln 2 - s_8| = \left|\ln 2 - \frac{533}{840}\right| \approx 0.059$$

This shows that the error is well under the estimate provided by upper bound (6).

Solution (b). For one decimal-place accuracy, we must choose n so that $|\ln 2 - s_n| \le 0.05$. However, it follows from (4) that

$$|\ln 2 - s_n| < a_{n+1}$$

so it suffices to choose n so that $a_{n+1} \le 0.05$.

One way to find n is to use a calculating utility to obtain numerical values for a_1, a_2, $a_3, \ldots$ until you encounter the first value that is less than or equal to 0.05. If you do this, you will find that it is $a_{20} = 0.05$; this tells us that partial sum s_{19} will provide the desired accuracy. Another way to find n is to solve the inequality

$$\frac{1}{n+1} \le 0.05$$

algebraically. We can do this by taking reciprocals, reversing the sense of the inequality, and then simplifying to obtain $n \ge 19$. Thus, s_{19} will provide the required accuracy, which is consistent with the previous result.

With the help of a calculating utility, the value of s_{19} is approximately $s_{19} \approx 0.7$ and the value of $\ln 2$ obtained directly is approximately $\ln 2 \approx 0.69$, which agrees with s_{19} when rounded to one decimal place. ◀

REMARK. As this example illustrates, the alternating harmonic series does not provide an efficient way to approximate $\ln 2$, since too much computation is required to achieve reasonable accuracy. Later, we will develop better ways to approximate logarithms.

ABSOLUTE CONVERGENCE

The series

$$1 - \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} - \frac{1}{2^5} - \frac{1}{2^6} + \cdots$$

does not fit in any of the categories studied so far—it has mixed signs, but is not alternating. We will now develop some convergence tests that can be applied to such series.

10.7.3 DEFINITION. A series

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + \cdots + u_k + \cdots$$

is said to ***converge absolutely*** if the series of absolute values

$$\sum_{k=1}^{\infty} |u_k| = |u_1| + |u_2| + \cdots + |u_k| + \cdots$$

converges and is said to ***diverge absolutely*** if the series of absolute values diverges.

Example 3 Determine whether the following series converge absolutely.

(a) $1 - \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} - \frac{1}{2^5} - \cdots$ (b) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$

Solution (a). The series of absolute values is the convergent geometric series

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \cdots$$

so the given series converges absolutely.

Solution (b). The series of absolute values is the divergent harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

so the given series diverges absolutely. ◀

It is important to distinguish between the notions of convergence and absolute convergence. For example, the series in part (b) of Example 3 converges, since it is the alternating harmonic series, yet we demonstrated that it does not converge absolutely. However, the following theorem shows that *if a series converges absolutely, then it converges.*

10.7.4 THEOREM. *If the series*

$$\sum_{k=1}^{\infty} |u_k| = |u_1| + |u_2| + \cdots + |u_k| + \cdots$$

converges, then so does the series

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + \cdots + u_k + \cdots$$

Proof. Our proof is based on a trick. We will write the series $\sum u_k$ as

$$\sum_{k=1}^{\infty} u_k = \sum_{k=1}^{\infty}[(u_k + |u_k|) - |u_k|] \tag{7}$$

We are assuming that $\sum |u_k|$ converges, so that if we can show that $\sum(u_k + |u_k|)$ converges, then it will follow from (7) and Theorem 10.5.3(a) that $\sum u_k$ converges. However, the value of $u_k + |u_k|$ is either 0 or $2|u_k|$, depending on the sign of u_k. Thus, in all cases it is true that

$$0 \le u_k + |u_k| \le 2|u_k|$$

But $\sum 2|u_k|$ converges, since it is a constant times the convergent series $\sum |u_k|$; hence $\sum(u_k + |u_k|)$ converges by the comparison test. ∎

Theorem 10.7.4 provides a way of inferring convergence of a series with positive and negative terms from the convergence of a series with nonnegative terms (the series of absolute values). This is important because most of the convergence tests we have developed apply only to series with nonnegative terms.

Example 4 Show that the following series converge.

(a) $1 - \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} - \frac{1}{2^5} - \frac{1}{2^6} + \cdots$ (b) $\sum_{k=1}^{\infty} \frac{\cos k}{k^2}$

Solution (a). Observe that this is not an alternating series because the signs alternate in pairs after the first term. Thus, we have no convergence test that can be applied directly. However, we showed in Example 3(a) that the series converges absolutely, so Theorem 10.7.4 implies that it converges.

Solution (b). With the help of a calculating utility, you will be able to verify that the signs of the terms in this series vary irregularly. Thus, we will test for absolute convergence. The series of absolute values is

$$\sum_{k=1}^{\infty} \left|\frac{\cos k}{k^2}\right|$$

However,

$$\left|\frac{\cos k}{k^2}\right| \le \frac{1}{k^2}$$

But $\sum 1/k^2$ is a convergent p-series ($p = 2$), so the series of absolute values converges by the comparison test. Thus, the given series converges absolutely and hence converges. ◀

CONDITIONAL CONVERGENCE

Although Theorem 10.7.4 is a useful tool for series that converge absolutely, it provides no information about the convergence or divergence of a series that diverges absolutely. For example, consider the two series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{k+1}\frac{1}{k} + \cdots \tag{8}$$

$$-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots - \frac{1}{k} - \cdots \tag{9}$$

Both of these series divergeabsolutely, since in each case the series of absolute values is the divergent harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} + \cdots$$

However, series (8) converges, since it is the alternating harmonic series, and series (9) diverges, since it is a constant times the divergent harmonic series. As a matter of terminology, a series that converges but diverges absolutely is said to ***converge conditionally*** (or to be ***conditionally convergent***). Thus, (8) is a conditionally convergent series.

THE RATIO TEST FOR ABSOLUTE CONVERGENCE

Although one cannot generally infer convergence or divergence of a series from absolute divergence, the following variation of the ratio test provides a way of deducing divergence from absolute divergence in certain situations. We omit the proof.

10.7.5 THEOREM (*Ratio Test for Absolute Convergence*). *Let $\sum u_k$ be a series with nonzero terms and suppose that*

$$\rho = \lim_{k\to+\infty} \frac{|u_{k+1}|}{|u_k|}$$

(a) *If $\rho < 1$, then the series $\sum u_k$ converges absolutely and therefore converges.*

(b) *If $\rho > 1$ or if $\rho = +\infty$, then the series $\sum u_k$ diverges.*

(c) *If $\rho = 1$, no conclusion about convergence or absolute convergence can be drawn from this test.*

Example 5 Use the ratio test for absolute convergence to determine whether the series converges.

(a) $\displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{2^k}{k!}$ (b) $\displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{(2k-1)!}{3^k}$

Solution (a). Taking the absolute value of the general term u_k we obtain

$$|u_k| = \left|(-1)^k \frac{2^k}{k!}\right| = \frac{2^k}{k!}$$

Thus,

$$\rho = \lim_{k\to+\infty} \frac{|u_{k+1}|}{|u_k|} = \lim_{k\to+\infty} \frac{2^{k+1}}{(k+1)!} \cdot \frac{k!}{2^k} = \lim_{k\to+\infty} \frac{2}{k+1} = 0 < 1$$

which implies that the series converges absolutely and therefore converges.

Solution (b). Taking the absolute value of the general term u_k we obtain

$$|u_k| = \left|(-1)^k \frac{(2k-1)!}{3^k}\right| = \frac{(2k-1)!}{3^k}$$

Thus,

$$\rho = \lim_{k\to+\infty} \frac{|u_{k+1}|}{|u_k|} = \lim_{k\to+\infty} \frac{[2(k+1)-1]!}{3^{k+1}} \cdot \frac{3^k}{(2k-1)!}$$

$$= \lim_{k\to+\infty} \frac{1}{3} \cdot \frac{(2k+1)!}{(2k-1)!} = \frac{1}{3} \lim_{k\to+\infty} (2k)(2k+1) = +\infty$$

which implies that the series diverges. ◀

SUMMARY OF CONVERGENCE TESTS

We conclude this section with a summary of convergence tests that can be used for reference.

Summary of Convergence Tests

NAME	STATEMENT	COMMENTS
Divergence Test (10.5.1)	If $\lim_{k\to+\infty} u_k \neq 0$, then $\sum u_k$ diverges.	If $\lim_{k\to+\infty} u_k = 0$, then $\sum u_k$ may or may not converge.
Integral Test (10.5.4)	Let $\sum u_k$ be a series with positive terms, and let $f(x)$ be the function that results when k is replaced by x in the general term of the series. If f is decreasing and continuous for $x \geq a$, then $\sum_{k=1}^{\infty} u_k$ and $\int_a^{+\infty} f(x)\,dx$ both converge or both diverge.	This test only applies to series that have positive terms. Try this test when $f(x)$ is easy to integrate.
Comparison Test (10.6.1)	Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be series with nonnegative terms such that $a_1 \leq b_1,\ a_2 \leq b_2, \ldots, a_k \leq b_k, \ldots$ If $\sum b_k$ converges, then $\sum a_k$ converges, and if $\sum a_k$ diverges, then $\sum b_k$ diverges.	This test only applies to series with nonnegative terms. Try this test as a last resort; other tests are often easier to apply.
Limit Comparison Test (10.6.4)	Let $\sum a_k$ and $\sum b_k$ be series with positive terms such that $\rho = \lim_{k\to+\infty} \frac{a_k}{b_k}$ If $0 < \rho < +\infty$, then both series converge or both diverge.	This is easier to apply than the comparison test, but still requires some skill in choosing the series $\sum b_k$ for comparison.
Ratio Test (10.6.5)	Let $\sum u_k$ be a series with positive terms and suppose that $\rho = \lim_{k\to+\infty} \frac{u_{k+1}}{u_k}$ (a) Series converges if $\rho < 1$. (b) Series diverges if $\rho > 1$ or $\rho = +\infty$. (c) The test is inconclusive if $\rho = 1$.	Try this test when u_k involves factorials or kth powers.
Root Test (10.6.6)	Let $\sum u_k$ be a series with positive terms such that $\rho = \lim_{k\to+\infty} \sqrt[k]{u_k}$ (a) The series converges if $\rho < 1$. (b) The series diverges if $\rho > 1$ or $\rho = +\infty$. (c) The test is inconclusive if $\rho = 1$.	Try this test when u_k involves kth powers.
Alternating Series Test (10.7.1)	If $a_k > 0$ for $k = 1, 2, 3, \ldots$, then the series $a_1 - a_2 + a_3 - a_4 + \cdots$ $-a_1 + a_2 - a_3 + a_4 - \cdots$ converge if the following conditions hold: (a) $a_1 \geq a_2 \geq a_3 \geq \cdots$ (b) $\lim_{k\to+\infty} a_k = 0$	This test applies only to alternating series.
Ratio Test for Absolute Convergence (10.7.5)	Let $\sum u_k$ be a series with nonzero terms such that $\rho = \lim_{k\to+\infty} \frac{\lvert u_{k+1}\rvert}{\lvert u_k\rvert}$ (a) The series converges absolutely if $\rho < 1$. (b) The series diverges if $\rho > 1$ or $\rho = +\infty$. (c) The test is inconclusive if $\rho = 1$.	The series need not have positive terms and need not be alternating to use this test.

EXERCISE SET 10.7 Graphing Utility C CAS

In Exercises 1 and 2 show that the series converges by confirming that it satisfies the hypotheses of the alternating series test (Theorem 10.7.1).

1. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k+1}$ **2.** $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k}{3^k}$

In Exercises 3–6, determine whether the alternating series converges, and justify your answer.

3. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+1}{3k+1}$ **4.** $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+1}{\sqrt{k}+1}$

5. $\sum_{k=1}^{\infty} (-1)^{k+1} e^{-k}$ **6.** $\sum_{k=3}^{\infty} (-1)^{k} \frac{\ln k}{k}$

In Exercises 7–12, use the ratio test for absolute convergence (Theorem 10.7.5) to determine whether the series converges or diverges. If the test is inconclusive, then say so.

7. $\sum_{k=1}^{\infty} \left(-\frac{3}{5}\right)^k$ **8.** $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^k}{k!}$

9. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{3^k}{k^2}$ **10.** $\sum_{k=1}^{\infty} (-1)^{k} \frac{k}{5^k}$

11. $\sum_{k=1}^{\infty} (-1)^{k} \frac{k^3}{e^k}$ **12.** $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k^k}{k!}$

In Exercises 13–30, classify the series as absolutely convergent, conditionally convergent, or divergent.

13. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{3k}$ **14.** $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{4/3}}$ **15.** $\sum_{k=1}^{\infty} \frac{(-4)^k}{k^2}$

16. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!}$ **17.** $\sum_{k=1}^{\infty} \frac{\cos k\pi}{k}$ **18.** $\sum_{k=3}^{\infty} \frac{(-1)^k \ln k}{k}$

19. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+2}{k(k+3)}$ **20.** $\sum_{k=1}^{\infty} \frac{(-1)^{k+1} k^2}{k^3+1}$

21. $\sum_{k=1}^{\infty} \sin \frac{k\pi}{2}$ **22.** $\sum_{k=1}^{\infty} \frac{\sin k}{k^3}$

23. $\sum_{k=2}^{\infty} \frac{(-1)^k}{k \ln k}$ **24.** $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k(k+1)}}$

25. $\sum_{k=2}^{\infty} \left(-\frac{1}{\ln k}\right)^k$ **26.** $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k+1}+\sqrt{k}}$

27. $\sum_{k=2}^{\infty} \frac{(-1)^k (k^2+1)}{k^3+2}$ **28.** $\sum_{k=1}^{\infty} \frac{k \cos k\pi}{k^2+1}$

29. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1} k!}{(2k-1)!}$ **30.** $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{3^{2k-1}}{k^2+1}$

In Exercises 31–34, the series satisfies the hypotheses of the alternating series test. For the stated value of n, find an upper bound on the absolute error that results if the sum of the series is approximated by the nth partial sum.

31. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$; $n = 7$ **32.** $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!}$; $n = 5$

33. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}$; $n = 99$

34. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k+1)\ln(k+1)}$; $n = 3$

In Exercises 35–38, the series satisfies the hypotheses of the alternating series test. Find a value of n for which the nth partial sum is ensured to approximate the sum of the series to the stated accuracy.

35. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$; $|\text{error}| < 0.0001$

36. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!}$; $|\text{error}| < 0.00001$

37. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}$; two decimal places

38. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k+1)\ln(k+1)}$; one decimal place

In Exercises 39 and 40, find an upper bound on the absolute error that results if s_{10} is used to approximate the sum of the given *geometric* series. Compute s_{10} rounded to four decimal places and compare this value with the exact sum of the series.

39. $\frac{3}{4} - \frac{3}{8} + \frac{3}{16} - \frac{3}{32} + \cdots$ **40.** $1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \cdots$

In Exercises 41–44, the series satisfies the hypotheses of the alternating series test. Approximate the sum of the series to two decimal-place accuracy.

41. $1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots$ **42.** $1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \cdots$

43. $\frac{1}{1 \cdot 2} - \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} - \frac{1}{4 \cdot 2^4} + \cdots$

44. $\frac{1}{1^5 + 4 \cdot 1} - \frac{1}{3^5 + 4 \cdot 3} + \frac{1}{5^5 + 4 \cdot 5} - \frac{1}{7^5 + 4 \cdot 7} + \cdots$

C **45.** The purpose of this exercise is to show that the error bound in part (*b*) of Theorem 10.7.2 can be overly conservative in certain cases.

(a) Use a CAS to confirm that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

(b) Use the CAS to show that $|(\pi/4) - s_{26}| < 10^{-2}$.

(c) According to the error bound in part (*b*) of Theorem 10.7.2, what value of n is required to ensure that $|(\pi/4) - s_n| < 10^{-2}$?

46. Show that the alternating p-series

$$1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots + (-1)^{k+1}\frac{1}{k^p} + \cdots$$

converges absolutely if $p > 1$, converges conditionally if $0 < p \le 1$, and diverges if $p \le 0$.

It can be proved that any series that is constructed from an absolutely convergent series by rearranging the terms is absolutely convergent and has the same sum as the original series. Use this fact together with parts (*a*) and (*b*) of Theorem 10.5.3 in Exercises 47 and 48.

47. It was stated in Exercise 27 of Section 10.5 that

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

Use this to show that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

48. It was stated in Exercise 27 of Section 10.5 that

$$\frac{\pi^4}{90} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots$$

Use this to show that

$$\frac{\pi^4}{96} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots$$

49. It can be proved that the terms of any conditionally convergent series can be rearranged to give either a divergent series or a conditionally convergent series whose sum is any given number S. For example, we stated in Example 2 that

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

Show that we can rearrange this series so that its sum is $\frac{1}{2}\ln 2$ by rewriting it as

$$\left(1 - \frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{6} - \frac{1}{8}\right) + \left(\frac{1}{5} - \frac{1}{10} - \frac{1}{12}\right) + \cdots$$

[*Hint:* Add the first two terms in each set of parentheses.]

50. (a) Use a graphing utility to graph

$$f(x) = \frac{4x-1}{4x^2-2x}, \quad x \ge 1$$

(b) Based on your graph, do think that the series

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{4k-1}{4k^2-2k}$$

converges? Explain your reasoning.

51. As illustrated in the accompanying figure, a bug, starting at point A on a 180-cm wire, walks the length of the wire, stops and walks in the opposite direction for half the length of the wire, stops again and walks in the opposite direction for one-third the length of the wire, stops again and walks in the opposite direction for one-fourth the length of the wire, and so forth until it stops for the 1000th time.

(a) Give upper and lower bounds on the distance between the bug and point A when it finally stops. [*Hint:* As stated in Example 2, assume that the sum of the alternating harmonic series is ln 2.]

(b) Give upper and lower bounds on the total distance that the bug has traveled when it finally stops. [*Hint:* Use inequality (2) of Section 10.5.]

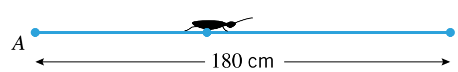


Figure Ex-51

52. (a) Prove that if $\sum a_k$ converges absolutely, then $\sum a_k^2$ converges.

(b) Show that the converse of part (a) is false by giving a counterexample.

10.8 MACLAURIN AND TAYLOR SERIES; POWER SERIES

In the last four sections we focused exclusively on series whose terms are numbers. In this section we will introduce Maclaurin and Taylor series, examples of series whose terms are functions. Our primary objective is to develop mathematical tools for the investigation of convergence of Maclaurin and Taylor series.

MACLAURIN AND TAYLOR SERIES

In Section 10.1 we defined the nth Maclaurin polynomial for a function f as

$$\sum_{k=0}^{n}\frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

and the nth Taylor polynomial for f about $x = x_0$ as

$$\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

Since then we have gone on to consider sums with an infinite number of terms, so it is not a big step to extend the notions of Maclaurin and Taylor polynomials to series by not stopping the summation index at n. Thus, we have the following definition.

10.8.1 DEFINITION. If f has derivatives of all orders at x_0, then we call the series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \cdots \quad (1)$$

the ***Taylor series for f about $x = x_0$***. In the special case where $x_0 = 0$, this series becomes

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(k)}(0)}{k!}x^k + \cdots \quad (2)$$

in which case we call it the ***Maclaurin series for f***.

Note that the nth Maclaurin and Taylor polynomials are the nth partial sums for the corresponding Maclaurin and Taylor series.

Example 1 Find the Maclaurin series for

(a) e^x (b) $\sin x$ (c) $\cos x$ (d) $\dfrac{1}{1-x}$

Solution (a). In Example 2 of Section 10.1 we found that the nth Maclaurin polynomial for e^x is

$$p_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

Thus, the Maclaurin series for e^x is

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \cdots$$

Solution (b). In Example 4(a) of Section 10.1 we found that the Maclaurin polynomials for $\sin x$ are given by

$$p_{2k+1}(x) = p_{2k+2}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^k \frac{x^{2k+1}}{(2k+1)!} \quad (k = 0, 1, 2, \ldots)$$

Thus, the Maclaurin series for $\sin x$ is

$$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^k \frac{x^{2k+1}}{(2k+1)!} + \cdots$$

Solution (c). In Example 4(b) of Section 10.1 we found that the Maclaurin polynomials for $\cos x$ are given by

$$p_{2k}(x) = p_{2k+1}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^k \frac{x^{2k}}{(2k)!} \quad (k = 0, 1, 2, \ldots)$$

Thus, the Maclaurin series for $\cos x$ is

$$\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!}=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots+(-1)^k\frac{x^{2k}}{(2k)!}+\cdots$$

Solution (d). In Example 4(c) of Section 10.1 we found that the nth Maclaurin polynomial for $1/(1-x)$ is

$$p_n(x)=\sum_{k=0}^{n}x^k=1+x+x^2+\cdots+x^n \quad (n=0,1,2,\ldots)$$

Thus, the Maclaurin series for $1/(1-x)$ is

$$\sum_{k=0}^{\infty}x^k=1+x+x^2+\cdots+x^k+\cdots$$ ◀

Example 2 Find the Taylor series for $1/x$ about $x=1$.

Solution. In Example 5 of Section 10.1 we found that the nth Taylor polynomial for $1/x$ about $x=1$ is

$$\sum_{k=0}^{n}(-1)^k(x-1)^k=1-(x-1)+(x-1)^2-(x-1)^3+\cdots+(-1)^n(x-1)^n$$

Thus, the Taylor series for $1/x$ about $x=1$ is

$$\sum_{k=0}^{\infty}(-1)^k(x-1)^k=1-(x-1)+(x-1)^2-(x-1)^3+\cdots+(-1)^k(x-1)^k+\cdots$$ ◀

POWER SERIES IN x

Maclaurin and Taylor series differ from the series that we have considered in the last four sections in that their terms are not merely constants, but instead involve a variable. These are examples of *power series*, which we now define.

If $c_0, c_1, c_2, \ldots$ are constants and x is a variable, then a series of the form

$$\sum_{k=0}^{\infty}c_kx^k=c_0+c_1x+c_2x^2+\cdots+c_kx^k+\cdots \tag{3}$$

is called a ***power series in x***. Some examples are

$$\sum_{k=0}^{\infty}x^k=1+x+x^2+x^3+\cdots$$

$$\sum_{k=0}^{\infty}\frac{x^k}{k!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$$

$$\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!}=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots$$

From Example 1, these are the Maclaurin series for the functions $1/(1-x)$, e^x, and $\cos x$, respectively. Indeed, every Maclaurin series

$$\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(k)}(0)}{k!}x^k+\cdots$$

is a power series in x.

RADIUS AND INTERVAL OF CONVERGENCE

If a numerical value is substituted for x in a power series $\sum c_kx^k$, then the resulting series of numbers may either converge or diverge. This leads to the problem of determining the set of x-values for which a given power series converges; this is called its ***convergence set***.

Observe that every power series in x converges at $x = 0$, since substituting this value in (3) produces the series

$$c_0 + 0 + 0 + 0 + \cdots + 0 + \cdots$$

whose sum is c_0. In rare cases $x = 0$ may be the only number in the convergence set, but more usually the convergence set is some finite or infinite interval containing $x = 0$. This is the content of the following theorem, whose proof will be omitted.

10.8.2 THEOREM. *For any power series in x, exactly one of the following is true:*

(a) *The series converges only for $x = 0$.*

(b) *The series converges absolutely (and hence converges) for all real values of x.*

(c) *The series converges absolutely (and hence converges) for all x in some finite open interval $(-R, R)$, and diverges if $x < -R$ or $x > R$. At either of the values $x = R$ or $x = -R$, the series may converge absolutely, converge conditionally, or diverge, depending on the particular series.*

This theorem states that the convergence set for a power series in x is always an interval centered at $x = 0$ (possibly just the value $x = 0$ itself or possibly infinite). For this reason, the convergence set of a power series in x is called the ***interval of convergence***. In the case where the convergence set is the single value $x = 0$ we say that the series has ***radius of convergence* 0**, in the case where the convergence set is $(-\infty, +\infty)$ we say that the series has ***radius of convergence* $+\infty$**, and in the case where the convergence set extends between $-R$ and R we say that the series has ***radius of convergence R*** (Figure 10.8.1).

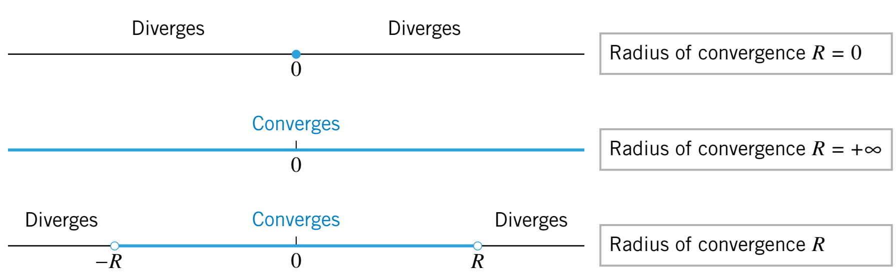


Figure 10.8.1

FINDING THE INTERVAL OF CONVERGENCE

The usual procedure for finding the interval of convergence of a power series is to apply the ratio test for absolute convergence (Theorem 10.7.5). The following example illustrates how this works.

Example 3 Find the interval of convergence and radius of convergence of the following power series.

(a) $\displaystyle\sum_{k=0}^{\infty} x^k$ (b) $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!}$ (c) $\displaystyle\sum_{k=0}^{\infty} k!x^k$ (d) $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^k}{3^k(k+1)}$

Solution (a). We apply the ratio test for absolute convergence. We have

$$\rho = \lim_{k\to+\infty}\left|\frac{u_{k+1}}{u_k}\right| = \lim_{k\to+\infty}\left|\frac{x^{k+1}}{x^k}\right| = \lim_{k\to+\infty}|x| = |x|$$

so the series converges absolutely if $\rho = |x| < 1$ and diverges if $\rho = |x| > 1$. The test is inconclusive if $|x| = 1$ (i.e., if $x = 1$ or $x = -1$), which means that we will have to investigate convergence at these values separately. At these values the series becomes

$$\sum_{k=0}^{\infty} 1^k = 1 + 1 + 1 + 1 + \cdots \qquad \boxed{x = 1}$$

$$\sum_{k=0}^{\infty} (-1)^k = 1 - 1 + 1 - 1 + \cdots \qquad \boxed{x = -1}$$

both of which diverge; thus, the interval of convergence for the given power series is $(-1, 1)$, and the radius of convergence is $R = 1$.

Solution (b). Applying the ratio test for absolute convergence, we obtain

$$\rho = \lim_{k\to+\infty} \left|\frac{u_{k+1}}{u_k}\right| = \lim_{k\to+\infty} \left|\frac{x^{k+1}}{(k+1)!} \cdot \frac{k!}{x^k}\right| = \lim_{k\to+\infty} \left|\frac{x}{k+1}\right| = 0$$

Since $\rho < 1$ for all x, the series converges absolutely for all x. Thus, the interval of convergence is $(-\infty, +\infty)$ and the radius of convergence is $R = +\infty$.

Solution (c). If $x \neq 0$, then the ratio test for absolute convergence yields

$$\rho = \lim_{k\to+\infty} \left|\frac{u_{k+1}}{u_k}\right| = \lim_{k\to+\infty} \left|\frac{(k+1)!x^{k+1}}{k!x^k}\right| = \lim_{k\to+\infty} |(k+1)x| = +\infty$$

Therefore, the series diverges for all nonzero values of x. Thus, the interval of convergence is the single value $x = 0$ and the radius of convergence is $R = 0$.

