13

VECTOR-VALUED FUNCTIONS

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In this chapter we will consider functions whose values are vectors. Such functions provide a unified way of studying parametric curves in 2-space and 3-space and are a basic tool for analyzing the motion of particles along curved paths. We will begin by developing the calculus of vector-valued functions—we will show how to differentiate and integrate such functions, and we will develop some of the basic properties of these operations. We will then apply these calculus tools to define three fundamental vectors that can be used to describe such basic characteristics of curves as curvature and twisting tendencies. Once this is done, we will develop the concepts of velocity and acceleration for such motion, and we will apply these concepts to explain various physical phenomena. Finally, we will use the calculus of vector-valued functions to develop basic principles of gravitational attraction and to derive Kepler's laws of planetary motion.

13.1 INTRODUCTION TO VECTOR-VALUED FUNCTIONS

In Section 12.5 we discussed parametric equations of lines in 3-space. In this section we will discuss more general parametric curves in 3-space, and we will show how vector notation can be used to express parametric equations in 2-space and 3-space in a more compact form. This will lead us to consider a new kind of function—namely, functions that associate vectors with real numbers. Such functions have many important applications in physics and engineering.

PARAMETRIC CURVES IN 3-SPACE

Recall from Section 1.8 that if f and g are well-behaved functions, then the pair of parametric equations

$$x = f(t), \quad y = g(t) \tag{1}$$

generates a curve in 2-space that is traced in a specific direction as the parameter t increases. We defined this direction to be the *orientation* of the curve or the *direction of increasing parameter*, and we called the curve together with its orientation the *graph* of the equations or the *parametric curve* represented by the equations. Analogously, if f, g, and h are three well-behaved functions, then the parametric equations

$$x = f(t), \quad y = g(t), \quad z = h(t) \tag{2}$$

generate a curve in 3-space that is traced in a specific direction as t increases. As in 2-space, this direction is called the ***orientation*** or ***direction of increasing parameter***, and the curve together with its orientation is called the ***graph*** of the equations or the ***parametric curve*** represented by the equations. If no restrictions are stated explicitly or are implied by the equations, then it will be understood that t varies over the interval $(-\infty, +\infty)$.

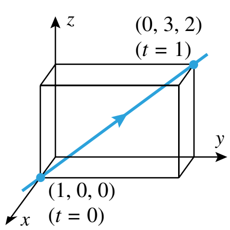


Figure 13.1.1

Example 1 The parametric equations

$$x = 1 - t, \quad y = 3t, \quad z = 2t$$

represent a line in 3-space that passes through the point $(1, 0, 0)$ and is parallel to the vector $\langle -1, 3, 2\rangle$. Since x, y, and z increase as t increases, the line has the orientation shown in Figure 13.1.1. ◀

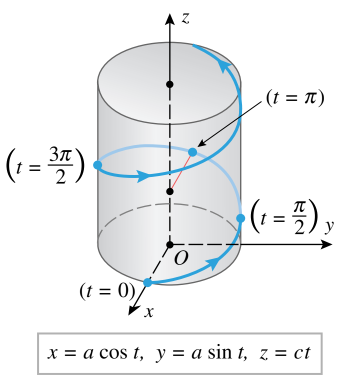


Figure 13.1.2

Example 2 Describe the parametric curve represented by the equations

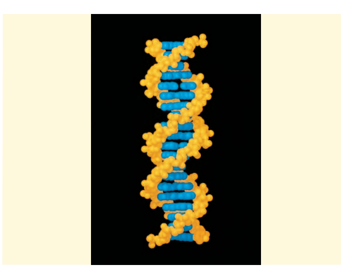
$$x = a\cos t, \quad y = a\sin t, \quad z = ct$$

where a and c are positive constants.

Solution. As the parameter t increases, the value of $z = ct$ also increases, so the point (x, y, z) moves upward. However, as t increases, the point (x, y, z) also moves in a path directly over the circle

$$x = a\cos t, \quad y = a\sin t$$

in the xy-plane. The combination of these upward and circular motions produces a corkscrew-shaped curve that wraps around a right circular cylinder of radius a centered on the z-axis (Figure 13.1.2). This curve is called a ***circular helix***. ◀



Computer representation of the twin helix DNA molecule (deoxyribonucleic acid). This structure contains all the inherited instructions necessary for the development of a living organism.

PARAMETRIC CURVES GENERATED WITH TECHNOLOGY

Except in the simplest cases, parametric curves in 3-space can be difficult to visualize and draw without the help of a graphing utility. For example, Figure 13.1.3*a* shows the graph of the parametric curve called a *torus knot* that was produced by a CAS. However, even this computer rendering is difficult to visualize because it is unclear whether the points of overlap are intersections or whether one portion of the curve is in front of the other. To resolve this visualization problem, some graphing utilities provide the capability of enclosing the curve within a thin tube, as in Figure 13.1.3*b*. Such graphs are called ***tube plots***.

FOR THE READER. If you have a CAS, read the documentation on graphing parametric curves in 3-space, and then use it to generate the line in Example 1 and the helix

$$x = 4\cos t, \quad y = 4\sin t, \quad z = t \qquad (0 \le t \le 3\pi)$$

shown in Figure 13.1.4.

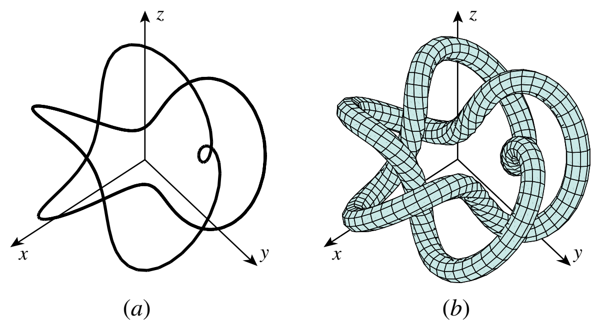


Figure 13.1.3

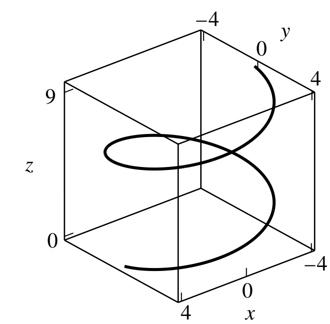


Figure 13.1.4

PARAMETRIC EQUATIONS FOR INTERSECTIONS OF SURFACES

Curves in 3-space often arise as intersections of surfaces. For example, Figure 13.1.5a shows a portion of the intersection of the cylinders $z = x^3$ and $y = x^2$. One method for finding parametric equations for the curve of intersection is to choose one of the variables as the parameter and use the two equations to express the remaining two variables in terms of that parameter. In particular, if we choose $x = t$ as the parameter and substitute this into the equations $z = x^3$ and $y = x^2$, we obtain the parametric equations

$$x = t, \quad y = t^2, \quad z = t^3 \tag{3}$$

This curve is called a ***twisted cubic***. The portion of the twisted cubic shown in Figure 13.1.5a corresponds to $t \ge 0$; a computer-generated graph of the twisted cubic for positive and negative values of t is shown in Figure 13.1.5b. Some other examples and techniques for finding intersections of surfaces are discussed in the exercises.

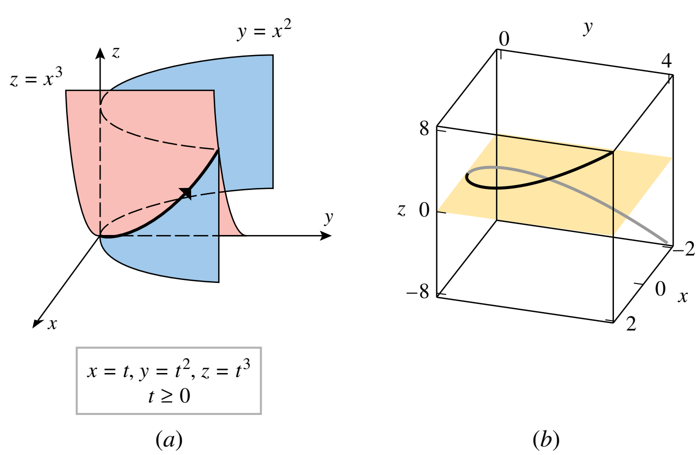


Figure 13.1.5

VECTOR-VALUED FUNCTIONS

The twisted cubic defined by the equations in (3) is the set of points of the form (t, t^2, t^3) for real values of t. If we view each of these points as a terminal point for a vector $\mathbf{r}$ whose initial point is at the origin,

$$\mathbf{r} = \langle x, y, z\rangle = \langle t, t^2, t^3\rangle = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$$

then we obtain $\mathbf{r}$ as a function of the parameter t, that is, $\mathbf{r} = \mathbf{r}(t)$. Since this function produces a *vector*, we say that $\mathbf{r} = \mathbf{r}(t)$ defines $\mathbf{r}$ as a ***vector-valued function of a real variable***, or more simply, a ***vector-valued function***. The vectors that we will consider in this text are either in 2-space or 3-space, so we will say that a vector-valued function is in 2-space or in 3-space according to the kind of vectors that it produces.

If $\mathbf{r}(t)$ is a vector-valued function in 2-space, then for each allowable value of t, the vector $\mathbf{r} = \mathbf{r}(t)$ can be represented in terms of components as

$$\mathbf{r} = \mathbf{r}(t) = \langle x(t), y(t) \rangle = x(t)\mathbf{i} + y(t)\mathbf{j}$$

As suggested by this notation, the vector-valued function $\mathbf{r}(t)$ defines a pair of real-valued functions, $x = x(t)$ and $y = y(t)$, which we call the ***component functions*** or the ***components*** of $\mathbf{r}(t)$. Similarly, a vector-valued function $\mathbf{r}(t)$ in 3-space defines three component functions, $x(t)$, $y(t)$, and $z(t)$, via

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

For example, the component functions of

$$\mathbf{r}(t) = \langle t, t^2, t^3 \rangle = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$$

are

$$x(t) = t, \quad y(t) = t^2, \quad z(t) = t^3$$

The ***domain*** of a vector-valued function $\mathbf{r}(t)$ is the set of allowable values of t. If $\mathbf{r}(t)$ is defined in terms of component functions and the domain is not specified explicitly, then it will be understood that the domain is the set of all values of t for which *every* component is defined and yields a real value; we call this the ***natural domain*** of $\mathbf{r}(t)$. That is, the natural domain of a vector-valued function is the intersection of the natural domains for its component functions. For example, the natural domain for

$$\mathbf{r}(t) = \langle \ln|t-1|, e^t, \sqrt{t} \rangle = (\ln|t-1|)\mathbf{i} + e^t\mathbf{j} + \sqrt{t}\mathbf{k}$$

is the set of values of t such that $0 \le t < 1$ or $1 < t$, since

$$((-\infty, 1) \cup (1, +\infty)) \cap (-\infty, +\infty) \cap [0, +\infty) = [0, 1) \cup (1, +\infty)$$

is the intersection of the natural domains of the component functions

$$x(t) = \ln|t-1|, \quad y(t) = e^t, \quad \text{and} \quad z(t) = \sqrt{t}$$

GRAPHS OF VECTOR-VALUED FUNCTIONS

If $\mathbf{r}(t)$ is a vector-valued function in 2-space or 3-space, then we define the ***graph*** of $\mathbf{r}(t)$ to be the parametric curve described by the component functions for $\mathbf{r}(t)$. For example, if

$$\mathbf{r}(t) = \langle 1-t, 3t, 2t \rangle = (1-t)\mathbf{i} + 3t\mathbf{j} + 2t\mathbf{k} \tag{4}$$

then the graph of $\mathbf{r} = \mathbf{r}(t)$ is the graph of the parametric equations

$$x = 1-t, \quad y = 3t, \quad z = 2t$$

Thus, the graph of (4) is the line in Figure 13.1.1.

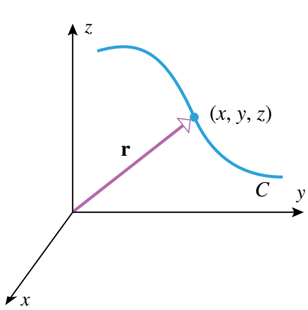
Example 3 Describe the graph of the vector-valued function

$$\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$$

Solution. The corresponding parametric equations are

$$x = \cos t, \quad y = \sin t, \quad z = t$$

Thus, as we saw in Example 2, the graph is a circular helix wrapped around a cylinder of radius 1. ◀



As t varies, the tip of the radius vector $\mathbf{r}$ traces out the curve C.

Figure 13.1.6

Up to now we have considered parametric curves to be paths traced by moving points. However, if a parametric curve is viewed as the graph of a vector-valued function, then we can also imagine the graph to be traced by the tip of a moving vector. For example, if the curve C in 3-space is the graph of

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

and if we position the vector $\mathbf{r} = \langle x, y, z \rangle$ with its initial point at the origin, then its terminal point will fall at the point (x, y, z) on the curve C (as shown in Figure 13.1.6). Thus, the terminal point of $\mathbf{r}(t)$ will trace out the curve C as the parameter t varies. We call $\mathbf{r}$ the ***radius vector*** or the ***position vector*** for C.

Example 4 Sketch the graph and a radius vector of

(a) $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}, \quad 0 \le t \le 2\pi$

(b) $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + 2\mathbf{k}, \quad 0 \le t \le 2\pi$

Solution (a). The corresponding parametric equations are

$$x = \cos t, \quad y = \sin t \qquad (0 \le t \le 2\pi)$$

so the graph is a circle of radius 1, centered at the origin, and oriented counterclockwise. The graph and a radius vector are shown in Figure 13.1.7.

Solution (b). The corresponding parametric equations are

$$x = \cos t, \quad y = \sin t, \quad z = 2 \qquad (0 \le t \le 2\pi)$$

From the third equation, the tip of the radius vector traces a curve in the plane $z = 2$, and from the first two equations, the curve is a circle of radius 1 centered on the z-axis and traced counterclockwise looking down the z-axis. The graph and a radius vector are shown in Figure 13.1.8. ◀

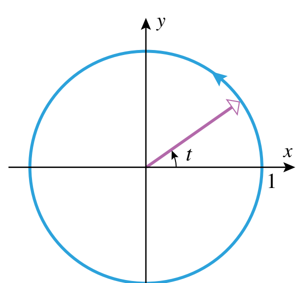


Figure 13.1.7

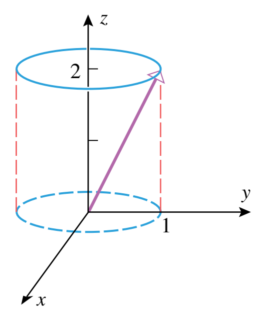


Figure 13.1.8

VECTOR FORM OF A LINE SEGMENT

Recall from Formula (9) of Section 12.5 that if $\mathbf{r}_0$ is a vector in 2-space or 3-space with its initial point at the origin, then the line that passes through the terminal point of $\mathbf{r}_0$ and is parallel to the vector $\mathbf{v}$ can be expressed in vector form as

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

In particular, if $\mathbf{r}_0$ and $\mathbf{r}_1$ are vectors in 2-space or 3-space with their initial points at the origin, then the line that passes through the terminal points of these vectors can be expressed in vector form as

$$\mathbf{r} = \mathbf{r}_0 + t(\mathbf{r}_1 - \mathbf{r}_0) \qquad \text{or} \qquad \mathbf{r} = (1-t)\mathbf{r}_0 + t\mathbf{r}_1 \tag{5–6}$$

as indicated in Figure 13.1.9.

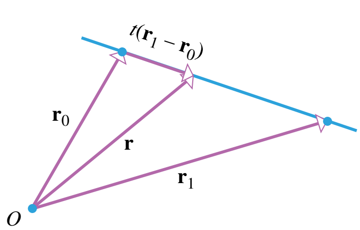


Figure 13.1.9

REMARK. It is common to call either (5) or (6) the ***two-point vector form of a line*** and to say, for simplicity, that the line passes through the *points* $\mathbf{r}_0$ and $\mathbf{r}_1$ (as opposed to saying that it passes through the *terminal points* of $\mathbf{r}_0$ and $\mathbf{r}_1$).

It is understood in (5) and (6) that t varies from $-\infty$ to $+\infty$. However, if we restrict t to vary over the interval $0 \le t \le 1$, then $\mathbf{r}$ will vary from $\mathbf{r}_0$ to $\mathbf{r}_1$. Thus, for example, the equation

$$\mathbf{r} = (1-t)\mathbf{r}_0 + t\mathbf{r}_1 \qquad (0 \le t \le 1) \tag{7}$$

represents the line segment in 2-space or 3-space that is traced from $\mathbf{r}_0$ to $\mathbf{r}_1$.

EXERCISE SET 13.1

Graphing Utility

In Exercises 1–4, find the domain of $\mathbf{r}(t)$ and the value of $\mathbf{r}(t_0)$.

1. $\mathbf{r}(t) = \cos t\mathbf{i} - 3t\mathbf{j}$; $t_0 = \pi$

2. $\mathbf{r}(t) = \langle \sqrt{3t+1}, t^2 \rangle$; $t_0 = 1$

3. $\mathbf{r}(t) = \cos \pi t\mathbf{i} - \ln t\,\mathbf{j} + \sqrt{t-2}\,\mathbf{k}$; $t_0 = 3$

4. $\mathbf{r}(t) = \langle 2e^{-t}, \sin^{-1} t, \ln(1-t) \rangle$; $t_0 = 0$

In Exercises 5–8, express the parametric equations as a single vector equation of the form $\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j}$ or $\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$.

5. $x = 3\cos t,\ \ y = t + \sin t$ **6.** $x = t^2 + 1,\ \ y = e^{-2t}$

7. $x = 2t,\ \ y = 2\sin 3t,\ \ z = 5\cos 3t$

8. $x = t\sin t,\ \ y = \ln t,\ \ z = \cos^2 t$

In Exercises 9–12, find the parametric equations that correspond to the given vector equation.

9. $\mathbf{r} = 3t^2\mathbf{i} - 2\mathbf{j}$ **10.** $\mathbf{r} = \sin^2 t\mathbf{i} + (1 - \cos 2t)\mathbf{j}$

11. $\mathbf{r} = (2t-1)\mathbf{i} - 3\sqrt{t}\,\mathbf{j} + \sin 3t\mathbf{k}$

12. $\mathbf{r} = te^{-t}\mathbf{i} - 5t^2\mathbf{k}$

In Exercises 13–18, describe the graph of the equation.

13. $\mathbf{r} = (2-3t)\mathbf{i} - 4t\mathbf{j}$ **14.** $\mathbf{r} = 3\sin 2t\mathbf{i} + 3\cos 2t\,\mathbf{j}$

15. $\mathbf{r} = 2t\mathbf{i} - 3\mathbf{j} + (1+3t)\mathbf{k}$ **16.** $\mathbf{r} = 3\mathbf{i} + 2\cos t\,\mathbf{j} + 2\sin t\mathbf{k}$

17. $\mathbf{r} = 3\cos t\mathbf{i} + 2\sin t\,\mathbf{j} - \mathbf{k}$ **18.** $\mathbf{r} = -2\mathbf{i} + t\mathbf{j} + (t^2 - 1)\mathbf{k}$

19. (a) Find the slope of the line in 2-space that is represented by the vector equation $\mathbf{r} = (1-2t)\mathbf{i} - (2-3t)\mathbf{j}$.
(b) Find the coordinates of the point where the line

$$\mathbf{r} = (2+t)\mathbf{i} + (1-2t)\mathbf{j} + 3t\mathbf{k}$$

intersects the xz-plane.