Solution (d). Since $|(-1)^k| = |(-1)^{k+1}| = 1$, we obtain

$$\begin{aligned}\rho = \lim_{k\to+\infty} \left|\frac{u_{k+1}}{u_k}\right| &= \lim_{k\to+\infty} \left|\frac{x^{k+1}}{3^{k+1}(k+2)} \cdot \frac{3^k(k+1)}{x^k}\right| \\ &= \lim_{k\to+\infty} \left[\frac{|x|}{3} \cdot \left(\frac{k+1}{k+2}\right)\right] \\ &= \frac{|x|}{3} \lim_{k\to+\infty} \left(\frac{1+(1/k)}{1+(2/k)}\right) = \frac{|x|}{3}\end{aligned}$$

The ratio test for absolute convergence implies that the series converges absolutely if $|x| < 3$ and diverges if $|x| > 3$. The ratio test fails to provide any information when $|x| = 3$, so the cases $x = -3$ and $x = 3$ need separate analyses. Substituting $x = -3$ in the given series yields

$$\sum_{k=0}^{\infty} \frac{(-1)^k(-3)^k}{3^k(k+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k(-1)^k 3^k}{3^k(k+1)} = \sum_{k=0}^{\infty} \frac{1}{k+1}$$

which is the divergent harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$. Substituting $x = 3$ in the given series yields

$$\sum_{k=0}^{\infty} \frac{(-1)^k 3^k}{3^k(k+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

which is the conditionally convergent alternating harmonic series. Thus, the interval of convergence for the given series is $(-3, 3]$ and the radius of convergence is $R = 3$. ◀

POWER SERIES IN $x - x_0$

If x_0 is a constant, and if x is replaced by $x - x_0$ in (3), then the resulting series has the form

$$\sum_{k=0}^{\infty} c_k(x-x_0)^k = c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + \cdots + c_k(x-x_0)^k + \cdots$$

This is called a ***power series in $x - x_0$***. Some examples are

$$\sum_{k=0}^{\infty} \frac{(x-1)^k}{k+1} = 1 + \frac{(x-1)}{2} + \frac{(x-1)^2}{3} + \frac{(x-1)^3}{4} + \cdots \qquad \boxed{x_0 = 1}$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k (x+3)^k}{k!} = 1 - (x+3) + \frac{(x+3)^2}{2!} - \frac{(x+3)^3}{3!} + \cdots \qquad \boxed{x_0 = -3}$$

The first of these is a power series in $x - 1$ and the second is a power series in $x + 3$. Note that a power series in x is a power series in $x - x_0$ in which $x_0 = 0$. More generally, the Taylor series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

is a power series in $x - x_0$.

The main result on convergence of a power series in $x - x_0$ can be obtained by substituting $x - x_0$ for x in Theorem 10.8.2. This leads to the following theorem.

10.8.3 THEOREM. *For a power series $\sum c_k (x - x_0)^k$, exactly one of the following statements is true:*

(*a*) *The series converges only for $x = x_0$.*

(*b*) *The series converges absolutely (and hence converges) for all real values of x.*

(*c*) *The series converges absolutely (and hence converges) for all x in some finite open interval $(x_0 - R, x_0 + R)$ and diverges if $x < x_0 - R$ or $x > x_0 + R$. At either of the values $x = x_0 - R$ or $x = x_0 + R$, the series may converge absolutely, converge conditionally, or diverge, depending on the particular series.*

It follows from this theorem that the set of values for which a power series in $x - x_0$ converges is always an interval centered at $x = x_0$; we call this the ***interval of convergence*** (Figure 10.8.2). In part (*a*) of Theorem 10.8.3 the interval of convergence reduces to the single value $x = x_0$, in which case we say that the series has ***radius of convergence $R = 0$***; in part (*b*) the interval of convergence is infinite (the entire real line), in which case we say that the series has ***radius of convergence $R = +\infty$***; and in part (*c*) the interval extends between $x_0 - R$ and $x_0 + R$, in which case we say that the series has ***radius of convergence R***.

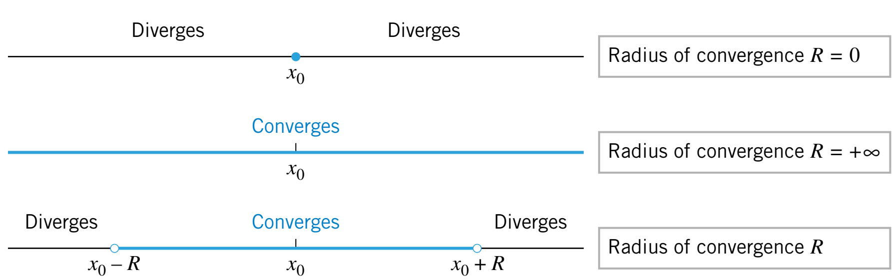


Figure 10.8.2

Example 4 Find the interval of convergence and radius of convergence of the series

$$\sum_{k=1}^{\infty} \frac{(x-5)^k}{k^2}$$

Solution. We apply the ratio test for absolute convergence.

$$\begin{aligned}\rho = \lim_{k\to+\infty}\left|\frac{u_{k+1}}{u_k}\right| &= \lim_{k\to+\infty}\left|\frac{(x-5)^{k+1}}{(k+1)^2}\cdot\frac{k^2}{(x-5)^k}\right| \\ &= \lim_{k\to+\infty}\left[|x-5|\left(\frac{k}{k+1}\right)^2\right] \\ &= |x-5|\lim_{k\to+\infty}\left(\frac{1}{1+(1/k)}\right)^2 = |x-5|\end{aligned}$$

Thus, the series converges absolutely if $|x-5| < 1$, or $-1 < x-5 < 1$, or $4 < x < 6$. The series diverges if $x < 4$ or $x > 6$.

To determine the convergence behavior at the endpoints $x = 4$ and $x = 6$, we substitute these values in the given series. If $x = 6$, the series becomes

$$\sum_{k=1}^{\infty}\frac{1^k}{k^2} = \sum_{k=1}^{\infty}\frac{1}{k^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

which is a convergent p-series ($p = 2$). If $x = 4$, the series becomes

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2} = -1 + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots$$

Since this series converges absolutely, the interval of convergence for the given series is [4, 6]. The radius of convergence is $R = 1$ (Figure 10.8.3). ◀

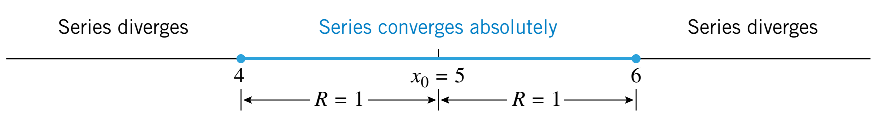


Figure 10.8.3

FOR THE READER. It will always be a waste of time to test for convergence at the endpoints of the interval of convergence using the ratio test, since ρ will always be 1 at those points if $\rho = \lim_{n\to+\infty}|a_{n+1}/a_n|$ exists. Explain why this must be so.

FUNCTIONS DEFINED BY POWER SERIES

If a function f is expressed as a power series on some interval, then we say that f is ***represented*** by the power series on that interval. For example, we saw in Example 4 of Section 10.4 that

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^k + \cdots$$

so that this power series represents the function $1/(1-x)$ on the interval $-1 < x < 1$.

Sometimes new functions actually originate as power series, and the properties of the functions are developed by working with their power series representations. For example, the functions

$$J_0(x) = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{2^{2k}(k!)^2} = 1 - \frac{x^2}{2^2(1!)^2} + \frac{x^4}{2^4(2!)^2} - \frac{x^6}{2^6(3!)^2} + \cdots \tag{4}$$

and

$$J_1(x) = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{2^{2k+1}(k!)(k+1)!} = \frac{x}{2} - \frac{x^3}{2^3(1!)(2!)} + \frac{x^5}{2^5(2!)(3!)} - \cdots \tag{5}$$

which are called ***Bessel functions*** in honor of the German mathematician and astronomer

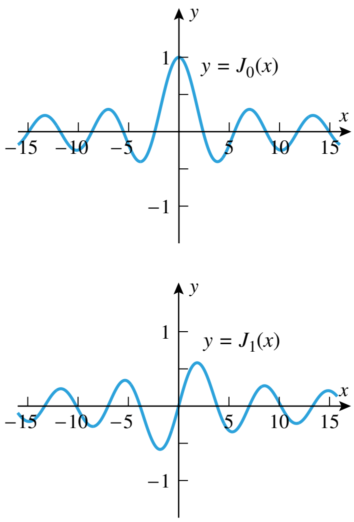


Figure 10.8.4

Friedrich Wilhelm Bessel (1784–1846), arise naturally in the study of planetary motion and in various problems that involve heat flow.

To find the domains of these functions, we must determine where their defining power series converge. For example, in the case $J_0(x)$ we have

$$\rho = \lim_{k\to+\infty}\left|\frac{u_{k+1}}{u_k}\right| = \lim_{k\to+\infty}\left|\frac{x^{2(k+1)}}{2^{2(k+1)}[(k+1)!]^2}\cdot\frac{2^{2k}(k!)^2}{x^{2k}}\right|$$

$$= \lim_{k\to+\infty}\left|\frac{x^2}{4(k+1)^2}\right| = 0 < 1$$

so that the series converges for all x; that is, the domain of $J_0(x)$ is $(-\infty, +\infty)$. We leave it as an exercise to show that the power series for $J_1(x)$ also converges for all x.

FOR THE READER. Many CAS programs have the Bessel functions as part of their libraries. If you have a CAS, read the documentation to determine whether it can graph $J_0(x)$ and $J_1(x)$; if so, generate the graphs shown in Figure 10.8.4.

EXERCISE SET 10.8 Graphing Utility

In Exercises 1–10, find the Maclaurin series for the function in sigma notation.

1. e^{-x} **2.** e^{ax} **3.** $\cos \pi x$ **4.** $\sin \pi x$

5. $\ln(1+x)$ **6.** $\dfrac{1}{1+x}$ **7.** $\cosh x$

8. $\sinh x$ **9.** $x \sin x$ **10.** xe^x

In Exercises 11–18, use sigma notation to write the Taylor series about $x = x_0$ for the given function.

11. $e^x;\ x_0 = 1$ **12.** $e^{-x};\ x_0 = \ln 2$

13. $\dfrac{1}{x};\ x_0 = -1$ **14.** $\dfrac{1}{x+2};\ x_0 = 3$

15. $\sin \pi x;\ x_0 = \dfrac{1}{2}$ **16.** $\cos x;\ x_0 = \dfrac{\pi}{2}$

17. $\ln x;\ x_0 = 1$ **18.** $\ln x;\ x_0 = e$

In Exercises 19–22, find the interval of convergence of the power series, and find a familiar function that is represented by the power series on that interval.

19. $1 - x + x^2 - x^3 + \cdots + (-1)^k x^k + \cdots$

20. $1 + x^2 + x^4 + \cdots + x^{2k} + \cdots$

21. $1 + (x-2) + (x-2)^2 + \cdots + (x-2)^k + \cdots$

22. $1 - (x+3) + (x+3)^2 - (x+3)^3 + \cdots + (-1)^k (x+3)^k + \cdots$

23. Suppose that the function f is represented by the power series

$$f(x) = 1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \cdots + (-1)^k \frac{x^k}{2^k} + \cdots$$

(a) Find the domain of f. (b) Find $f(0)$ and $f(1)$.

24. Suppose that the function f is represented by the power series

$$f(x) = 1 - \frac{x-5}{3} + \frac{(x-5)^2}{3^2} - \frac{(x-5)^3}{3^3} + \cdots$$

(a) Find the domain of f.
(b) Find $f(3)$ and $f(6)$.

In Exercises 25–48, find the radius of convergence and the interval of convergence.

25. $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k+1}$ **26.** $\displaystyle\sum_{k=0}^{\infty} 3^k x^k$ **27.** $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!}$

28. $\displaystyle\sum_{k=0}^{\infty} \frac{k!}{2^k} x^k$ **29.** $\displaystyle\sum_{k=1}^{\infty} \frac{5^k}{k^2} x^k$ **30.** $\displaystyle\sum_{k=2}^{\infty} \frac{x^k}{\ln k}$

31. $\displaystyle\sum_{k=1}^{\infty} \frac{x^k}{k(k+1)}$ **32.** $\displaystyle\sum_{k=0}^{\infty} \frac{(-2)^k x^{k+1}}{k+1}$

33. $\displaystyle\sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{\sqrt{k}}$ **34.** $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$

35. $\displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$ **36.** $\displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{x^{3k}}{k^{3/2}}$

37. $\displaystyle\sum_{k=0}^{\infty} \frac{3^k}{k!} x^k$ **38.** $\displaystyle\sum_{k=2}^{\infty} (-1)^{k+1} \frac{x^k}{k(\ln k)^2}$

39. $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{1+k^2}$ **40.** $\displaystyle\sum_{k=0}^{\infty} \frac{(x-3)^k}{2^k}$

41. $\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \frac{(x+1)^k}{k}$ **42.** $\displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{(x-4)^k}{(k+1)^2}$

43. $\displaystyle\sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^k (x+5)^k$ **44.** $\displaystyle\sum_{k=1}^{\infty} \frac{(2k+1)!}{k^3} (x-2)^k$

45. $\displaystyle\sum_{k=1}^{\infty}(-1)^k\frac{(x+1)^{2k+1}}{k^2+4}$

46. $\displaystyle\sum_{k=1}^{\infty}\frac{(\ln k)(x-3)^k}{k}$

47. $\displaystyle\sum_{k=0}^{\infty}\frac{\pi^k(x-1)^{2k}}{(2k+1)!}$

48. $\displaystyle\sum_{k=0}^{\infty}\frac{(2x-3)^k}{4^{2k}}$

49. Use the root test to find the interval of convergence of

$$\sum_{k=2}^{\infty}\frac{x^k}{(\ln k)^k}$$

50. Find the domain of the function

$$f(x)=\sum_{k=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2k-1)}{(2k-2)!}x^k$$

51. If a function f is represented by a power series on an interval, then the graphs of the partial sums can be used as approximations to the graph of f.

(a) Use a graphing utility to generate the graph of $1/(1-x)$ together with the graphs of the first four partial sums of its Maclaurin series over the interval $(-1, 1)$.

(b) In general terms, where are the graphs of the partial sums the most accurate?

52. Show that the power series representation of the Bessel function $J_1(x)$ converges for all x [Formula (5)].

53. Show that if p is a positive integer, then the power series

$$\sum_{k=0}^{\infty}\frac{(pk)!}{(k!)^p}x^k$$

has a radius of convergence of $1/p^p$.

54. Show that if p and q are positive integers, then the power series

$$\sum_{k=0}^{\infty}\frac{(k+p)!}{k!(k+q)!}x^k$$

has a radius of convergence of $+\infty$.

55. (a) Suppose that the power series $\sum c_k(x-x_0)^k$ has radius of convergence R and p is a nonzero constant. What can you say about the radius of convergence of the power series $\sum pc_k(x-x_0)^k$? Explain your reasoning. [*Hint:* See Theorem 10.5.3.]