20. (a) Find the y-intercept of the line in 2-space that is represented by the vector equation $\mathbf{r} = (3+2t)\mathbf{i} + 5t\mathbf{j}$.
(b) Find the coordinates of the point where the line

$$\mathbf{r} = t\mathbf{i} + (1+2t)\mathbf{j} - 3t\mathbf{k}$$

intersects the plane $3x - y - z = 2$.

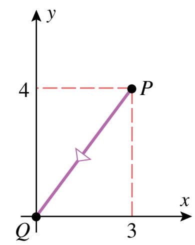
In Exercises 21 and 22, sketch the line segment represented by the vector equation.

21. (a) $\mathbf{r} = (1-t)\mathbf{i} + t\mathbf{j}$; $0 \le t \le 1$
(b) $\mathbf{r} = (1-t)(\mathbf{i} + \mathbf{j}) + t(\mathbf{i} - \mathbf{j})$; $0 \le t \le 1$

22. (a) $\mathbf{r} = (1-t)(\mathbf{i} + \mathbf{j}) + t\mathbf{k}$; $0 \le t \le 1$
(b) $\mathbf{r} = (1-t)(\mathbf{i} + \mathbf{j} + \mathbf{k}) + t(\mathbf{i} + \mathbf{j})$; $0 \le t \le 1$

In Exercises 23 and 24, write a vector equation for the line segment from P to Q.

23.



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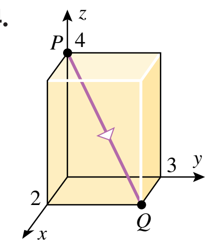


In Exercises 25–34, sketch the graph of $\mathbf{r}(t)$ and show the direction of increasing t.

25. $\mathbf{r}(t) = 2\mathbf{i} + t\mathbf{j}$ **26.** $\mathbf{r}(t) = \langle 3t-4, 6t+2 \rangle$

27. $\mathbf{r}(t) = (1 + \cos t)\mathbf{i} + (3 - \sin t)\mathbf{j}$; $0 \le t \le 2\pi$

28. $\mathbf{r}(t) = \langle 2\cos t, 5\sin t \rangle$; $0 \le t \le 2\pi$

29. $\mathbf{r}(t) = \cosh t\mathbf{i} + \sinh t\,\mathbf{j}$ **30.** $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + (2t+4)\mathbf{j}$

31. $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\,\mathbf{j} + t\mathbf{k}$

32. $\mathbf{r}(t) = 9\cos t\mathbf{i} + 4\sin t\,\mathbf{j} + t\mathbf{k}$

33. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + 2\mathbf{k}$

34. $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + \sin t\mathbf{k}$; $0 \le t \le 2\pi$

In Exercises 35 and 36, sketch the curve of intersection of the surfaces, and find parametric equations for the intersection in terms of parameter $x = t$. Check your work with a graphing utility by generating the parametric curve over the interval $-1 \le t \le 1$.

35. $z = x^2 + y^2,\ \ x - y = 0$

36. $y + x = 0,\ \ z = \sqrt{2 - x^2 - y^2}$

In Exercises 37 and 38, sketch the curve of intersection of the surfaces, and find a vector equation for the curve in terms of the parameter $x = t$.

37. $9x^2 + y^2 + 9z^2 = 81,\ \ y = x^2 \quad (z > 0)$

38. $y = x,\ \ x + y + z = 1$

39. Show that the graph of

$$\mathbf{r} = t\sin t\mathbf{i} + t\cos t\,\mathbf{j} + t^2\mathbf{k}$$

lies on the paraboloid $z = x^2 + y^2$.

40. Show that the graph of

$$\mathbf{r} = t\mathbf{i} + \frac{1+t}{t}\mathbf{j} + \frac{1-t^2}{t}\mathbf{k}, \quad t > 0$$

lies in the plane $x - y + z + 1 = 0$.

41. Show that the graph of

$$\mathbf{r} = \sin t\mathbf{i} + 2\cos t\,\mathbf{j} + \sqrt{3}\sin t\mathbf{k}$$

is a circle, and find its center and radius. [*Hint:* Show that the curve lies on both a sphere and a plane.]

42. Show that the graph of

$$\mathbf{r} = 3\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j} + 3\sin t\,\mathbf{k}$$

is an ellipse, and find the lengths of the major and minor axes. [*Hint:* Show that the graph lies on both a circular cylinder and a plane and use the result in Exercise 60 of Section 11.4.]

43. For the helix $\mathbf{r} = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} + ct\mathbf{k}$, find c ($c > 0$) so that the helix will make one complete turn in a distance of 3 units measured along the z-axis.

44. How many revolutions will the circular helix

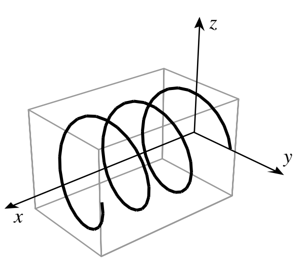
$$\mathbf{r} = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} + 0.2t\mathbf{k}$$

make in a distance of 10 units measured along the z-axis?

45. Show that the curve $\mathbf{r} = t\cos t\,\mathbf{i} + t\sin t\,\mathbf{j} + t\mathbf{k}$, $t \geq 0$, lies on the cone $z = \sqrt{x^2 + y^2}$. Describe the curve.

46. Describe the curve $\mathbf{r} = a\cos t\,\mathbf{i} + b\sin t\,\mathbf{j} + ct\mathbf{k}$, where a, b, and c are positive constants such that $a \neq b$.

47. In each part, match the vector equation with one of the accompanying graphs, and explain your reasoning.
(a) $\mathbf{r} = t\mathbf{i} - t\mathbf{j} + \sqrt{2 - t^2}\,\mathbf{k}$
(b) $\mathbf{r} = \sin \pi t\,\mathbf{i} - t\mathbf{j} + t\mathbf{k}$
(c) $\mathbf{r} = \sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \sin 2t\mathbf{k}$
(d) $\mathbf{r} = \frac{1}{2}t\mathbf{i} + \cos 3t\,\mathbf{j} + \sin 3t\mathbf{k}$



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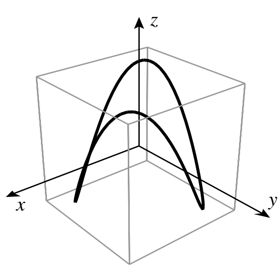

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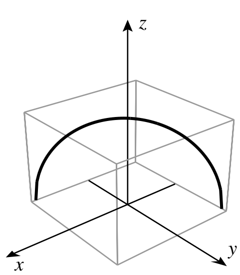

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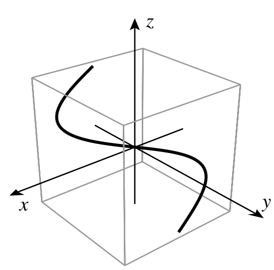


IV

48. Check your conclusions in Exercise 47 by generating the curves with a graphing utility. [*Note:* Your graphing utility may look at the curve from a different viewpoint. Read the documentation for your graphing utility to determine how to control the viewpoint, and see if you can generate a reasonable facsimile of the graphs shown in the figure by adjusting the viewpoint and choosing the interval of t-values appropriately.]

49. (a) Find parametric equations for the curve of intersection of the circular cylinder $x^2 + y^2 = 9$ and the parabolic cylinder $z = x^2$ in terms of a parameter t for which $x = 3\cos t$.
(b) Use a graphing utility to generate the curve of intersection in part (a).

50. Use a graphing utility to generate the intersection of the cone $z = \sqrt{x^2 + y^2}$ and the plane $z = y + 2$. Identify the curve and explain your reasoning.

51. (a) Sketch the graph of

$$\mathbf{r}(t) = \left\langle 2t, \frac{2}{1 + t^2} \right\rangle$$

(b) Prove that the curve in part (a) is also the graph of the function

$$y = \frac{8}{4 + x^2}$$

[The graphs of $y = a^3/(a^2 + x^2)$, where a denotes a constant, were first studied by the French mathematician Pierre de Fermat, and later by the Italian mathematicians Guido Grandi and Maria Agnesi. Any such curve is now known as a "witch of Agnesi." There are a number of theories for the origin of this name. Some suggest there was a mistranslation by either Grandi or Agnesi of some less colorful Latin name into Italian. Others lay the blame on a translation into English of Agnesi's 1748 treatise, *Analytical Institutions*.]

13.2 CALCULUS OF VECTOR-VALUED FUNCTIONS

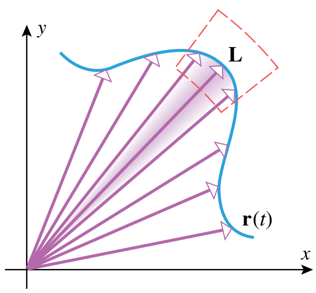
In this section we will define limits, derivatives, and integrals of vector-valued functions and discuss their properties.

LIMITS AND CONTINUITY

Our first goal in this section is to develop a notion of what it means for a vector-valued function $\mathbf{r}(t)$ to approach a limiting vector $\mathbf{L}$ as t approaches a number a. That is, we want to define

$$\lim_{t\to a} \mathbf{r}(t) = \mathbf{L} \tag{1}$$

In the introduction to Chapter 12 we mentioned that vectors are useful in many physical contexts because they encapsulate both magnitude (or length) and direction. Equation (1) can be interpreted intuitively through this geometric perspective: as t approaches a, the limit of the length of $\mathbf{r}(t)$ must match the length of $\mathbf{L}$, and the limit of the direction of $\mathbf{r}(t)$ must match the direction of $\mathbf{L}$.



$\mathbf{r}(t)$ approaches $\mathbf{L}$ in length and direction if $\lim_{t\to a} \mathbf{r}(t) = \mathbf{L}$.

Figure 13.2.1

13.2.1 GEOMETRIC INTERPRETATION OF LIMITS. If $\mathbf{r}(t)$ is a vector-valued function in 2-space or 3-space, then

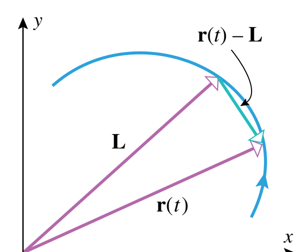
$$\lim_{t\to a} \mathbf{r}(t) = \mathbf{L}$$

if and only if the radius vector $\mathbf{r} = \mathbf{r}(t)$ approaches $\mathbf{L}$ in both length and direction as $t \to a$ (Figure 13.2.1).

Although saying that $\mathbf{r}(t)$ approaches $\mathbf{L}$ in both length and direction may be helpful for visualizing a limit, it is difficult to use the statement in 13.2.1 to establish a limit. Instead, let us go back to Chapter 2 and use the definition of the limit of a real-valued function as a guide. Recall from Section 2.1 that the limit

$$\lim_{x\to a} f(x) = L$$

was defined informally as the assertion that values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but not equal to a). This was formalized in Section 2.4 to the assertion that for any given $\epsilon > 0$, we can find a number $\delta > 0$ such that $|f(x) - L| < \epsilon$ if $0 < |x - a| < \delta$.



$\|\mathbf{r}(t) - \mathbf{L}\|$ is the distance between terminal points for vectors $\mathbf{r}(t)$ and $\mathbf{L}$ when positioned with the same initial points.

Figure 13.2.2

To adapt the notion of limits of a real-valued function $y = f(x)$ to limits of a vector-valued function $\mathbf{r} = \mathbf{r}(t)$, we need to replace the notion of "closeness" of the real numbers $f(x)$ and L by a corresponding notion for the vectors $\mathbf{r}(t)$ and $\mathbf{L}$. But how do we measure how close (or how far apart) two vectors $\mathbf{r}(t)$ and $\mathbf{L}$ are? We can look at the difference between the vectors, $\mathbf{r}(t) - \mathbf{L}$ (Figure 13.2.2), but this is a vector. What we need is the length of this vector, $\|\mathbf{r}(t) - \mathbf{L}\|$, which gives the distance between the terminal points of $\mathbf{r}(t)$ and $\mathbf{L}$ when they are positioned with the same initial point.

To say that a vector $\mathbf{r}(t)$ is close to the vector $\mathbf{L}$ is to say that $\|\mathbf{r}(t) - \mathbf{L}\|$ is small, say less than some positive number ϵ. In 2-space, the set of all vectors $\mathbf{r}$ satisfying $\|\mathbf{r} - \mathbf{L}\| < \epsilon$ can be described geometrically as those vectors that, when positioned with the same initial point as $\mathbf{L}$, have terminal points lying within a disk of radius ϵ centered at the terminal point for $\mathbf{L}$. In 3-space, this set is those vectors with terminal points lying within a ball of radius ϵ centered at the terminal point for $\mathbf{L}$ (Figure 13.2.3).

We can now transform Definition 2.4.1 into a definition for (1).

13.2.2 DEFINITION. Let $\mathbf{r}(t)$ be a vector-valued function defined for all t in some open interval containing the number a, except that $\mathbf{r}(t)$ need not be defined at a. We will write

$$\lim_{t\to a} \mathbf{r}(t) = \mathbf{L}$$

if given any number $\epsilon > 0$ we can find a number $\delta > 0$ such that

$$\|\mathbf{r}(t) - \mathbf{L}\| < \epsilon \quad \text{if} \quad 0 < |t - a| < \delta$$

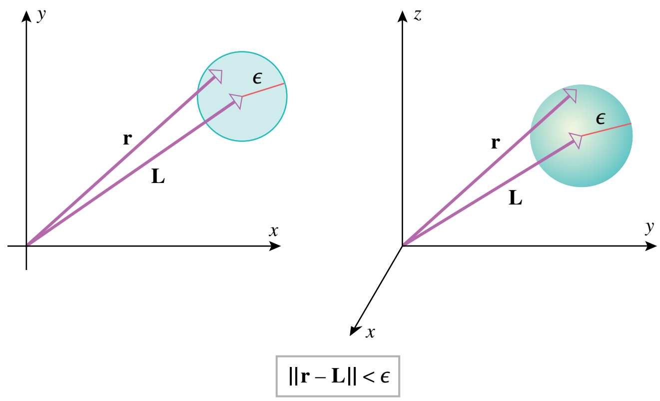


Figure 13.2.3

Similarly, we mirror Definition 2.5.1 to define continuity of a vector-valued function.

13.2.3 DEFINITION. A vector-valued function $\mathbf{r}(t)$ is ***continuous at*** $t = c$ provided the following conditions are satisfied:

1. $\mathbf{r}(c)$ is defined.
2. $\lim_{t \to c} \mathbf{r}(t)$ exists.
3. $\lim_{t \to c} \mathbf{r}(t) = \mathbf{r}(c)$.

As before, we say that $\mathbf{r}(t)$ is ***continuous*** on an interval I if it is continuous at each value of t in I (with only the appropriate one-sided limit results required at any endpoints of I that are included in I).

In practice, limits of vector-valued functions are frequently computed using components. For example, if

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

then

$$\begin{aligned} \lim_{t \to a} \mathbf{r}(t) &= \left\langle \lim_{t \to a} x(t), \lim_{t \to a} y(t), \lim_{t \to a} z(t) \right\rangle \\ &= \left(\lim_{t \to a} x(t) \right) \mathbf{i} + \left(\lim_{t \to a} y(t) \right) \mathbf{j} + \left(\lim_{t \to a} z(t) \right) \mathbf{k} \end{aligned} \tag{2}$$

provided each of the component limits exists. Furthermore, it follows immediately that $\mathbf{r}(t)$ is continuous at $t = c$ if and only if its component functions $x(t)$, $y(t)$, and $z(t)$ are each continuous at $t = c$.

• FOR THE READER. Write the corresponding statement to (2) when $\mathbf{r}(t)$ is in 2-space.

Example 1 Let $\mathbf{r}(t) = t^2\mathbf{i} + e^t\mathbf{j} - (2\cos \pi t)\mathbf{k}$. Then

$$\lim_{t \to 0} \mathbf{r}(t) = \left(\lim_{t \to 0} t^2 \right) \mathbf{i} + \left(\lim_{t \to 0} e^t \right) \mathbf{j} - \left(\lim_{t \to 0} 2\cos \pi t \right) \mathbf{k} = \mathbf{j} - 2\mathbf{k}$$

Alternatively, using the angle bracket notation for vectors,

$$\lim_{t \to 0} \mathbf{r}(t) = \lim_{t \to 0} \langle t^2, e^t, -2\cos \pi t \rangle = \left\langle \lim_{t \to 0} t^2, \lim_{t \to 0} e^t, \lim_{t \to 0} (-2\cos \pi t) \right\rangle = \langle 0, 1, -2 \rangle \quad \blacktriangleleft$$

DERIVATIVES

Following the lead of the discussion above, we consider substituting a vector-valued function for the real-valued function in the definition of the derivative (Definition 3.2.3). Note that the numerator in the resulting difference quotient is now a difference of vectors, which results in a vector, whereas the denominator is a difference of scalars. Thus, the difference quotient is a scalar multiple of a vector, so it is also a vector.

13.2.4 DEFINITION. If $\mathbf{r}(t)$ is a vector-valued function, we define the ***derivative of* r *with respect to t*** to be the vector-valued function $\mathbf{r}'$ given by

$$\mathbf{r}'(t) = \lim_{w \to t} \frac{\mathbf{r}(w) - \mathbf{r}(t)}{w - t} \tag{3}$$

The domain of $\mathbf{r}'$ consists of all values of t in the domain of $\mathbf{r}(t)$ for which the limit exists.