(b) Suppose that the power series $\sum c_k(x-x_0)^k$ has a finite radius of convergence R, and the power series $\sum d_k(x-x_0)^k$ has a radius of convergence of $+\infty$. What can you say about the radius of convergence of $\sum(c_k+d_k)(x-x_0)^k$? Explain your reasoning.

(c) Suppose that the power series $\sum c_k(x-x_0)^k$ has a finite radius of convergence R_1 and the power series $\sum d_k(x-x_0)^k$ has a finite radius of convergence R_2. What can you say about the radius of convergence of $\sum(c_k+d_k)(x-x_0)^k$? Explain your reasoning.

56. Prove: If $\lim_{k\to+\infty}|c_k|^{1/k}=L$, where $L\neq 0$, then $1/L$ is the radius of convergence of the power series $\sum_{k=0}^{\infty}c_kx^k$.

57. Prove: If the power series $\sum_{k=0}^{\infty}c_kx^k$ has radius of convergence R, then the series $\sum_{k=0}^{\infty}c_kx^{2k}$ has radius of convergence $\sqrt{R}$.

58. Prove: If the interval of convergence of the series $\sum_{k=0}^{\infty}c_k(x-x_0)^k$ is $(x_0-R, x_0+R]$, then the series converges conditionally at x_0+R.

10.9 CONVERGENCE OF TAYLOR SERIES; COMPUTATIONAL METHODS

In the last section we introduced power series and intervals of convergence for power series. In this section we focus in particular on Taylor series, and we demonstrate the use of the Remainder Estimation Theorem from Section 10.1 as a tool to determine whether the Taylor series of a function converges to the function on some interval. We will also show how Taylor series can be used to approximate values of trigonometric, exponential, and logarithmic functions.

THE *n*TH REMAINDER

Recall that the nth Taylor polynomial for a function f about $x=x_0$ has the property that its value and the values of its first n derivatives match those of f at x_0. As n increases, more and more derivatives match up, so it is reasonable to hope that for values of x near x_0 the values of the Taylor polynomials might converge to the value of $f(x)$; that is,

$$\sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k\to f(x)\quad\text{as } n\to+\infty \tag{1}$$

However, the nth Taylor polynomial for f is the nth partial sum of the Taylor series for f, so (1) is equivalent to stating that the Taylor series for f converges at x, and its sum is $f(x)$. Thus, we are led to consider the following problem.

10.9.1 PROBLEM. Given a function f that has derivatives of all orders at $x = x_0$, determine whether there is an open interval containing x_0 such that $f(x)$ is the sum of its Taylor series about $x = x_0$ at each number in the interval; that is,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \tag{2}$$

for all values of x in the interval.

• FOR THE READER. Show that (2) holds at $x = x_0$, regardless of the function f.

To determine whether (2) holds on some open interval containing x_0, recall the nth remainder for f about $x = x_0$ as given in Formula (13) of Section 10.1,

$$R_n(x) = f(x) - p_n(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \tag{3}$$

where $p_n(x)$ is the nth Taylor polynomial for f about $x = x_0$.

One can think of $R_n(x)$ as the error that results at the domain value x when f is approximated by $p_n(x)$. Thus, for a particular value of x, if $p_n(x)$ converges to $f(x)$ as $n \to +\infty$, the error $R_n(x)$ must approach 0; conversely, if $R_n(x) \to 0$ as $n \to +\infty$, then the Taylor polynomials converge to f at x. More precisely:

10.9.2 THEOREM. *The equality*

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

holds at a number x if and only if $\lim_{n \to +\infty} R_n(x) = 0$.

ESTIMATING THE *n*TH REMAINDER

It is relatively rare that one can prove directly that $R_n(x) \to 0$ as $n \to +\infty$. Usually, this is proved indirectly by finding appropriate bounds on $|R_n(x)|$ and applying the Squeezing Theorem for Sequences. The Remainder Estimation Theorem (Theorem 10.1.4) provides a useful bound for this purpose. Recall that this theorem asserts that if M is an upper bound for $|f^{(n+1)}(x)|$ on an interval I containing x_0, then

$$|R_n(x)| \le \frac{M}{(n+1)!}|x - x_0|^{n+1} \tag{4}$$

for all x in I.

The following example illustrates how the Remainder Estimation Theorem is applied.

Example 1 Show that the Maclaurin series for $\cos x$ converges to $\cos x$ for all x; that is,

$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad (-\infty < x < +\infty)$$

Solution. From Theorem 10.9.2 we must show that $R_n(x) \to 0$ for all x as $n \to +\infty$. For this purpose let $f(x) = \cos x$, so that for all x we have

$$f^{(n+1)}(x) = \pm\cos x \quad \text{or} \quad f^{(n+1)}(x) = \pm\sin x$$

In all cases we have

$$|f^{(n+1)}(x)| \le 1$$

so we can apply (4) with $M = 1$ and $x_0 = 0$ to conclude that

$$0 \le |R_n(x)| \le \frac{|x|^{n+1}}{(n+1)!} \tag{5}$$

However, it follows from Formula (5) of Section 10.3 with $n+1$ in place of n and $|x|$ in place of x that

$$\lim_{n\to+\infty} \frac{|x|^{n+1}}{(n+1)!} = 0 \tag{6}$$

Thus, it follows from (5) and the Squeezing Theorem for Sequences (Theorem 10.2.5) that $|R_n(x)| \to 0$ as $n \to +\infty$; this implies that $R_n(x) \to 0$ as $n \to +\infty$ by Theorem 10.2.6. Since this is true for all x, we have proved that the Maclaurin series for $\cos x$ converges to $\cos x$ for all x. This is illustrated in Figure 10.9.1, where we can see how successive partial sums approximate the cosine curve more and more closely. ◀

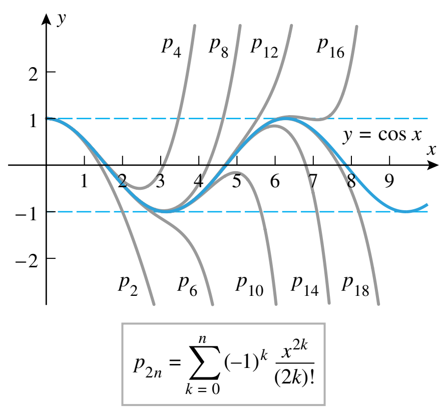


Figure 10.9.1

REMARK. The method used in Example 1 can be easily modified to prove that the Taylor series for $\cos x$ about any value $x = x_0$ converges to $\cos x$ for all x, and similarly that the Taylor series for $\sin x$ about any value $x = x_0$ converges to $\sin x$ for all x (Exercises 21 and 22). For reference, there is a list of some of the most important Maclaurin series in Table 10.9.1 at the end of this section.

APPROXIMATING TRIGONOMETRIC FUNCTIONS

In general, to approximate the value of a function f at a number x using a Taylor series, there are two basic questions that must be answered:

- About what domain value x_0 should the Taylor series be expanded?
- How many terms in the series should be used to achieve the desired accuracy?

In response to the first question, x_0 needs to be a number at which the derivatives of f can be evaluated easily, since these values are needed for the coefficients in the Taylor series. Furthermore, if the function f is being evaluated at x, then x_0 should be chosen as close as possible to x, since Taylor series tend to converge more rapidly near x_0. For example, to approximate $\sin 3^\circ$ ($= \pi/60$ radians), it would be reasonable to take $x_0 = 0$, since $\pi/60$ is close to 0 and the derivatives of $\sin x$ are easy to evaluate at 0. On the other hand, to approximate $\sin 85^\circ$ ($= 17\pi/36$ radians), it would be more natural to take $x_0 = \pi/2$, since $17\pi/36$ is close to $\pi/2$ and the derivatives of $\sin x$ are easy to evaluate at $\pi/2$.

In response to the second question posed above, the number of terms required to achieve a specific accuracy needs to be determined on a problem-by-problem basis. The next example gives two methods for doing this.

Example 2 Use the Maclaurin series for $\sin x$ to approximate $\sin 3^\circ$ to five decimal-place accuracy.

Solution. In the Maclaurin series

$$\sin x = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \tag{7}$$

the angle x is assumed to be in radians (because the differentiation formulas for the trigonometric functions were derived with this assumption). Since $3^\circ = \pi/60$ radians, it follows from (7) that

$$\sin 3^\circ = \sin\frac{\pi}{60} = \left(\frac{\pi}{60}\right) - \frac{(\pi/60)^3}{3!} + \frac{(\pi/60)^5}{5!} - \frac{(\pi/60)^7}{7!} + \cdots \tag{8}$$

We must now determine how many terms in the series are required to achieve five decimal-place accuracy. We will consider two possible approaches, one using the Remainder Estimation Theorem (Theorem 10.1.4) and the other using the fact that (8) satisfies the hypotheses of the alternating series test (Theorem 10.7.1).

Method 1 (The Remainder Estimation Theorem). Since we want to achieve five decimal-place accuracy, our goal is to choose n so that the absolute value of the nth remainder at $x = \pi/60$ does not exceed $0.000005 = 5 \times 10^{-6}$; that is,

$$\left|R_n\left(\frac{\pi}{60}\right)\right| \le 0.000005 \tag{9}$$

However, if we let $f(x) = \sin x$, then $f^{(n+1)}(x)$ is either $\pm\sin x$ or $\pm\cos x$, and in either case $|f^{(n+1)}(x)| \le 1$ for all x. Thus, it follows from the Remainder Estimation Theorem with $M = 1$, $x_0 = 0$, and $x = \pi/60$ that

$$\left|R_n\left(\frac{\pi}{60}\right)\right| \le \frac{|\pi/60|^{n+1}}{(n+1)!}$$

Thus, we can satisfy (9) by choosing n so that

$$\frac{|\pi/60|^{n+1}}{(n+1)!} \le 0.000005$$

With the help of a calculating utility you can verify that the smallest value of n that meets this criterion is $n = 3$. Thus, to achieve five decimal-place accuracy we need only keep terms up to the third power in (8). This yields

$$\sin 3^\circ \approx \left(\frac{\pi}{60}\right) - \frac{(\pi/60)^3}{3!} \approx 0.05234 \tag{10}$$

(verify). As a check, a calculator gives $\sin 3^\circ \approx 0.05233595624$, which agrees with (10) when rounded to five decimal places.

Method 2 (The Alternating Series Test). We leave it for you to check that (8) satisfies the hypotheses of the alternating series test (Theorem 10.7.1).

Let s_n denote the sum of the terms in (8) up to and including the nth power of $\pi/60$. Since the exponents in the series are odd integers, the integer n must be odd, and the exponent of the first term *not* included in the sum s_n must be $n + 2$. Thus, it follows from part (b) of Theorem 10.7.2 that

$$|\sin 3^\circ - s_n| < \frac{(\pi/60)^{n+2}}{(n+2)!}$$

This means that for five decimal-place accuracy we must look for the first positive odd integer n such that

$$\frac{(\pi/60)^{n+2}}{(n+2)!} \le 0.000005$$

With the help of a calculating utility you can verify that the smallest value of n that meets this criterion is $n = 3$. This agrees with the result obtained above using the Remainder Estimation Theorem and hence leads to approximation (10) as before. ◀

ROUNDOFF AND TRUNCATION ERROR

There are two types of errors that occur when computing with series. The first, called ***truncation error***, is the error that results when a series is approximated by a partial sum; and the second, called ***roundoff error***, is the error that arises from approximations in numerical computations. For example, in our derivation of (10) we took $n = 3$ to keep the truncation error below 0.000005. However, to evaluate the partial sum we had to approximate π, thereby introducing roundoff error. Had we not exercised some care in choosing this approximation, the roundoff error could easily have degraded the final result.

Methods for estimating and controlling roundoff error are studied in a branch of mathematics called ***numerical analysis***. However, as a rule of thumb, to achieve n decimal-place accuracy in a final result, all intermediate calculations must be accurate to at least $n + 1$ decimal places. Thus, in (10) at least six decimal-place accuracy in π is required to achieve the five decimal-place accuracy in the final numerical result. As a practical matter, a good working procedure is to perform all intermediate computations with the maximum number of digits that your calculating utility can handle and then round at the end.

APPROXIMATING EXPONENTIAL FUNCTIONS

Example 3 Show that the Maclaurin series for e^x converges to e^x for all x; that is,

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^k}{k!} + \cdots \qquad (-\infty < x < +\infty)$$

Solution. Let $f(x) = e^x$, so that

$$f^{(n+1)}(x) = e^x$$

We want to show that $R_n(x) \to 0$ as $n \to +\infty$ for all x in the interval $-\infty < x < +\infty$. However, it will be helpful here to consider the cases $x \le 0$ and $x > 0$ separately. If $x \le 0$, then we will take the interval I in the Remainder Estimation Theorem (Theorem 10.1.4) to be $[x, 0]$, and if $x > 0$, then we will take it to be $[0, x]$. Since $f^{(n+1)}(x) = e^x$ is an increasing function, it follows that if c is in the interval $[x, 0]$, then

$$|f^{(n+1)}(c)| \le |f^{(n+1)}(0)| = e^0 = 1$$

and if c is in the interval $[0, x]$, then

$$|f^{(n+1)}(c)| \le |f^{(n+1)}(x)| = e^x$$

Thus, we can apply Theorem 10.1.4 with $M = 1$ in the case where $x \le 0$ and with $M = e^x$ in the case where $x > 0$. This yields

$$0 \le |R_n(x)| \le \frac{|x|^{n+1}}{(n+1)!} \qquad \text{if } x \le 0$$

$$0 \le |R_n(x)| \le e^x \frac{|x|^{n+1}}{(n+1)!} \qquad \text{if } x > 0$$

Thus, in both cases it follows from (6) and the Squeezing Theorem for Sequences that $|R_n(x)| \to 0$ as $n \to +\infty$, which in turn implies that $R_n(x) \to 0$ as $n \to +\infty$. Since this is true for all x, we have proved that the Maclaurin series for e^x converges to e^x for all x. ◀

Since the Maclaurin series for e^x converges to e^x for all x, we can use partial sums of the Maclaurin series to approximate powers of e to arbitrary precision. Recall that in Example 6 of Section 10.1 we were able to use the Remainder Estimation Theorem to determine that evaluating the ninth Maclaurin polynomial for e^x at $x = 1$ yields an approximation for e with five decimal-place accuracy:

$$e \approx 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!} + \frac{1}{9!} \approx 2.71828$$

APPROXIMATING LOGARITHMS

The Maclaurin series

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad (-1 < x \le 1) \tag{11}$$

is the starting point for the approximation of natural logarithms. Unfortunately, the usefulness of this series is limited because of its slow convergence and the restriction $-1 < x \le 1$. However, if we replace x by $-x$ in this series, we obtain

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots \qquad (-1 \le x < 1) \tag{12}$$

and on subtracting (12) from (11) we obtain

$$\ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots\right) \qquad (-1 < x < 1) \tag{13}$$

Series (13), first obtained by James Gregory* in 1668, can be used to compute the natural logarithm of any positive number y by letting

$$y = \frac{1+x}{1-x}$$

or, equivalently,

$$x = \frac{y-1}{y+1} \tag{14}$$

and noting that $-1 < x < 1$. For example, to compute $\ln 2$ we let $y = 2$ in (14), which yields $x = \frac{1}{3}$. Substituting this value in (13) gives

$$\ln 2 = 2\left[\frac{1}{3} + \frac{\left(\frac{1}{3}\right)^3}{3} + \frac{\left(\frac{1}{3}\right)^5}{5} + \frac{\left(\frac{1}{3}\right)^7}{7} + \cdots\right] \tag{15}$$

In Exercise 19 we will ask you to show that five decimal-place accuracy can be achieved using the partial sum with terms up to and including the 13th power of $\frac{1}{3}$. Thus, to five decimal-place accuracy

$$\ln 2 \approx 2\left[\frac{1}{3} + \frac{\left(\frac{1}{3}\right)^3}{3} + \frac{\left(\frac{1}{3}\right)^5}{5} + \frac{\left(\frac{1}{3}\right)^7}{7} + \cdots + \frac{\left(\frac{1}{3}\right)^{13}}{13}\right] \approx 0.69315$$

(verify). As a check, a calculator gives $\ln 2 \approx 0.69314718056$, which agrees with the preceding approximation when rounded to five decimal places.