The function $\mathbf{r}(t)$ is ***differentiable*** at t if the limit in (3) exists. All of the standard notations for derivatives continue to apply. For example, the derivative of $\mathbf{r}(t)$ can be expressed as

$$\frac{d}{dt}[\mathbf{r}(t)], \quad \frac{d\mathbf{r}}{dt}, \quad \mathbf{r}'(t), \quad \text{or} \quad \mathbf{r}'$$

It is important to remember that for a given value of t the derivative $\mathbf{r}'(t)$ is a *vector*, not a number. Since $\mathbf{r}'(t)$ is a vector, it has both magnitude and [if $\mathbf{r}'(t)$ is nonzero] direction. Our next goal is to relate the *direction* of $\mathbf{r}'(t)$ to the graph of $\mathbf{r}(t)$. [We will study the significance of the *magnitude* of $\mathbf{r}'(t)$ in the next section.] To do this, consider parts (*a*) and (*b*) of Figure 13.2.4. These illustrations show the graph C of $\mathbf{r}(t)$ (with its orientation) and the vectors $\mathbf{r}(w)$, $\mathbf{r}(t)$, and $\mathbf{r}(w) - \mathbf{r}(t)$ for $w > t$ and for $w < t$. In both cases, the vector $\mathbf{r}(w) - \mathbf{r}(t)$ runs along the secant line joining the terminal points of $\mathbf{r}(t)$ and $\mathbf{r}(w)$, but with opposite directions in the two cases. In the case where $w > t$, the vector $\mathbf{r}(w) - \mathbf{r}(t)$ points in the direction of increasing parameter; and in the case where $w < t$, the vector $\mathbf{r}(w) - \mathbf{r}(t)$ points in the opposite direction. However, if $w < t$, the direction is reversed when we multiply by the negative value $1/(w - t)$, so that in both cases the vector

$$\frac{1}{w - t}[\mathbf{r}(w) - \mathbf{r}(t)] = \frac{\mathbf{r}(w) - \mathbf{r}(t)}{w - t}$$

points in the direction of increasing parameter and runs along the secant line. As $w \to t$ the secant line approaches the tangent line at the terminal point of $\mathbf{r}(t)$, so we can conclude that the limit

$$\mathbf{r}'(t) = \lim_{w \to t} \frac{\mathbf{r}(w) - \mathbf{r}(t)}{w - t}$$

(if it exists and is nonzero) is a vector that is tangent to the curve C at the tip of $\mathbf{r}(t)$ and points in the direction of increasing parameter (Figure 13.2.4*c*).

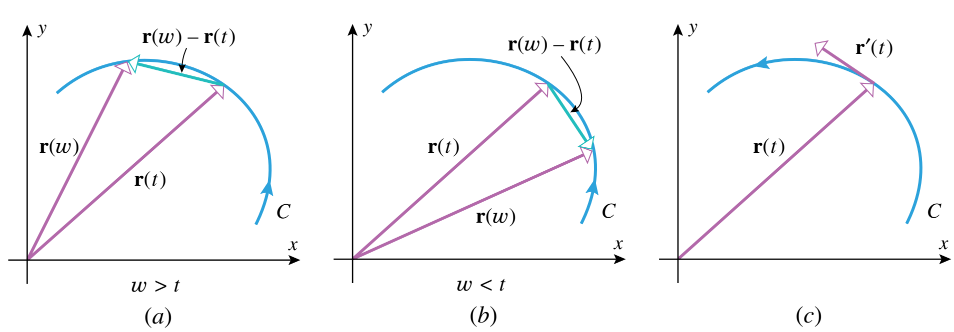


Figure 13.2.4

13.2.5 GEOMETRIC INTERPRETATION OF THE DERIVATIVE. Suppose that C is the graph of a vector-valued function $\mathbf{r}(t)$ in 2-space or 3-space and that $\mathbf{r}'(t)$ exists and is nonzero for a given value of t. If the vector $\mathbf{r}'(t)$ is positioned with its initial point at the terminal point of the radius vector $\mathbf{r}(t)$, then $\mathbf{r}'(t)$ is tangent to C and points in the direction of increasing parameter.

Since limits of vector-valued functions can be computed componentwise, it seems reasonable that we should be able to compute derivatives in terms of component functions as well. This is the result of the next theorem.

13.2.6 THEOREM. *If* $\mathbf{r}(t)$ *is a vector-valued function, then*

$$\mathbf{r}'(t) = \lim_{w\to t} \frac{\mathbf{r}(w) - \mathbf{r}(t)}{w - t}$$

exists if and only if each of the component functions for $\mathbf{r}(t)$ *is differentiable at* t, *in which case the component functions for* $\mathbf{r}'(t)$ *are the derivatives of the component functions for* $\mathbf{r}(t)$.

Proof. For simplicity, we give the proof in 2-space; the proof in 3-space is identical, except for the additional component. Assume that $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, so

$$\begin{aligned}
\mathbf{r}'(t) &= \lim_{w\to t} \frac{\mathbf{r}(w) - \mathbf{r}(t)}{w - t} = \lim_{w\to t} \frac{[x(w)\mathbf{i} + y(w)\mathbf{j}] - [x(t)\mathbf{i} + y(t)\mathbf{j}]}{w - t} \\
&= \lim_{w\to t} \frac{[x(w) - x(t)]\mathbf{i} + [y(w) - y(t)]\mathbf{j}}{w - t} \\
&= \lim_{w\to t} \left[\left(\frac{x(w) - x(t)}{w - t}\right)\mathbf{i} + \left(\frac{y(w) - y(t)}{w - t}\right)\mathbf{j}\right] \\
&= \left(\lim_{w\to t} \frac{x(w) - x(t)}{w - t}\right)\mathbf{i} + \left(\lim_{w\to t} \frac{y(w) - y(t)}{w - t}\right)\mathbf{j} \\
&= x'(t)\mathbf{i} + y'(t)\mathbf{j}
\end{aligned}$$

Example 2 Let $\mathbf{r}(t) = t^2\mathbf{i} + e^t\mathbf{j} - (2\cos\pi t)\mathbf{k}$. Then

$$\mathbf{r}'(t) = \frac{d}{dt}(t^2)\mathbf{i} + \frac{d}{dt}(e^t)\mathbf{j} - \frac{d}{dt}(2\cos\pi t)\mathbf{k} = 2t\mathbf{i} + e^t\mathbf{j} + (2\pi\sin\pi t)\mathbf{k}$$

and

$$\mathbf{r}'(1) = 2\mathbf{i} + e\mathbf{j}$$

◀

DERIVATIVE RULES

Many of the rules for differentiating real-valued functions have analogs in the context of differentiating vector-valued functions. We state some of these in the following theorem.

13.2.7 THEOREM (*Rules of Differentiation*). *Let* $\mathbf{r}(t)$, $\mathbf{r}_1(t)$, *and* $\mathbf{r}_2(t)$ *be vector-valued functions that are all in 2-space or all in 3-space, and let* $f(t)$ *be a real-valued function,* k *a scalar, and* $\mathbf{c}$ *a constant vector (that is, a vector whose value does not depend on* t). *Then the following rules of differentiation hold:*

(*a*) $\dfrac{d}{dt}[\mathbf{c}] = \mathbf{0}$

(*b*) $\dfrac{d}{dt}[k\mathbf{r}(t)] = k\dfrac{d}{dt}[\mathbf{r}(t)]$

(*c*) $\dfrac{d}{dt}[\mathbf{r}_1(t) + \mathbf{r}_2(t)] = \dfrac{d}{dt}[\mathbf{r}_1(t)] + \dfrac{d}{dt}[\mathbf{r}_2(t)]$

(*d*) $\dfrac{d}{dt}[\mathbf{r}_1(t) - \mathbf{r}_2(t)] = \dfrac{d}{dt}[\mathbf{r}_1(t)] - \dfrac{d}{dt}[\mathbf{r}_2(t)]$

(*e*) $\dfrac{d}{dt}[f(t)\mathbf{r}(t)] = f(t)\dfrac{d}{dt}[\mathbf{r}(t)] + \dfrac{d}{dt}[f(t)]\mathbf{r}(t)$

The proofs of most of these rules are immediate consequences of Definition 13.2.4, although the last rule can be seen more easily by application of the product rule for real-valued functions to the component functions. The proof of Theorem 13.2.7 is left as an exercise.

TANGENT LINES TO GRAPHS OF VECTOR-VALUED FUNCTIONS

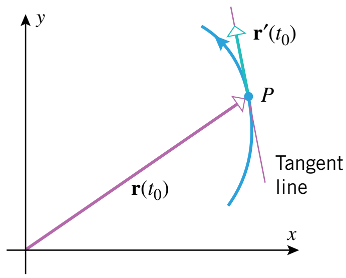


Figure 13.2.5

Motivated by the discussion of the geometric interpretation of the derivative of a vector-valued function, we make the following definition.

> **13.2.8 DEFINITION.** Let P be a point on the graph of a vector-valued function $\mathbf{r}(t)$, and let $\mathbf{r}(t_0)$ be the radius vector from the origin to P (Figure 13.2.5). If $\mathbf{r}'(t_0)$ exists and $\mathbf{r}'(t_0) \neq \mathbf{0}$, then we call $\mathbf{r}'(t_0)$ the ***tangent vector*** to the graph of $\mathbf{r}(t)$ at $\mathbf{r}(t_0)$, and we call the line through P that is parallel to the tangent vector the ***tangent line*** to the graph of $\mathbf{r}(t)$ at $\mathbf{r}(t_0)$.

Let $\mathbf{r}_0 = \mathbf{r}(t_0)$ and $\mathbf{v}_0 = \mathbf{r}'(t_0)$. It follows from Formula (9) of Section 12.5 that the tangent line to the graph of $\mathbf{r}(t)$ at $\mathbf{r}_0$ is given by the vector equation

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}_0 \tag{4}$$

Example 3 Find parametric equations of the tangent line to the circular helix

$$x = \cos t, \quad y = \sin t, \quad z = t$$

where $t = t_0$, and use that result to find parametric equations for the tangent line at the point where $t = \pi$.