REMARK. In Example 2 of Section 10.7, we stated without proof that

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

This result can be obtained letting $x = 1$ in (11). However, this series converges too slowly to be of practical value.

APPROXIMATING π

In the next section we will show that

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad (-1 \le x \le 1) \tag{16}$$

Letting $x = 1$, we obtain

$$\frac{\pi}{4} = \tan^{-1} 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

*JAMES GREGORY (1638–1675). Scottish mathematician and astronomer. Gregory, the son of a minister, was famous in his time as the inventor of the Gregorian reflecting telescope, so named in his honor. Although he is not generally ranked with the great mathematicians, much of his work relating to calculus was studied by Leibniz and Newton and undoubtedly influenced some of their discoveries. There is a manuscript, discovered posthumously, which shows that Gregory had anticipated Taylor series well before Taylor.

or

$$\pi = 4\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right]$$

This famous series, obtained by Leibniz in 1674, converges too slowly to be of computational value. A more practical procedure for approximating π uses the identity

$$\frac{\pi}{4} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} \tag{17}$$

which was derived in Exercise 79 of Section 7.6. By using this identity and series (16) to approximate $\tan^{-1}\frac{1}{2}$ and $\tan^{-1}\frac{1}{3}$, the value of π can be approximated efficiently to any degree of accuracy.

BINOMIAL SERIES

If m is a real number, then the Maclaurin series for $(1+x)^m$ is called the ***binomial series***; it is given by (verify)

$$1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots + \frac{m(m-1)\cdots(m-k+1)}{k!}x^k + \cdots$$

In the case where m is a nonnegative integer, the function $f(x) = (1+x)^m$ is a polynomial of degree m, so

$$f^{(m+1)}(0) = f^{(m+2)}(0) = f^{(m+3)}(0) = \cdots = 0$$

and the binomial series reduces to the familiar binomial expansion

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots + x^m$$

which is valid for $-\infty < x < +\infty$.

It can be proved that if m is not a nonnegative integer, then the binomial series converges to $(1+x)^m$ if $|x| < 1$. Thus, for such values of x

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \cdots + \frac{m(m-1)\cdots(m-k+1)}{k!}x^k + \cdots \tag{18}$$

or in sigma notation,

$$(1+x)^m = 1 + \sum_{k=1}^{\infty} \frac{m(m-1)\cdots(m-k+1)}{k!}x^k \quad \text{if } |x| < 1 \tag{19}$$

Example 4 Find binomial series for

(a) $\dfrac{1}{(1+x)^2}$ (b) $\dfrac{1}{\sqrt{1+x}}$

Solution (a). Since the general term of the binomial series is complicated, you may find it helpful to write out some of the beginning terms of the series, as in Formula (18), to see developing patterns. Substituting $m = -2$ in this formula yields

$$\begin{aligned}
\frac{1}{(1+x)^2} = (1+x)^{-2} &= 1 + (-2)x + \frac{(-2)(-3)}{2!}x^2 \\
&\quad + \frac{(-2)(-3)(-4)}{3!}x^3 + \frac{(-2)(-3)(-4)(-5)}{4!}x^4 + \cdots \\
&= 1 - 2x + \frac{3!}{2!}x^2 - \frac{4!}{3!}x^3 + \frac{5!}{4!}x^4 + \cdots \\
&= 1 - 2x + 3x^2 - 4x^3 + 5x^4 + \cdots \\
&= \sum_{k=0}^{\infty} (-1)^k (k+1)x^k
\end{aligned}$$

Solution (b). Substituting $m = -\frac{1}{2}$ in (18) yields

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}x^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{3!}x^3 - \cdots$$

$$= 1 - \frac{1}{2}x + \frac{1\cdot 3}{2^2\cdot 2!}x^2 - \frac{1\cdot 3\cdot 5}{2^3\cdot 3!}x^3 + \cdots$$

$$= 1 + \sum_{k=1}^{\infty}(-1)^k\frac{1\cdot 3\cdot 5\cdots(2k-1)}{2^k k!}x^k$$

◀

For reference, Table 10.9.1 lists the Maclaurin series for some of the most important functions, together with a specification of the intervals over which the Maclaurin series converge to those functions. Some of these results are derived in the exercises and others will be derived in the next section using some special techniques that we will develop.

Table 10.9.1

MACLAURIN SERIES	INTERVAL OF CONVERGENCE
$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots$	$-1 < x < 1$
$\frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k} = 1 - x^2 + x^4 - x^6 + \cdots$	$-1 < x < 1$
$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$	$-\infty < x < +\infty$
$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	$-\infty < x < +\infty$
$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	$-\infty < x < +\infty$
$\ln(1+x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	$-1 < x \le 1$
$\tan^{-1} x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$	$-1 \le x \le 1$
$\sinh x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$	$-\infty < x < +\infty$
$\cosh x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$	$-\infty < x < +\infty$
$(1+x)^m = 1 + \sum_{k=1}^{\infty} \frac{m(m-1)\cdots(m-k+1)}{k!} x^k$	$-1 < x < 1$* $(m \ne 0, 1, 2, \ldots)$

*The behavior at the endpoints depends on m: For $m > 0$ the series converges absolutely at both endpoints; for $m \le -1$ the series diverges at both endpoints; and for $-1 < m < 0$ the series converges conditionally at $x = 1$ and diverges at $x = -1$.

EXERCISE SET 10.9 Graphing Utility C CAS

1. Use both of the methods given in Example 2 to approximate $\sin 4°$ to five decimal-place accuracy, and check your work by comparing your answer to that produced directly by your calculating utility.

2. Use both of the methods given in Example 2 to approximate $\cos 3°$ to three decimal-place accuracy, and check your work by comparing your answer to that produced directly by your calculating utility.

3. Use the Maclaurin series for $\cos x$ to approximate $\cos 0.1$ to five decimal-place accuracy, and check your work by comparing your answer to that produced directly by your calculating utility.

4. Use the Maclaurin series for $\tan^{-1} x$ to approximate $\tan^{-1} 0.1$ to three decimal-place accuracy, and check your work by comparing your answer to that produced directly by your calculating utility.

5. Use an appropriate Taylor series to approximate $\sin 85°$ to four decimal-place accuracy, and check your work by comparing your answer to that produced directly by your calculating utility.

6. Use a Taylor series to approximate $\cos(-175°)$ to four decimal-place accuracy, and check your work by comparing your answer to that produced directly by your calculating utility.

7. Use the Maclaurin series for $\sinh x$ to approximate $\sinh 0.5$ to three decimal-place accuracy. Check your work by computing $\sinh 0.5$ with a calculating utility.

8. Use the Maclaurin series for $\cosh x$ to approximate $\cosh 0.1$ to three decimal-place accuracy. Check your work by computing $\cosh 0.1$ with a calculating utility.

9. Use the Remainder Estimation Theorem and the method of Example 1 to prove that the Taylor series for $\sin x$ about $x = \pi/4$ converges to $\sin x$ for all x.

10. Use the Remainder Estimation Theorem and the method of Example 3 to prove that the Taylor series for e^x about $x = 1$ converges to e^x for all x.

11. (a) Use Formula (13) in the text to find a series that converges to ln 1.25.
    (b) Approximate ln 1.25 using the first two terms of the series. Round your answer to three decimal places, and compare the result to that produced directly by your calculating utility.

12. (a) Use Formula (13) to find a series that converges to ln 3.
    (b) Approximate ln 3 using the first two terms of the series. Round your answer to three decimal places, and compare the result to that produced directly by your calculating utility.

13. (a) Use the Maclaurin series for $\tan^{-1} x$ to approximate $\tan^{-1} \frac{1}{2}$ and $\tan^{-1} \frac{1}{3}$ to three decimal-place accuracy.
    (b) Use the results in part (a) and Formula (17) to approximate π.
    (c) Would you be willing to guarantee that your answer in part (b) is accurate to three decimal places? Explain your reasoning.
    (d) Compare your answer in part (b) to that produced by your calculating utility.

14. Use an appropriate Taylor series for $\sqrt[3]{x}$ to approximate $\sqrt[3]{28}$ to three decimal-place accuracy, and check your answer by comparing it to that produced directly by your calculating utility.

15. (a) Find an upper bound on the error that can result if $\cos x$ is approximated by $1 - (x^2/2!) + (x^4/4!)$ over the interval $[-0.2, 0.2]$.
    (b) Check your answer in part (a) by graphing

$$\left|\cos x - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\right)\right|$$

over the interval.

16. (a) Find an upper bound on the error that can result if $\ln(1 + x)$ is approximated by x over the interval $[-0.01, 0.01]$.
    (b) Check your answer in part (a) by graphing

$$|\ln(1 + x) - x|$$

over the interval.

17. Use Formula (18) for the binomial series to obtain the Maclaurin series for

(a) $\dfrac{1}{1 + x}$ (b) $\sqrt[3]{1 + x}$ (c) $\dfrac{1}{(1 + x)^3}$.

18. If m is any real number, and k is a nonnegative integer, then we define the ***binomial coefficient***

$$\binom{m}{k} \text{ by the formulas } \binom{m}{0} = 1 \text{ and}$$

$$\binom{m}{k} = \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!}$$

for $k \geq 1$. Express Formula (18) in the text in terms of binomial coefficients.

19. In this exercise we will use the Remainder Estimation Theorem to determine the number of terms that are required in Formula (15) to approximate ln 2 to five decimal-place accuracy. For this purpose let

$$f(x) = \ln \frac{1 + x}{1 - x} = \ln(1 + x) - \ln(1 - x) \quad (-1 < x < 1)$$

(a) Show that

$$f^{(n+1)}(x) = n!\left[\frac{(-1)^n}{(1 + x)^{n+1}} + \frac{1}{(1 - x)^{n+1}}\right]$$

(b) Use the triangle inequality [Theorem 1.2.2(*d*)] to show that

$$|f^{(n+1)}(x)| \le n!\left[\frac{1}{(1+x)^{n+1}} + \frac{1}{(1-x)^{n+1}}\right]$$

(c) Since we want to achieve five decimal-place accuracy, our goal is to choose n so that the absolute value of the nth remainder at $x = \frac{1}{3}$ does not exceed the value $0.000005 = 0.5 \times 10^{-5}$; that is, $\left|R_n\left(\frac{1}{3}\right)\right| \le 0.000005$. Use the Remainder Estimation Theorem to show that this condition will be satisfied if n is chosen so that

$$\frac{M}{(n+1)!}\left(\frac{1}{3}\right)^{n+1} \le 0.000005$$

where $|f^{(n+1)}(x)| \le M$ on the interval $\left[0, \frac{1}{3}\right]$.

(d) Use the result in part (b) to show that M can be taken as

$$M = n!\left[1 + \frac{1}{\left(\frac{2}{3}\right)^{n+1}}\right]$$

(e) Use the results in parts (c) and (d) to show that five decimal-place accuracy will be achieved if n satisfies

$$\frac{1}{n+1}\left[\left(\frac{1}{3}\right)^{n+1} + \left(\frac{1}{2}\right)^{n+1}\right] \le 0.000005$$

and then show that the smallest value of n that satisfies this condition is $n = 13$.

20. Use Formula (13) and the method of Exercise 19 to approximate $\ln\left(\frac{5}{3}\right)$ to five decimal-place accuracy. Then check your work by comparing your answer to that produced directly by your calculating utility.

21. Prove: The Taylor series for $\cos x$ about any value $x = x_0$ converges to $\cos x$ for all x.

22. Prove: The Taylor series for $\sin x$ about any value $x = x_0$ converges to $\sin x$ for all x.

C **23.** (a) In 1706 the British astronomer and mathematician John Machin discovered the following formula for $\pi/4$, called ***Machin's formula***:

$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$$

Use a CAS to approximate $\pi/4$ using Machin's formula to 25 decimal places.

(b) In 1914 the brilliant Indian mathematician Srinivasa Ramanujan (1887–1920) showed that

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801}\sum_{k=0}^{\infty}\frac{(4k)!(1103 + 26{,}390k)}{(k!)^4 396^{4k}}$$

Use a CAS to compute the first four partial sums in ***Ramanujan's formula***.

24. The purpose of this exercise is to show that the Taylor series of a function f may possibly converge to a value different from $f(x)$ for certain x. Let

$$f(x) = \begin{cases} e^{-1/x^2}, & x \ne 0 \\ 0, & x = 0 \end{cases}$$

(a) Use the definition of a derivative to show that $f'(0) = 0$.

(b) With some difficulty it can be shown that $f^{(n)}(0) = 0$ for $n \ge 2$. Accepting this fact, show that the Maclaurin series of f converges for all x, but converges to $f(x)$ only at $x = 0$.

10.10 DIFFERENTIATING AND INTEGRATING POWER SERIES; MODELING WITH TAYLOR SERIES

In this section we will discuss methods for finding power series for derivatives and integrals of functions, and we will discuss some practical methods for finding Taylor series that can be used in situations where it is difficult or impossible to find the series directly.

DIFFERENTIATING POWER SERIES

We begin by considering the following problem:

10.10.1 PROBLEM. Suppose that a function f is represented by a power series on an open interval. How can we use the power series to find the derivative of f on that interval?

The solution to this problem can be motivated by considering the Maclaurin series for $\sin x$:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad (-\infty < x < +\infty)$$

Of course, we already know that the derivative of $\sin x$ is $\cos x$; however, we are concerned here with using the Maclaurin series to deduce this. The solution is easy—all we need to do is differentiate the Maclaurin series term by term and observe that the resulting series is the Maclaurin series for $\cos x$:

$$\frac{d}{dx}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right] = 1 - 3\frac{x^2}{3!} + 5\frac{x^4}{5!} - 7\frac{x^6}{7!} + \cdots$$
$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \cos x$$

Here is another example.

$$\frac{d}{dx}[e^x] = \frac{d}{dx}\left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right]$$
$$= 1 + 2\frac{x}{2!} + 3\frac{x^2}{3!} + 4\frac{x^3}{4!} + \cdots = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x$$

• FOR THE READER. See whether you can use this method to find the derivative of $\cos x$.

The preceding computations suggest that if a function f is represented by a power series on an open interval, then a power series representation of f' on that interval can be obtained by differentiating the power series for f term by term. This is stated more precisely in the following theorem, which we give without proof.

10.10.2 THEOREM **(*Differentiation of Power Series*).** *Suppose that a function f is represented by a power series in $x - x_0$ that has a nonzero radius of convergence R; that is,*

$$f(x) = \sum_{k=0}^{\infty} c_k(x - x_0)^k \qquad (x_0 - R < x < x_0 + R)$$

Then:

(*a*) *The function f is differentiable on the interval $(x_0 - R, x_0 + R)$.*

(*b*) *If the power series representation for f is differentiated term by term, then the resulting series has radius of convergence R and converges to f' on the interval $(x_0 - R, x_0 + R)$; that is,*

$$f'(x) = \sum_{k=0}^{\infty} \frac{d}{dx}[c_k(x - x_0)^k] \qquad (x_0 - R < x < x_0 + R)$$

This theorem has an important implication about the differentiability of functions that are represented by power series. According to the theorem, the power series for f' has the same radius of convergence as the power series for f, and this means that the theorem can be applied to f' as well as f. However, if we do this, then we conclude that f' is differentiable on the interval $(x_0 - R, x_0 + R)$, and the power series for f'' has the same radius of convergence as the power series for f and f'. We can now repeat this process ad infinitum, applying the theorem successively to $f'', f''', \ldots, f^{(n)}, \ldots$ to conclude that f has derivatives of all orders on the interval $(x_0 - R, x_0 + R)$. Thus, we have established the following result.

10.10.3 THEOREM. *If a function f can be represented by a power series in $x - x_0$ with a nonzero radius of convergence R, then f has derivatives of all orders on the interval $(x_0 - R, x_0 + R)$.*

In short, it is only the most "well-behaved" functions that can be represented by power series; that is, if a function f does not possess derivatives of all orders on an interval $(x_0 - R, x_0 + R)$, then it cannot be represented by a power series in $x - x_0$ on that interval.

Example 1 In Section 10.8, we showed that the Bessel function $J_0(x)$ is represented by the power series

$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}(k!)^2} \tag{1}$$

with radius of convergence $+\infty$ [see Formula (4) of that section and the related discussion]. Thus, $J_0(x)$ has derivatives of all orders on the interval $(-\infty, +\infty)$, and these can be obtained by differentiating the series term by term. For example, if we write (1) as

$$J_0(x) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}(k!)^2}$$

and differentiate term by term, we obtain

$$J_0'(x) = \sum_{k=1}^{\infty} \frac{(-1)^k (2k) x^{2k-1}}{2^{2k}(k!)^2} = \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k-1}}{2^{2k-1}k!(k-1)!}$$ ◀

REMARK. The computations in this example use some techniques that are worth noting. First, when a power series is expressed in sigma notation, the formula for the general term of the series will often not be of a form that can be used for differentiating the constant term. Thus, if the series has a nonzero constant term, as here, it is usually a good idea to split it off from the summation before differentiating. Second, observe how we simplified the final formula by canceling the factor k from one of the factorials in the denominator. This is a standard simplification technique.

INTEGRATING POWER SERIES

Since the derivative of a function that is represented by a power series can be obtained by differentiating the series term by term, it should not be surprising that an antiderivative of a function represented by a power series can be obtained by integrating the series term by term. For example, we know that $\sin x$ is an antiderivative of $\cos x$. Here is how this result can be obtained by integrating the Maclaurin series for $\cos x$ term by term:

$$\begin{aligned}
\int \cos x \, dx &= \int \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right] dx \\
&= \left[x - \frac{x^3}{3(2!)} + \frac{x^5}{5(4!)} - \frac{x^7}{7(6!)} + \cdots\right] + C \\
&= \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right] + C = \sin x + C
\end{aligned}$$

The same idea applies to definite integrals. For example, by direct integration we have

$$\int_0^1 \frac{dx}{1+x^2} = \tan^{-1} x \Big]_0^1 = \tan^{-1} 1 - \tan 0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

and we will show later in this section that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \tag{2}$$

Thus,

$$\int_0^1 \frac{dx}{1+x^2} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Here is how this result can be obtained by integrating the Maclaurin series for $1/(1+x^2)$ term by term (see Table 10.9.1):

$$\begin{aligned}
\int_0^1 \frac{dx}{1+x^2} &= \int_0^1 [1 - x^2 + x^4 - x^6 + \cdots] \, dx \\
&= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \Big]_0^1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots
\end{aligned}$$

The preceding computations are justified by the following theorem, which we give without proof.

10.10.4 THEOREM (*Integration of Power Series*). *Suppose that a function f is represented by a power series in $x - x_0$ that has a nonzero radius of convergence R; that is,*

$$f(x) = \sum_{k=0}^{\infty} c_k (x - x_0)^k \qquad (x_0 - R < x < x_0 + R)$$

(*a*) *If the power series representation of f is integrated term by term, then the resulting series has radius of convergence R and converges to an antiderivative for $f(x)$ on the interval $(x_0 - R, x_0 + R)$; that is,*

$$\int f(x)\,dx = \sum_{k=0}^{\infty} \left[\frac{c_k}{k+1}(x - x_0)^{k+1}\right] + C \qquad (x_0 - R < x < x_0 + R)$$

(*b*) *If α and β are points in the interval $(x_0 - R, x_0 + R)$, and if the power series representation of f is integrated term by term from α to β, then the resulting series converges absolutely on the interval $(x_0 - R, x_0 + R)$ and*

$$\int_{\alpha}^{\beta} f(x)\,dx = \sum_{k=0}^{\infty} \left[\int_{\alpha}^{\beta} c_k (x - x_0)^k\,dx\right]$$

POWER SERIES REPRESENTATIONS MUST BE TAYLOR SERIES

For many functions it is difficult or impossible to find the derivatives that are required to obtain a Taylor series. For example, to find the Maclaurin series for $1/(1 + x^2)$ directly would require some tedious derivative computations (try it). A more practical approach is to substitute $-x^2$ for x in the geometric series

$$\frac{1}{1 - x} = 1 + x + x^2 + x^3 + x^4 + \cdots \qquad (-1 < x < 1)$$

to obtain

$$\frac{1}{1 + x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots$$

However, there are two questions of concern with this procedure:

- Where does the power series that we obtained for $1/(1 + x^2)$ actually converge to $1/(1 + x^2)$?
- How do we know that the power series we have obtained is actually the Maclaurin series for $1/(1 + x^2)$?

The first question is easy to resolve. Since the geometric series converges to $1/(1 - x)$ if $|x| < 1$, the second series will converge to $1/(1 + x^2)$ if $|-x^2| < 1$ or $|x^2| < 1$. However, this is true if and only if $|x| < 1$, so the power series we obtained for the function $1/(1 + x^2)$ converges to this function if $-1 < x < 1$.

The second question is more difficult to answer and leads us to the following general problem.

10.10.5 PROBLEM. Suppose that a function f is represented by a power series in $x - x_0$ that has a nonzero radius of convergence. What relationship exists between the given power series and the Taylor series for f about $x = x_0$?

The answer is that they are the same; and here is the theorem that proves it.

10.10.6 THEOREM. *If a function f is represented by a power series in $x - x_0$ on some open interval containing x_0, then that power series is the Taylor series for f about $x = x_0$.*

Proof. Suppose that

$$f(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + \cdots + c_k(x - x_0)^k + \cdots$$

for all x in some open interval containing x_0. To prove that this is the Taylor series for f about $x = x_0$, we must show that

$$c_k = \frac{f^{(k)}(x_0)}{k!} \quad \text{for} \quad k = 0, 1, 2, 3, \ldots$$

However, the assumption that the series converges to $f(x)$ on an open interval containing x_0 ensures that it has a nonzero radius of convergence R; hence we can differentiate term by term in accordance with Theorem 10.10.2. Thus,

$$\begin{aligned}
f(x) &= c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + c_3(x - x_0)^3 + c_4(x - x_0)^4 + \cdots \\
f'(x) &= c_1 + 2c_2(x - x_0) + 3c_3(x - x_0)^2 + 4c_4(x - x_0)^3 + \cdots \\
f''(x) &= 2!c_2 + (3 \cdot 2)c_3(x - x_0) + (4 \cdot 3)c_4(x - x_0)^2 + \cdots \\
f'''(x) &= 3!c_3 + (4 \cdot 3 \cdot 2)c_4(x - x_0) + \cdots \\
&\vdots
\end{aligned}$$

On substituting $x = x_0$, all the powers of $x - x_0$ drop out, leaving

$$f(x_0) = c_0, \quad f'(x_0) = c_1, \quad f''(x_0) = 2!c_2, \quad f'''(x_0) = 3!c_3, \ldots$$

from which we obtain

$$c_0 = f(x_0), \quad c_1 = f'(x_0), \quad c_2 = \frac{f''(x_0)}{2!}, \quad c_3 = \frac{f'''(x_0)}{3!}, \ldots$$

which shows that the coefficients $c_0, c_1, c_2, c_3, \ldots$ are precisely the coefficients in the Taylor series about x_0 for $f(x)$. ∎

REMARK. This theorem tells us that no matter how we arrive at a power series representation of a function f, be it by substitution, by differentiation, by integration, or by some sort of algebraic manipulation, that series will be the Taylor series for f about $x = x_0$, provided that it converges to f on some open interval containing x_0.

SOME PRACTICAL WAYS TO FIND TAYLOR SERIES

Example 2 Find the Maclaurin series for $\tan^{-1} x$.

Solution. It would be tedious to find the Maclaurin series directly. A better approach is to start with the formula

$$\int \frac{1}{1 + x^2}\, dx = \tan^{-1} x + C$$

and integrate the Maclaurin series

$$\frac{1}{1 + x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots \qquad (-1 < x < 1)$$

term by term. This yields

$$\tan^{-1} x + C = \int \frac{1}{1 + x^2}\, dx = \int [1 - x^2 + x^4 - x^6 + x^8 - \cdots]\, dx$$

or

$$\tan^{-1} x = \left[x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots\right] - C$$

The constant of integration can be evaluated by substituting $x = 0$ and using the condition $\tan^{-1} 0 = 0$. This gives $C = 0$, so that

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots \qquad (-1 < x < 1) \tag{3}$$

◀

REMARK. Observe that neither Theorem 10.10.2 nor Theorem 10.10.3 addresses what happens at the endpoints of the interval of convergence. However, it can be proved that if the Taylor series for f about $x = x_0$ converges to $f(x)$ for all x in the interval $(x_0 - R, x_0 + R)$, and if the Taylor series converges at the right endpoint $x_0 + R$, then the value that it converges to at that point is the limit of $f(x)$ as $x \to x_0 + R$ from the left; and if the Taylor series converges at the left endpoint $x_0 - R$, then the value that it converges to at that point is the limit of $f(x)$ as $x \to x_0 - R$ from the right.

For example, the Maclaurin series for $\tan^{-1} x$ given in (3) converges at both $x = -1$ and $x = 1$, since the hypotheses of the alternating series test (Theorem 10.7.1) are satisfied at those points. Thus, the continuity of $\tan^{-1} x$ on the interval $[-1, 1]$ implies that at $x = 1$ the Maclaurin series converges to

$$\lim_{x \to 1^-} \tan^{-1} x = \tan^{-1} 1 = \frac{\pi}{4}$$

and at $x = -1$ it converges to

$$\lim_{x \to -1^+} \tan^{-1} x = \tan^{-1}(-1) = -\frac{\pi}{4}$$

This shows that the Maclaurin series for $\tan^{-1} x$ actually converges to $\tan^{-1} x$ on the interval $-1 \le x \le 1$. Moreover, the convergence at $x = 1$ establishes Formula (2).

Taylor series provide an alternative to Simpson's rule and other numerical methods for approximating definite integrals.

Example 3 Approximate the integral

$$\int_0^1 e^{-x^2}\, dx$$

to three decimal-place accuracy by expanding the integrand in a Maclaurin series and integrating term by term.