Solution. The vector equation of the helix is

$$\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$$

so we have

$$\mathbf{r}_0 = \mathbf{r}(t_0) = \cos t_0\mathbf{i} + \sin t_0\mathbf{j} + t_0\mathbf{k}$$

$$\mathbf{v}_0 = \mathbf{r}'(t_0) = (-\sin t_0)\mathbf{i} + \cos t_0\mathbf{j} + \mathbf{k}$$

It follows from (4) that the vector equation of the tangent line at $t = t_0$ is

$$\begin{aligned}\mathbf{r} &= \cos t_0\mathbf{i} + \sin t_0\mathbf{j} + t_0\mathbf{k} + t[(-\sin t_0)\mathbf{i} + \cos t_0\mathbf{j} + \mathbf{k}]\\ &= (\cos t_0 - t\sin t_0)\mathbf{i} + (\sin t_0 + t\cos t_0)\mathbf{j} + (t_0 + t)\mathbf{k}\end{aligned}$$

Thus, the parametric equations of the tangent line at $t = t_0$ are

$$x = \cos t_0 - t\sin t_0, \quad y = \sin t_0 + t\cos t_0, \quad z = t_0 + t$$

In particular, the tangent line at the point where $t = \pi$ has parametric equations

$$x = -1, \quad y = -t, \quad z = \pi + t$$

The graph of the helix and this tangent line are shown in Figure 13.2.6. ◀

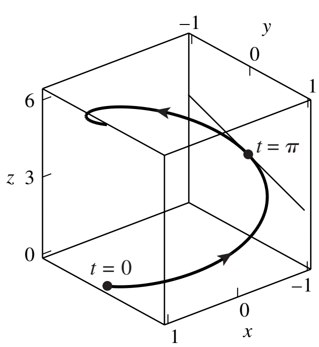


Figure 13.2.6

Example 4 Let

$$\mathbf{r}_1(t) = (\tan^{-1} t)\mathbf{i} + (\sin t)\mathbf{j} + t^2\mathbf{k}$$

and

$$\mathbf{r}_2(t) = (t^2 - t)\mathbf{i} + (2t - 2)\mathbf{j} + (\ln t)\mathbf{k}$$

The graphs of $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ intersect at the origin. Find the degree measure of the acute angle between the tangent lines to the graphs of $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ at the origin.

Solution. The graph of $\mathbf{r}_1(t)$ passes through the origin at $t = 0$, where its tangent vector is

$$\mathbf{r}_1'(0) = \left\langle \frac{1}{1+t^2}, \cos t, 2t \right\rangle\bigg|_{t=0} = \langle 1, 1, 0\rangle$$

The graph of $\mathbf{r}_2(t)$ passes through the origin at $t = 1$ (verify), where its tangent vector is

$$\mathbf{r}_2'(1) = \left\langle 2t - 1, 2, \frac{1}{t} \right\rangle \bigg|_{t=1} = \langle 1, 2, 1 \rangle$$

By Theorem 12.3.3, the angle θ between these two tangent vectors satisfies

$$\cos\theta = \frac{\langle 1, 1, 0\rangle \cdot \langle 1, 2, 1\rangle}{\|\langle 1, 1, 0\rangle\| \, \|\langle 1, 2, 1\rangle\|} = \frac{3}{\sqrt{12}} = \frac{\sqrt{3}}{2}$$

It follows that $\theta = \pi/6$ radians, or 30°. ◀

DERIVATIVES OF DOT AND CROSS PRODUCTS

The following rules, which are derived in the exercises, provide a method for differentiating dot products in 2-space and 3-space and cross products in 3-space.

$$\frac{d}{dt}[\mathbf{r}_1(t) \cdot \mathbf{r}_2(t)] = \mathbf{r}_1(t) \cdot \frac{d\mathbf{r}_2}{dt} + \frac{d\mathbf{r}_1}{dt} \cdot \mathbf{r}_2(t) \tag{5}$$

$$\frac{d}{dt}[\mathbf{r}_1(t) \times \mathbf{r}_2(t)] = \mathbf{r}_1(t) \times \frac{d\mathbf{r}_2}{dt} + \frac{d\mathbf{r}_1}{dt} \times \mathbf{r}_2(t) \tag{6}$$

REMARK. In (5) the order of the factors in each term on the right does not matter, but in (6) it does.

In plane geometry one learns that a tangent line to a circle is perpendicular to the radius at the point of tangency. Consequently, if a point moves along a circle in 2-space that is centered at the origin, then one would expect the radius vector and the tangent vector at any point on the circle to be orthogonal. This is the motivation for the following useful theorem, which is applicable in both 2-space and 3-space.

13.2.9 THEOREM. *If $\mathbf{r}(t)$ is a vector-valued function in 2-space or 3-space and $\|\mathbf{r}(t)\|$ is constant for all t, then*

$$\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0 \tag{7}$$

that is, $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are orthogonal vectors for all t.

Proof. It follows from (5) with $\mathbf{r}_1(t) = \mathbf{r}_2(t) = \mathbf{r}(t)$ that

$$\frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{r}(t)] = \mathbf{r}(t) \cdot \frac{d\mathbf{r}}{dt} + \frac{d\mathbf{r}}{dt} \cdot \mathbf{r}(t)$$

or, equivalently,

$$\frac{d}{dt}[\|\mathbf{r}(t)\|^2] = 2\mathbf{r}(t) \cdot \frac{d\mathbf{r}}{dt} \tag{8}$$

But $\|\mathbf{r}(t)\|^2$ is constant, so its derivative is zero. Thus

$$2\mathbf{r}(t) \cdot \frac{d\mathbf{r}}{dt} = 0$$

from which (7) follows. ■

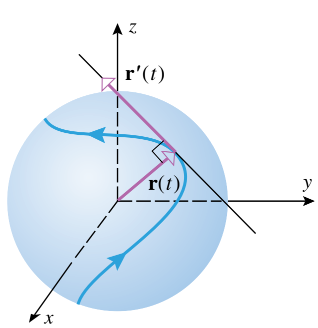


Figure 13.2.7

Example 5 Just as a tangent line to a circle in 2-space is perpendicular to the radius at the point of tangency, so a tangent vector to a curve on the surface of a sphere in 3-space that is centered at the origin is orthogonal to the radius vector at the point of tangency (Figure 13.2.7). To see that this is so, suppose that the graph of $\mathbf{r}(t)$ lies on the surface of a sphere of positive radius k centered at the origin. For each value of t we have $\|\mathbf{r}(t)\| = k$, so by Theorem 13.2.9

$$\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$$

and hence the radius vector $\mathbf{r}(t)$ and the tangent vector $\mathbf{r}'(t)$ are orthogonal. ◀

INTEGRALS OF VECTOR-VALUED FUNCTIONS

If $\mathbf{r}(t) = \langle x(t), y(t)\rangle = x(t)\mathbf{i} + y(t)\mathbf{j}$ is a vector-valued function in 2-space, we can define the definite integral of $\mathbf{r}(t)$ from $t = a$ to $t = b$ via a Riemann sum, as was done for real-valued functions in Definition 5.5.1. It follows immediately that a definite integral of $\mathbf{r}(t)$ can be expressed as a vector whose components are the definite integrals of the component functions for $\mathbf{r}(t)$.

$$\begin{aligned}
\int_a^b \mathbf{r}(t)\,dt &= \lim_{\max \Delta t_k \to 0} \sum_{k=1}^{n} \mathbf{r}(t_k^*)\Delta t_k \\
&= \lim_{\max \Delta t_k \to 0} \left[\left(\sum_{k=1}^{n} x(t_k^*)\Delta t_k\right)\mathbf{i} + \left(\sum_{k=1}^{n} y(t_k^*)\Delta t_k\right)\mathbf{j}\right] \\
&= \left(\lim_{\max \Delta t_k \to 0} \sum_{k=1}^{n} x(t_k^*)\Delta t_k\right)\mathbf{i} + \left(\lim_{\max \Delta t_k \to 0} \sum_{k=1}^{n} y(t_k^*)\Delta t_k\right)\mathbf{j} \\
&= \left(\int_a^b x(t)\,dt\right)\mathbf{i} + \left(\int_a^b y(t)\,dt\right)\mathbf{j}
\end{aligned}$$

Alternatively,

$$\int_a^b \langle x(t), y(t)\rangle\,dt = \left\langle \int_a^b x(t)\,dt, \int_a^b y(t)\,dt \right\rangle$$

For vector-valued functions in 3-space this becomes

$$\begin{aligned}
\int_a^b \langle x(t), y(t), z(t)\rangle\,dt &= \left\langle \int_a^b x(t)\,dt, \int_a^b y(t)\,dt, \int_a^b z(t)\,dt \right\rangle \\
&= \left(\int_a^b x(t)\,dt\right)\mathbf{i} + \left(\int_a^b y(t)\,dt\right)\mathbf{j} + \left(\int_a^b z(t)\,dt\right)\mathbf{k}
\end{aligned}$$

Example 6 Let $\mathbf{r}(t) = t^2\mathbf{i} + e^t\mathbf{j} - (2\cos \pi t)\mathbf{k}$. Then

$$\begin{aligned}
\int_0^1 \mathbf{r}(t)\,dt &= \left(\int_0^1 t^2\,dt\right)\mathbf{i} + \left(\int_0^1 e^t\,dt\right)\mathbf{j} - \left(\int_0^1 2\cos \pi t\,dt\right)\mathbf{k} \\
&= \frac{t^3}{3}\bigg]_0^1 \mathbf{i} + e^t\bigg]_0^1 \mathbf{j} - \frac{2}{\pi}\sin \pi t\bigg]_0^1 \mathbf{k} = \frac{1}{3}\mathbf{i} + (e-1)\mathbf{j}
\end{aligned}$$ ◀

An ***antiderivative*** for a vector-valued function $\mathbf{r}(t)$ is a vector-valued function $\mathbf{R}(t)$ such that

$$\mathbf{R}'(t) = \mathbf{r}(t) \tag{9}$$

As in Chapter 5, we recast Equation (9) using integral notation as

$$\int \mathbf{r}(t)\,dt = \mathbf{R}(t) + \mathbf{C} \tag{10}$$

where $\mathbf{C}$ is understood to represent an arbitrary constant *vector*.

Note that since differentiation of vector-valued functions can be done componentwise, antidifferentiation can also be done componentwise. This is illustrated in the next example.

Example 7

$$\begin{aligned}
\int (2t\mathbf{i} + 3t^2\mathbf{j})\,dt &= \left(\int 2t\,dt\right)\mathbf{i} + \left(\int 3t^2\,dt\right)\mathbf{j} \\
&= (t^2 + C_1)\mathbf{i} + (t^3 + C_2)\mathbf{j} \\
&= (t^2\mathbf{i} + t^3\mathbf{j}) + (C_1\mathbf{i} + C_2\mathbf{j}) = (t^2\mathbf{i} + t^3\mathbf{j}) + \mathbf{C}
\end{aligned}$$

where $\mathbf{C} = C_1\mathbf{i} + C_2\mathbf{j}$ is an arbitrary vector constant of integration. ◀

Most of the familiar integration properties have vector counterparts. For example, vector differentiation and integration are inverse operations in the sense that

$$\frac{d}{dt}\left[\int \mathbf{r}(t)\,dt\right] = \mathbf{r}(t) \tag{11}$$

and

$$\int \mathbf{r}'(t)\,dt = \mathbf{r}(t) + \mathbf{C} \tag{12}$$

If $\mathbf{R}(t)$ is an antiderivative of $\mathbf{r}(t)$ on an interval containing a and b, then

$$\int_a^b \mathbf{r}(t)\,dt = \mathbf{R}(t)\Big]_a^b = \mathbf{R}(b) - \mathbf{R}(a) \tag{13}$$

Example 8 Evaluate the definite integral $\displaystyle\int_0^2 (2t\mathbf{i} + 3t^2\mathbf{j})\,dt$.

Solution. Integrating the components yields

$$\int_0^2 (2t\mathbf{i} + 3t^2\mathbf{j})\,dt = t^2\Big]_0^2\,\mathbf{i} + t^3\Big]_0^2\,\mathbf{j} = 4\mathbf{i} + 8\mathbf{j}$$

Alternative Solution. The function $\mathbf{R}(t) = t^2\mathbf{i} + t^3\mathbf{j}$ is an antiderivative of the integrand since $\mathbf{R}'(t) = 2t\mathbf{i} + 3t^2\mathbf{j}$. Thus, it follows from (13) that

$$\int_0^2 (2t\mathbf{i} + 3t^2\mathbf{j})\,dt = \mathbf{R}(t)\Big]_0^2 = t^2\mathbf{i} + t^3\mathbf{j}\Big]_0^2 = (4\mathbf{i} + 8\mathbf{j}) - (0\mathbf{i} + 0\mathbf{j}) = 4\mathbf{i} + 8\mathbf{j}$$ ◀

Example 9 Find $\mathbf{r}(t)$ given that $\mathbf{r}'(t) = \langle 3, 2t\rangle$ and $\mathbf{r}(1) = \langle 2, 5\rangle$.

Solution. Integrating $\mathbf{r}'(t)$ to obtain $\mathbf{r}(t)$ yields

$$\mathbf{r}(t) = \int \mathbf{r}'(t)\,dt = \int \langle 3, 2t\rangle\,dt = \langle 3t, t^2\rangle + \mathbf{C}$$

where $\mathbf{C}$ is a vector constant of integration. To find the value of $\mathbf{C}$ we substitute $t = 1$ and use the given value of $\mathbf{r}(1)$ to obtain

$$\mathbf{r}(1) = \langle 3, 1\rangle + \mathbf{C} = \langle 2, 5\rangle$$

so that $\mathbf{C} = \langle -1, 4\rangle$. Thus,

$$\mathbf{r}(t) = \langle 3t, t^2\rangle + \langle -1, 4\rangle = \langle 3t - 1, t^2 + 4\rangle$$ ◀

INTEGRAL RULES

As with differentiation, many of the rules for integrating real-valued functions have analogs in the context of integrating vector-valued functions.

13.2.10 THEOREM (*Rules of Integration*). *Let k be a scalar and let $\mathbf{r}(t)$, $\mathbf{r}_1(t)$, $\mathbf{r}_2(t)$, $\mathbf{R}(t)$, $\mathbf{R}_1(t)$, and $\mathbf{R}_2(t)$ be vector-valued functions, all in 2-space or all in 3-space, such that $\mathbf{R}$, $\mathbf{R}_1$, and $\mathbf{R}_2$ are antiderivatives of $\mathbf{r}$, $\mathbf{r}_1$, and $\mathbf{r}_2$, respectively; that is, $\mathbf{R}'(t) = \mathbf{r}(t)$, $\mathbf{R}_1'(t) = \mathbf{r}_1(t)$, and $\mathbf{R}_2'(t) = \mathbf{r}_2(t)$. Then*

(a) $\displaystyle\int k\mathbf{r}(t)\,dt = k\mathbf{R}(t) + \mathbf{C}$

(b) $\displaystyle\int [\mathbf{r}_1(t) + \mathbf{r}_2(t)]\,dt = \mathbf{R}_1(t) + \mathbf{R}_2(t) + \mathbf{C}$

(c) $\displaystyle\int [\mathbf{r}_1(t) - \mathbf{r}_2(t)]\,dt = \mathbf{R}_1(t) - \mathbf{R}_2(t) + \mathbf{C}$

The proofs of these rules are left as an exercise.

EXERCISE SET 13.2 Graphing Utility

In Exercises 1–6, find the limit.

1. $\lim_{t\to 3}(t^2\mathbf{i}+2t\mathbf{j})$

2. $\lim_{t\to \pi/4}\langle \cos t, \sin t\rangle$

3. $\lim_{t\to +\infty}\left\langle \dfrac{t^2+1}{3t^2+2}, \dfrac{1}{t}\right\rangle$

4. $\lim_{t\to 0^+}\left(\sqrt{t}\,\mathbf{i}+\dfrac{\sin t}{t}\mathbf{j}\right)$

5. $\lim_{t\to 2}(t\mathbf{i}-3\mathbf{j}+t^2\mathbf{k})$

6. $\lim_{t\to 1}\left\langle \dfrac{3}{t^2}, \dfrac{\ln t}{t^2-1}, \sin 2t\right\rangle$

In Exercises 7 and 8, determine whether $\mathbf{r}(t)$ is continuous at $t=0$. Explain your reasoning.

7. (a) $\mathbf{r}(t)=3\sin t\,\mathbf{i}-2t\mathbf{j}$ (b) $\mathbf{r}(t)=t^2\mathbf{i}+\dfrac{1}{t}\mathbf{j}+t\mathbf{k}$

8. (a) $\mathbf{r}(t)=e^t\mathbf{i}+\mathbf{j}+\csc t\,\mathbf{k}$
(b) $\mathbf{r}(t)=5\mathbf{i}-\sqrt{3t+1}\,\mathbf{j}+e^{2t}\mathbf{k}$

9. Sketch the circle $\mathbf{r}(t)=\cos t\,\mathbf{i}+\sin t\,\mathbf{j}$, and in each part draw the vector with its correct length.
(a) $\mathbf{r}'(\pi/4)$ (b) $\mathbf{r}''(\pi)$ (c) $\mathbf{r}(2\pi)-\mathbf{r}(3\pi/2)$

10. Sketch the circle $\mathbf{r}(t)=\cos t\,\mathbf{i}-\sin t\,\mathbf{j}$, and in each part draw the vector with its correct length.
(a) $\mathbf{r}'(\pi/4)$ (b) $\mathbf{r}''(\pi)$ (c) $\mathbf{r}(2\pi)-\mathbf{r}(3\pi/2)$

In Exercises 11–14, find $\mathbf{r}'(t)$.

11. $\mathbf{r}(t)=(4+5t)\mathbf{i}+(t-t^2)\mathbf{j}$

12. $\mathbf{r}(t)=4\mathbf{i}-\cos t\,\mathbf{j}$

13. $\mathbf{r}(t)=\dfrac{1}{t}\mathbf{i}+\tan t\,\mathbf{j}+e^{2t}\mathbf{k}$

14. $\mathbf{r}(t)=(\tan^{-1} t)\mathbf{i}+t\cos t\,\mathbf{j}-\sqrt{t}\,\mathbf{k}$

In Exercises 15–18, find the vector $\mathbf{r}'(t_0)$; then sketch the graph of $\mathbf{r}(t)$ in 2-space and draw the tangent vector $\mathbf{r}'(t_0)$.

15. $\mathbf{r}(t)=\langle t, t^2\rangle;\ t_0=2$

16. $\mathbf{r}(t)=t^3\mathbf{i}+t^2\mathbf{j};\ t_0=1$

17. $\mathbf{r}(t)=\sec t\,\mathbf{i}+\tan t\,\mathbf{j};\ t_0=0$

18. $\mathbf{r}(t)=2\sin t\,\mathbf{i}+3\cos t\,\mathbf{j};\ t_0=\pi/6$

In Exercises 19 and 20, find the vector $\mathbf{r}'(t_0)$; then sketch the graph of $\mathbf{r}(t)$ in 3-space and draw the tangent vector $\mathbf{r}'(t_0)$.

19. $\mathbf{r}(t)=2\sin t\,\mathbf{i}+\mathbf{j}+2\cos t\,\mathbf{k};\ t_0=\pi/2$

20. $\mathbf{r}(t)=\cos t\,\mathbf{i}+\sin t\,\mathbf{j}+t\mathbf{k};\ t_0=\pi/4$

In Exercises 21 and 22, use a graphing utility to generate the graph of $\mathbf{r}(t)$ and the graph of the tangent line at t_0 on the same screen.