Solution. The simplest way to obtain the Maclaurin series for e^{-x^2} is to replace x by $-x^2$ in the Maclaurin series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

to obtain

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots$$

Therefore,

$$\begin{aligned}\int_0^1 e^{-x^2}\, dx &= \int_0^1 \left[1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots\right] dx \\ &= \left[x - \frac{x^3}{3} + \frac{x^5}{5(2!)} - \frac{x^7}{7(3!)} + \frac{x^9}{9(4!)} - \cdots\right]_0^1 \\ &= 1 - \frac{1}{3} + \frac{1}{5 \cdot 2!} - \frac{1}{7 \cdot 3!} + \frac{1}{9 \cdot 4!} - \cdots \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)k!}\end{aligned}$$

Since this series clearly satisfies the hypotheses of the alternating series test (Theorem 10.7.1), it follows from Theorem 10.7.2 that if we approximate the integral by s_n (the nth partial sum of the series), then

$$\left|\int_0^1 e^{-x^2}\,dx - s_n\right| < \frac{1}{[2(n+1)+1](n+1)!} = \frac{1}{(2n+3)(n+1)!}$$

Thus, for three decimal-place accuracy we must choose n such that

$$\frac{1}{(2n+3)(n+1)!} \le 0.0005 = 5 \times 10^{-4}$$

With the help of a calculating utility you can show that the smallest value of n that satisfies this condition is $n = 5$. Thus, the value of the integral to three decimal-place accuracy is

$$\int_0^1 e^{-x^2}\,dx \approx 1 - \frac{1}{3} + \frac{1}{5\cdot 2!} - \frac{1}{7\cdot 3!} + \frac{1}{9\cdot 4!} - \frac{1}{11\cdot 5!} \approx 0.747$$

As a check, a calculator with a built-in numerical integration capability produced the approximation 0.746824, which agrees with our result when rounded to three decimal places. ◀

FOR THE READER. What advantages does the method in this example have over Simpson's rule? What are its disadvantages?

FINDING MACLAURIN SERIES BY MULTIPLICATION AND DIVISION

The following examples illustrate some algebraic techniques that are sometimes useful for finding Taylor series.

Example 4 Find the first three nonzero terms in the Maclaurin series for the function $f(x) = e^{-x^2}\tan^{-1} x$.

$$\begin{array}{rl} & 1 - x^2 + \frac{x^4}{2} - \cdots \\ \times & x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \\ \hline & x - x^3 + \frac{x^5}{2} - \cdots \\ & -\frac{x^3}{3} + \frac{x^5}{3} - \frac{x^7}{6} + \cdots \\ & \frac{x^5}{5} - \frac{x^7}{5} + \cdots \\ \hline & x - \frac{4}{3}x^3 + \frac{31}{30}x^5 - \cdots \end{array}$$

Solution. Using the series for e^{-x^2} and $\tan^{-1} x$ obtained in Examples 2 and 3 gives

$$e^{-x^2}\tan^{-1} x = \left(1 - x^2 + \frac{x^4}{2} - \cdots\right)\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots\right)$$

Multiplying, as shown in the margin, we obtain

$$e^{-x^2}\tan^{-1} x = x - \frac{4}{3}x^3 + \frac{31}{30}x^5 - \cdots$$

More terms in the series can be obtained by including more terms in the factors. Moreover, one can prove that a series obtained by this method converges at each point in the intersection of the intervals of convergence of the factors (and possibly on a larger interval). Thus, we can be certain that the series we have obtained converges for all x in the interval $-1 \le x \le 1$ (why?). ◀

$$\begin{array}{r|l} & x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots \\ \hline 1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots & x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots \\ & x - \frac{x^3}{2} + \frac{x^5}{24} - \cdots \\ \hline & \frac{x^3}{3} - \frac{x^5}{30} + \cdots \\ & \frac{x^3}{3} - \frac{x^5}{6} + \cdots \\ \hline & \frac{2x^5}{15} + \cdots \end{array}$$

FOR THE READER. If you have a CAS, read the documentation about multiplying polynomials, and then use the CAS to duplicate the result in the last example.

Example 5 Find the first three nonzero terms in the Maclaurin series for $\tan x$.

Solution. Using the first three terms in the Maclaurin series for $\sin x$ and $\cos x$, we can express $\tan x$ as

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots}$$

Dividing, as shown in the margin, we obtain

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$$ ◀

MODELING PHYSICAL LAWS WITH TAYLOR SERIES

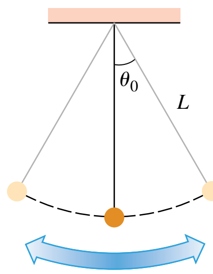


Figure 10.10.1

Taylor series provide an important way of modeling physical laws. To illustrate the idea we will consider the problem of modeling the period of a simple pendulum (Figure 10.10.1). As explained in Exercise 38 of the supplementary exercises to Chapter 8, the period T of such a pendulum is given by

$$T = 4\sqrt{\frac{L}{g}}\int_0^{\pi/2} \frac{1}{\sqrt{1-k^2\sin^2\phi}}\,d\phi \tag{4}$$

where

L = length of the supporting rod

g = acceleration due to gravity

$k = \sin(\theta_0/2)$, where θ_0 is the initial angle of displacement from the vertical

The integral, which is called a ***complete elliptic integral of the first kind***, cannot be expressed in terms of elementary functions and is often approximated by numerical methods. Unfortunately, numerical values are so specific that they often give little insight into general physical principles. However, if we expand the integrand of (4) in a Maclaurin series and integrate term by term, then we can generate an infinite series that can be used to construct various mathematical models for the period T that give a deeper understanding of the behavior of the pendulum.

To obtain the Maclaurin series for the integrand, we will substitute $-k^2\sin^2\phi$ for x in the binomial series for $1/\sqrt{1+x}$ that we derived in Example 4 of Section 10.9. If we do this, then we can rewrite (4) as

$$T = 4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\left[1+\frac{1}{2}k^2\sin^2\phi+\frac{1\cdot 3}{2^2 2!}k^4\sin^4\phi+\frac{1\cdot 3\cdot 5}{2^3 3!}k^6\sin^6\phi+\cdots\right]d\phi \tag{5}$$

If we integrate term by term, then we can produce a Maclaurin series that converges to the period T. However, one of the most important cases of pendulum motion occurs when the initial displacement is small, in which case all subsequent displacements are small, and we can assume that $k = \sin(\theta_0/2) \approx 0$. In this case we expect the convergence of the Maclaurin series for T to be rapid, and we can approximate the sum of the series by dropping all but the constant term in (5). This yields

$$T = 2\pi\sqrt{\frac{L}{g}} \tag{6}$$

which is called the ***first-order model*** of T or the model for ***small vibrations***. This model can be improved on by using more terms in the series. For example, if we use the first two terms in the Maclaurin series, we obtain the ***second-order model***

$$T = 2\pi\sqrt{\frac{L}{g}}\left(1+\frac{k^2}{4}\right) \tag{7}$$

(verify).

EXERCISE SET 10.10 C CAS

1. In each part, obtain the Maclaurin series for the function by making an appropriate substitution in the Maclaurin series for $1/(1-x)$. Include the general term in your answer, and state the radius of convergence of the series.

(a) $\dfrac{1}{1+x}$ (b) $\dfrac{1}{1-x^2}$ (c) $\dfrac{1}{1-2x}$ (d) $\dfrac{1}{2-x}$

2. In each part, obtain the Maclaurin series for the function by making an appropriate substitution in the Maclaurin series for $\ln(1+x)$. Include the general term in your answer, and state the radius of convergence of the series.

(a) $\ln(1-x)$ (b) $\ln(1+x^2)$

(c) $\ln(1+2x)$ (d) $\ln(2+x)$

3. In each part, obtain the first four nonzero terms of the Maclaurin series for the function by making an appropriate substitution in one of the binomial series obtained in Example 4 of Section 10.9.
(a) $(2+x)^{-1/2}$ (b) $(1-x^2)^{-2}$

4. (a) Use the Maclaurin series for $1/(1-x)$ to find the Maclaurin series for $1/(a-x)$, where $a \neq 0$, and state the radius of convergence of the series.
(b) Use the binomial series for $1/(1+x)^2$ obtained in Example 4 of Section 10.9 to find the first four nonzero terms in the Maclaurin series for $1/(a+x)^2$, where $a \neq 0$, and state the radius of convergence of the series.

In Exercises 5–8, obtain the first four nonzero terms of the Maclaurin series for the function by making an appropriate substitution in a known Maclaurin series and performing any algebraic operations that are required. State the radius of convergence of the series.

5. (a) $\sin 2x$ (b) e^{-2x} (c) e^{x^2} (d) $x^2 \cos \pi x$

6. (a) $\cos 2x$ (b) $x^2 e^x$ (c) xe^{-x} (d) $\sin(x^2)$

7. (a) $\dfrac{x^2}{1+3x}$ (b) $x \sinh 2x$ (c) $x(1-x^2)^{3/2}$

8. (a) $\dfrac{x}{x-1}$ (b) $3\cosh(x^2)$ (c) $\dfrac{x}{(1+2x)^3}$

In Exercises 9 and 10, find the first four nonzero terms of the Maclaurin series for the function by using an appropriate trigonometric identity or property of logarithms and then substituting in a known Maclaurin series.

9. (a) $\sin^2 x$ (b) $\ln[(1+x^3)^{12}]$

10. (a) $\cos^2 x$ (b) $\ln\left(\dfrac{1-x}{1+x}\right)$

11. (a) Use a known Maclaurin series to find the Taylor series of $1/x$ about $x=1$ by expressing this function as
$$\frac{1}{x} = \frac{1}{1-(1-x)}$$
(b) Find the interval of convergence of the Taylor series.

12. Use the method of Exercise 11 to find the Taylor series of $1/x$ about $x = x_0$, and state the interval of convergence of the Taylor series.

In Exercises 13 and 14, find the first four nonzero terms of the Maclaurin series for the function by multiplying the Maclaurin series of the factors.

13. (a) $e^x \sin x$ (b) $\sqrt{1+x}\ln(1+x)$

14. (a) $e^{-x^2}\cos x$ (b) $(1+x^2)^{4/3}(1+x)^{1/3}$

In Exercises 15 and 16, find the first four nonzero terms of the Maclaurin series for the function by dividing appropriate Maclaurin series.

15. (a) $\sec x \left(= \dfrac{1}{\cos x}\right)$ (b) $\dfrac{\sin x}{e^x}$

16. (a) $\dfrac{\tan^{-1} x}{1+x}$ (b) $\dfrac{\ln(1+x)}{1-x}$

17. Use the Maclaurin series for e^x and e^{-x} to derive the Maclaurin series for $\sinh x$ and $\cosh x$. Include the general terms in your answers and state the radius of convergence of each series.

18. Use the Maclaurin series for $\sinh x$ and $\cosh x$ to obtain the first four nonzero terms in the Maclaurin series for $\tanh x$.

In Exercises 19 and 20, find the first five nonzero terms of the Maclaurin series for the function by using partial fractions and a known Maclaurin series.

19. $\dfrac{4x-2}{x^2-1}$ **20.** $\dfrac{x^3+x^2+2x-2}{x^2-1}$

In Exercises 21 and 22, confirm the derivative formula by differentiating the appropriate Maclaurin series term by term.

21. (a) $\dfrac{d}{dx}[\cos x] = -\sin x$ (b) $\dfrac{d}{dx}[\ln(1+x)] = \dfrac{1}{1+x}$

22. (a) $\dfrac{d}{dx}[\sinh x] = \cosh x$ (b) $\dfrac{d}{dx}[\tan^{-1} x] = \dfrac{1}{1+x^2}$

In Exercises 23 and 24, confirm the integration formula by integrating the appropriate Maclaurin series term by term.

23. (a) $\displaystyle\int e^x\,dx = e^x + C$ (b) $\displaystyle\int \sinh x\,dx = \cosh x + C$

24. (a) $\displaystyle\int \sin x\,dx = -\cos x + C$
(b) $\displaystyle\int \frac{1}{1+x}\,dx = \ln(1+x) + C$

25. (a) Use the Maclaurin series for $1/(1-x)$ to find the Maclaurin series for
$$f(x) = \frac{x}{1-x^2}$$
(b) Use the Maclaurin series obtained in part (a) to find $f^{(5)}(0)$ and $f^{(6)}(0)$.
(c) What can you say about the value of $f^{(n)}(0)$?

26. Let $f(x) = x^2 \cos 2x$. Use the method of Exercise 25 to find $f^{(99)}(0)$.

The limit of an indeterminate form as $x \to x_0$ can sometimes be found without using L'Hôpital's rule by expanding the functions involved in Taylor series about $x = x_0$ and taking the limit of the series term by term. Use this method to find the limits in Exercises 27 and 28.

27. (a) $\displaystyle\lim_{x\to 0} \frac{\sin x}{x}$ (b) $\displaystyle\lim_{x\to 0} \frac{\tan^{-1} x - x}{x^3}$

28. (a) $\lim_{x\to 0}\dfrac{1-\cos x}{\sin x}$ (b) $\lim_{x\to 0}\dfrac{\ln\sqrt{1+x}-\sin 2x}{x}$

In Exercises 29–32, use Maclaurin series to approximate the integral to three decimal-place accuracy.

29. $\int_0^1 \sin(x^2)\,dx$ **30.** $\int_0^{1/2} \tan^{-1}(2x^2)\,dx$

31. $\int_0^{0.2} \sqrt[3]{1+x^4}\,dx$ **32.** $\int_0^{1/2} \dfrac{dx}{\sqrt[4]{x^2+1}}$

33. (a) Differentiate the Maclaurin series for $1/(1-x)$, and use the result to show that

$$\sum_{k=1}^{\infty} kx^k = \frac{x}{(1-x)^2} \quad \text{for } -1<x<1$$

(b) Integrate the Maclaurin series for $1/(1-x)$, and use the result to show that

$$\sum_{k=1}^{\infty} \frac{x^k}{k} = -\ln(1-x) \quad \text{for } -1<x<1$$

(c) Use the result in part (b) to show that

$$\sum_{k=1}^{\infty} (-1)^{k+1}\frac{x^k}{k} = \ln(1+x) \quad \text{for } -1<x<1$$

(d) Show that the series in part (c) converges if $x=1$.

(e) Use the remark following Example 2 to show that

$$\sum_{k=1}^{\infty} (-1)^{k+1}\frac{x^k}{k} = \ln(1+x) \quad \text{for } -1<x\le 1$$

34. In each part, use the results in Exercise 33 to find the sum of the series.

(a) $\sum_{k=1}^{\infty} \dfrac{k}{3^k} = \dfrac{1}{3}+\dfrac{2}{3^2}+\dfrac{3}{3^3}+\dfrac{4}{3^4}+\cdots$

(b) $\sum_{k=1}^{\infty} \dfrac{1}{k(4^k)} = \dfrac{1}{4}+\dfrac{1}{2(4^2)}+\dfrac{1}{3(4^3)}+\dfrac{1}{4(4^4)}+\cdots$

(c) $\sum_{k=1}^{\infty} (-1)^{k+1}\dfrac{1}{k} = 1-\dfrac{1}{2}+\dfrac{1}{3}-\dfrac{1}{4}+\cdots$

35. (a) Use the relationship

$$\int \frac{1}{\sqrt{1+x^2}}\,dx = \sinh^{-1} x + C$$

to find the first four nonzero terms in the Maclaurin series for $\sinh^{-1} x$.

(b) Express the series in sigma notation.

(c) What is the radius of convergence?

36. (a) Use the relationship

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$$

to find the first four nonzero terms in the Maclaurin series for $\sin^{-1} x$.

(b) Express the series in sigma notation.

(c) What is the radius of convergence?

37. We showed by Formula (12) of Section 9.3 that if there are y_0 units of radioactive carbon-14 present at time $t=0$, then the number of units present t years later is

$$y(t) = y_0 e^{-0.000121t}$$

(a) Express $y(t)$ as a Maclaurin series.

(b) Use the first two terms in the series to show that the number of units present after 1 year is approximately $(0.999879)y_0$.

(c) Compare this to the value produced by the formula for $y(t)$.

38. In Section 9.1 we studied the motion of a falling object that has mass m and is retarded by air resistance. We showed that if the initial velocity is v_0 and the drag force F_R is proportional to the velocity, that is, $F_R=-cv$, then the velocity of the object at time t is

$$v(t) = e^{-ct/m}\left(v_0+\frac{mg}{c}\right)-\frac{mg}{c}$$

where g is the acceleration due to gravity [see Formula (23) of Section 9.1].

(a) Use a Maclaurin series to show that if $ct/m \approx 0$, then the velocity can be approximated as

$$v(t) \approx v_0 - \left(\frac{cv_0}{m}+g\right)t$$

(b) Improve on the approximation in part (a).

C **39.** Suppose that a simple pendulum with a length of $L=1$ meter is given an initial displacement of $\theta_0 = 5^\circ$ from the vertical.

(a) Approximate the period of the pendulum using Formula (6) for the first-order model. [Take $g=9.8\text{ m/s}^2$.]

(b) Approximate the period of the pendulum using Formula (7) for the second-order model.

(c) Use the numerical integration capability of a CAS to approximate the period of the pendulum from Formula (4), and compare it to the values obtained in parts (a) and (b).

40. Use the first three nonzero terms in Formula (5) and the Wallis sine formula in the Endpaper Integral Table (Formula 122) to obtain a model for the period of a simple pendulum.

41. Recall that the gravitational force exerted by the Earth on an object is called the object's *weight* (or more precisely, its *Earth weight*). We noted in statement 9.4.3 that if an object has mass m, then the magnitude of its weight is mg. However, this result presumes that the object is on the surface of the Earth (mean sea level). A more general formula for the magnitude of the gravitational force that the Earth exerts on an object of mass m is

$$F = \frac{mgR^2}{(R+h)^2}$$

where R is the radius of the Earth and h is the height of the object above the Earth's surface.

(a) Use the binomial series for $1/(1+x)^2$ obtained in Example 4 of Section 10.9 to express F as a Maclaurin series in powers of h/R.

(b) Show that if $h = 0$, then $F = mg$.
(c) Show that if $h/R \approx 0$, then $F \approx mg - (2mgh/R)$. [*Note:* The quantity $2mgh/R$ can be thought of as a "correction term" for the weight that takes the object's height above the Earth's surface into account.]
(d) If we assume that the Earth is a sphere of radius $R = 4000$ mi at mean sea level, by approximately what percentage does a person's weight change in going from mean sea level to the top of Mt. Everest (29,028 ft)?

42. (a) Show that the Bessel function $J_0(x)$ given by Formula (4) of Section 10.8 satisfies the differential equation $xy'' + y' + xy = 0$. (This is called the ***Bessel equation of order zero***.)
(b) Show that the Bessel function $J_1(x)$ given by Formula (5) of Section 10.8 satisfies the differential equation $x^2y'' + xy' + (x^2 - 1)y = 0$. (This is called the ***Bessel equation of order one***.)
(c) Show that $J_0'(x) = -J_1(x)$.

43. Prove: If the power series $\sum_{k=0}^{\infty} a_k x^k$ and $\sum_{k=0}^{\infty} b_k x^k$ have the same sum on an interval $(-r, r)$, then $a_k = b_k$ for all values of k.

SUPPLEMENTARY EXERCISES

C CAS

1. What is the difference between an infinite sequence and an infinite series?

2. What is meant by the sum of an infinite series?

3. (a) What is a geometric series? Give some examples of convergent and divergent geometric series.
(b) What is a p-series? Give some examples of convergent and divergent p-series.

4. (a) Write down the formula for the Maclaurin series for f in sigma notation.
(b) Write down the formula for the Taylor series for f about $x = x_0$ in sigma notation.

5. State conditions under which an alternating series is guaranteed to converge.

6. (a) What does it mean to say that an infinite series converges absolutely?
(b) What relationship exists between convergence and absolute convergence of an infinite series?

7. If a power series in $x - x_0$ has radius of convergence R, what can you say about the set of x-values at which it converges?

8. State the Remainder Estimation Theorem, and describe some of its uses.

9. Are the following statements true or false? If true, state a theorem to justify your conclusion; if false, then give a counterexample.
(a) If $\sum u_k$ converges, then $u_k \to 0$ as $k \to +\infty$.
(b) If $u_k \to 0$ as $k \to +\infty$, then $\sum u_k$ converges.
(c) If $f(n) = a_n$ for $n = 1, 2, 3, \ldots$, and if $a_n \to L$ as $n \to +\infty$, then $f(x) \to L$ as $x \to +\infty$.
(d) If $f(n) = a_n$ for $n = 1, 2, 3, \ldots$, and if $f(x) \to L$ as $x \to +\infty$, then $a_n \to L$ as $n \to +\infty$.
(e) If $0 < a_n < 1$, then $\{a_n\}$ converges.
(f) If $0 < u_k < 1$, then $\sum u_k$ converges.
(g) If $\sum u_k$ and $\sum v_k$ converge, then $\sum(u_k + v_k)$ diverges.
(h) If $\sum u_k$ and $\sum v_k$ diverge, then $\sum(u_k - v_k)$ converges.
(i) If $0 \le u_k \le v_k$ and $\sum v_k$ converges, then $\sum u_k$ converges.
(j) If $0 \le u_k \le v_k$ and $\sum u_k$ diverges, then $\sum v_k$ diverges.
(k) If an infinite series converges, then it converges absolutely.
(l) If an infinite series diverges absolutely, then it diverges.

10. State whether each of the following is true or false. Justify your answers.
(a) The function $f(x) = x^{1/3}$ has a Maclaurin series.
(b) $1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \frac{1}{4} - \frac{1}{4} + \cdots = 1$
(c) $1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{2} - \frac{1}{2} + \frac{1}{2} - \frac{1}{2} + \cdots = 1$

In Exercises 11–14, use any method to determine whether the series converge.

11. (a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{5^k}$ (b) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{5^k + 1}$ (c) $\displaystyle\sum_{k=1}^{\infty} \frac{9}{\sqrt{k} + 1}$

12. (a) $\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+4}{k^2 + k}$ (b) $\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \left(\frac{k+2}{3k-1}\right)^k$
(c) $\displaystyle\sum_{k=1}^{\infty} \frac{k^{-1/2}}{2 + \sin^2 k}$

13. (a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^3 + 2k + 1}$ (b) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(3+k)^{2/5}}$
(c) $\displaystyle\sum_{k=1}^{\infty} \frac{\cos(1/k)}{k^2}$

14. (a) $\displaystyle\sum_{k=1}^{\infty} \frac{\ln k}{k\sqrt{k}}$ (b) $\displaystyle\sum_{k=1}^{\infty} \frac{k^{4/3}}{8k^2 + 5k + 1}$ (c) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2 + 1}$

15. Find a formula for the exact error that results when the sum of the geometric series $\sum_{k=0}^{\infty} (1/5)^k$ is approximated by the sum of the first 100 terms in the series.

16. Does the series $1 - \frac{2}{3} + \frac{3}{5} - \frac{4}{7} + \frac{5}{9} + \cdots$ converge? Justify your answer.

17. (a) Find the first five Maclaurin polynomials of the function $p(x) = 1 - 7x + 5x^2 + 4x^3$.

(b) Make a general statement about the Maclaurin polynomials of a polynomial of degree n.

18. Use a Maclaurin series and properties of alternating series to show that $|\ln(1+x) - x| \le x^2/2$ if $0 < x < 1$.

19. Show that the approximation

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

is accurate to four decimal places if $0 \le x \le \pi/4$.

20. Use Maclaurin series to approximate the integral

$$\int_0^1 \frac{1-\cos x}{x}\, dx$$

to three decimal-place accuracy.

21. It can be proved that

$$\lim_{n\to+\infty} \sqrt[n]{n!} = +\infty \quad \text{and} \quad \lim_{n\to+\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}$$

In each part, use these limits and the root test to determine whether the series converges.

(a) $\displaystyle\sum_{k=0}^{\infty} \frac{2^k}{k!}$ (b) $\displaystyle\sum_{k=0}^{\infty} \frac{k^k}{k!}$

22. (a) Show that $k^k \ge k!$.

(b) Use the comparison test to show that $\displaystyle\sum_{k=1}^{\infty} k^{-k}$ converges.

(c) Use the root test to show that the series converges.

23. Suppose that $\displaystyle\sum_{k=1}^{n} u_k = 2 - \frac{1}{n}$. Find

(a) u_{100} (b) $\displaystyle\lim_{k\to+\infty} u_k$ (c) $\displaystyle\sum_{k=1}^{\infty} u_k$.

24. In each part, determine whether the series converges; if so, find its sum.

(a) $\displaystyle\sum_{k=1}^{\infty}\left(\frac{3}{2^k} - \frac{2}{3^k}\right)$ (b) $\displaystyle\sum_{k=1}^{\infty}[\ln(k+1) - \ln k]$

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+2)}$ (d) $\displaystyle\sum_{k=1}^{\infty}[\tan^{-1}(k+1) - \tan^{-1} k]$

25. In each part, find the sum of the series by associating it with some Maclaurin series.

(a) $2 + \frac{4}{2!} + \frac{8}{3!} + \frac{16}{4!} + \cdots$

(b) $\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \cdots$

(c) $1 - \frac{e^2}{2!} + \frac{e^4}{4!} - \frac{e^6}{6!} + \cdots$

(d) $1 - \ln 3 + \frac{(\ln 3)^2}{2!} - \frac{(\ln 3)^3}{3!} + \cdots$

26. Suppose that the sequence $\{a_k\}$ is defined recursively by

$$a_0 = c, \quad a_{k+1} = \sqrt{a_k}$$

Assuming that the sequence converges, find its limit if

(a) $c = \frac{1}{2}$ (b) $c = \frac{3}{2}$.

27. Research has shown that the proportion p of the population with IQs (intelligence quotients) between α and β is approximately

$$p = \frac{1}{16\sqrt{2\pi}} \int_\alpha^\beta e^{-\frac{1}{2}\left(\frac{x-100}{16}\right)^2} dx$$

Use the first three terms of an appropriate Maclaurin series to estimate the proportion of the population that has IQs between 100 and 110.

28. Differentiate the Maclaurin series for xe^x and use the result to show that

$$\sum_{k=0}^{\infty} \frac{k+1}{k!} = 2e$$

29. Given: $\dfrac{\pi^2}{6} = 1 + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \dfrac{1}{4^2} + \cdots$.

Show: $\dfrac{\pi^2}{12} = 1 - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \cdots$.

30. Let a, b, and p be positive constants. For which values of p does the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(a+bk)^p}$ converge?

31. In each part, write out the first four terms of the series, and then find the radius of convergence.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{1\cdot 2\cdot 3\cdots k}{1\cdot 4\cdot 7\cdots(3k-2)} x^k$

(b) $\displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{1\cdot 2\cdot 3\cdots k}{1\cdot 3\cdot 5\cdots(2k-1)} x^{2k+1}$

32. Find the interval of convergence of

$$\sum_{k=0}^{\infty} \frac{(x-x_0)^k}{b^k} \quad (b > 0)$$

33. Show that the series

$$1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \cdots$$

converges to the function

$$f(x) = \begin{cases} \cos\sqrt{x}, & x \ge 0 \\ \cosh\sqrt{-x}, & x < 0 \end{cases}$$

[*Hint:* Use the Maclaurin series for $\cos x$ and $\cosh x$ to obtain series for $\cos\sqrt{x}$, where $x \ge 0$, and $\cosh\sqrt{-x}$, where $x \le 0$.]

34. Prove:

(a) If f is an even function, then all odd powers of x in its Maclaurin series have coefficient 0.

(b) If f is an odd function, then all even powers of x in its Maclaurin series have coefficient 0.

35. In Section 6.6 we defined the kinetic energy K of a particle with mass m and velocity v to be $K = \frac{1}{2}mv^2$ [see Formula (6) of that section]. In this formula the mass m is assumed to be constant, and K is called the ***Newtonian kinetic energy***. However, in Albert Einstein's relativity theory the mass m increases with the velocity and the kinetic energy K is given by the formula

$$K = m_0 c^2 \left[\frac{1}{\sqrt{1-(v/c)^2}} - 1 \right]$$

in which m_0 is the mass of the particle when its velocity is zero, and c is the speed of light. This is called the ***relativistic kinetic energy***. Use an appropriate binomial series to show that if the velocity is small compared to the speed of light (i.e., $v/c \approx 0$), then the Newtonian and relativistic kinetic energies are in close agreement.

C **36.** If the constant p in the general p-series is replaced by a variable x for $x > 1$, then the resulting function is called the ***Riemann zeta function*** and is denoted by

$$\zeta(x) = \sum_{k=1}^{\infty} \frac{1}{k^x}$$

(a) Let s_n be the nth partial sum of the series for $\zeta(3.7)$. Find n such that s_n approximates $\zeta(3.7)$ to two decimal-place accuracy, and calculate s_n using this value of n. [*Hint:* Use the right inequality in Exercise 30(b) of Section 10.5 with $f(x) = 1/x^{3.7}$.]

(b) Determine whether your CAS can evaluate the Riemann zeta function directly. If so, compare the value produced by the CAS to the value of s_n obtained in part (a).