21. $\mathbf{r}(t)=\sin \pi t\,\mathbf{i}+t^2\mathbf{j};\ t_0=\frac{1}{2}$

22. $\mathbf{r}(t)=3\sin t\,\mathbf{i}+4\cos t\,\mathbf{j};\ t_0=\pi/4$

In Exercises 23–26, find parametric equations of the line tangent to the graph of $\mathbf{r}(t)$ at the point where $t=t_0$.

23. $\mathbf{r}(t)=t^2\mathbf{i}+(2-\ln t)\mathbf{j};\ t_0=1$

24. $\mathbf{r}(t)=e^{2t}\mathbf{i}-2\cos 3t\,\mathbf{j};\ t_0=0$

25. $\mathbf{r}(t)=2\cos \pi t\,\mathbf{i}+2\sin \pi t\,\mathbf{j}+3t\mathbf{k};\ t_0=\frac{1}{3}$

26. $\mathbf{r}(t)=\ln t\,\mathbf{i}+e^{-t}\mathbf{j}+t^3\mathbf{k};\ t_0=2$

In Exercises 27–30, find a vector equation of the line tangent to the graph of $\mathbf{r}(t)$ at the point P_0 on the curve.

27. $\mathbf{r}(t)=(2t-1)\mathbf{i}+\sqrt{3t+4}\,\mathbf{j};\ P_0(-1,2)$

28. $\mathbf{r}(t)=4\cos t\,\mathbf{i}-3t\mathbf{j};\ P_0(2,-\pi)$

29. $\mathbf{r}(t)=t^2\mathbf{i}-\dfrac{1}{t+1}\mathbf{j}+(4-t^2)\mathbf{k};\ P_0(4,1,0)$

30. $\mathbf{r}(t)=\sin t\,\mathbf{i}+\cosh t\,\mathbf{j}+(\tan^{-1} t)\mathbf{k};\ P_0(0,1,0)$

31. Let $\mathbf{r}(t)=\cos t\,\mathbf{i}+\sin t\,\mathbf{j}+\mathbf{k}$. Find
(a) $\lim_{t\to 0}(\mathbf{r}(t)-\mathbf{r}'(t))$ (b) $\lim_{t\to 0}(\mathbf{r}(t)\times\mathbf{r}'(t))$
(c) $\lim_{t\to 0}(\mathbf{r}(t)\cdot\mathbf{r}'(t))$.

32. Let $\mathbf{r}(t)=t\mathbf{i}+t^2\mathbf{j}+t^3\mathbf{k}$. Find

$$\lim_{t\to 1}\mathbf{r}(t)\cdot(\mathbf{r}'(t)\times\mathbf{r}''(t))$$

In Exercises 33 and 34, calculate

$$\frac{d}{dt}[\mathbf{r}_1(t)\cdot\mathbf{r}_2(t)] \quad\text{and}\quad \frac{d}{dt}[\mathbf{r}_1(t)\times\mathbf{r}_2(t)]$$

first by differentiating the product directly and then by applying Formulas (5) and (6).

33. $\mathbf{r}_1(t)=2t\mathbf{i}+3t^2\mathbf{j}+t^3\mathbf{k},\ \mathbf{r}_2(t)=t^4\mathbf{k}$

34. $\mathbf{r}_1(t)=\cos t\,\mathbf{i}+\sin t\,\mathbf{j}+t\mathbf{k},\ \mathbf{r}_2(t)=\mathbf{i}+t\mathbf{k}$

In Exercises 35–40, evaluate the indefinite integral.

35. $\int(3\mathbf{i}+4t\mathbf{j})\,dt$

36. $\int(\cos t\,\mathbf{i}+\sin t\,\mathbf{j})\,dt$

37. $\int(t\sin \mathbf{i}+\mathbf{j})\,dt$

38. $\int\langle te^t, \ln t\rangle\,dt$

39. $\int\left(t^2\mathbf{i}-2t\mathbf{j}+\dfrac{1}{t}\mathbf{k}\right)dt$

40. $\int\langle e^{-t}, e^t, 3t^2\rangle\,dt$

In Exercises 41– 46, evaluate the definite integral.

41. $\int_0^{\pi/3}\langle \cos 3t, -\sin 3t\rangle\,dt$

42. $\int_0^1(t^2\mathbf{i}+t^3\mathbf{j})\,dt$

43. $\int_0^2\|t\mathbf{i}+t^2\mathbf{j}\|\,dt$

44. $\int_{-3}^3\langle(3-t)^{3/2}, (3+t)^{3/2}, 1\rangle\,dt$

45. $\displaystyle\int_1^9 (t^{1/2}\mathbf{i} + t^{-1/2}\mathbf{j})\,dt$ **46.** $\displaystyle\int_0^1 (e^{2t}\mathbf{i} + e^{-t}\mathbf{j} + t\mathbf{k})\,dt$

In Exercises 47–50, solve the vector initial-value problem for $\mathbf{y}(t)$ by integrating and using the initial conditions to find the constants of integration.

47. $\mathbf{y}'(t) = t^2\mathbf{i} + 2t\mathbf{j},\ \ \mathbf{y}(0) = \mathbf{i} + \mathbf{j}$

48. $\mathbf{y}'(t) = \cos t\mathbf{i} + \sin t\mathbf{j},\ \ \mathbf{y}(0) = \mathbf{i} - \mathbf{j}$

49. $\mathbf{y}''(t) = \mathbf{i} + e^t\mathbf{j},\ \ \mathbf{y}(0) = 2\mathbf{i},\ \ \mathbf{y}'(0) = \mathbf{j}$

50. $\mathbf{y}''(t) = 12t^2\mathbf{i} - 2t\mathbf{j},\ \ \mathbf{y}(0) = 2\mathbf{i} - 4\mathbf{j},\ \ \mathbf{y}'(0) = \mathbf{0}$

In Exercises 51 and 52, let $\theta(t)$ be the angle between $\mathbf{r}(t)$ and $\mathbf{r}'(t)$. Use a graphing calculator to generate the graph of θ versus t, and make rough estimates of the t-values at which t-intercepts or relative extrema occur. What do these values tell you about the vectors $\mathbf{r}(t)$ and $\mathbf{r}'(t)$?

51. $\mathbf{r}(t) = 4\cos t\mathbf{i} + 3\sin t\mathbf{j};\ \ 0 \le t \le 2\pi$

52. $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j};\ \ 0 \le t \le 1$

53. (a) Find the points where the curve

$$\mathbf{r} = t\mathbf{i} + t^2\mathbf{j} - 3t\mathbf{k}$$

intersects the plane $2x - y + z = -2$.

(b) For the curve and plane in part (a), find, to the nearest degree, the acute angle that the tangent line to the curve makes with a line normal to the plane at each point of intersection.

54. Find where the tangent line to the curve

$$\mathbf{r} = e^{-2t}\mathbf{i} + \cos t\mathbf{j} + 3\sin t\mathbf{k}$$

at the point $(1, 1, 0)$ intersects the yz-plane.

In Exercises 55 and 56, show that the graphs of $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ intersect at the point P. Find, to the nearest degree, the acute angle between the tangent lines to the graphs of $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ at the point P.

55. $\mathbf{r}_1(t) = t^2\mathbf{i} + t\mathbf{j} + 3t^3\mathbf{k}$
$\mathbf{r}_2(t) = (t-1)\mathbf{i} + \frac{1}{4}t^2\mathbf{j} + (5-t)\mathbf{k};\ \ P(1, 1, 3)$

56. $\mathbf{r}_1(t) = 2e^{-t}\mathbf{i} + \cos t\mathbf{j} + (t^2+3)\mathbf{k}$
$\mathbf{r}_2(t) = (1-t)\mathbf{i} + t^2\mathbf{j} + (t^3+4)\mathbf{k};\ \ P(2, 1, 3)$

57. Use Formula (6) to derive the differentiation formula

$$\frac{d}{dt}[\mathbf{r}(t) \times \mathbf{r}'(t)] = \mathbf{r}(t) \times \mathbf{r}''(t)$$

58. Let $\mathbf{u} = \mathbf{u}(t)$, $\mathbf{v} = \mathbf{v}(t)$, and $\mathbf{w} = \mathbf{w}(t)$ be differentiable vector-valued functions. Use Formulas (5) and (6) to show that

$$\frac{d}{dt}[\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})] = \frac{d\mathbf{u}}{dt} \cdot [\mathbf{v} \times \mathbf{w}] + \mathbf{u} \cdot \left[\frac{d\mathbf{v}}{dt} \times \mathbf{w}\right] + \mathbf{u} \cdot \left[\mathbf{v} \times \frac{d\mathbf{w}}{dt}\right]$$

59. Let u_1, u_2, u_3, v_1, v_2, v_3, w_1, w_2, and w_3 be differentiable functions of t. Use Exercise 58 to show that

$$\frac{d}{dt}\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} u_1' & u_2' & u_3' \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} + \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1' & v_2' & v_3' \\ w_1 & w_2 & w_3 \end{vmatrix} + \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1' & w_2' & w_3' \end{vmatrix}$$

60. Prove Theorem 13.2.7 for 2-space.

61. Derive Formulas (5) and (6) for 3-space.

62. Prove Theorem 13.2.10 for 2-space.

13.3 CHANGE OF PARAMETER; ARC LENGTH

We observed in earlier sections that a curve in 2-space or 3-space can be represented parametrically in more than one way. For example, in Section 1.8 we gave two parametric representations of a circle—one in which the circle was traced clockwise and the other in which it was traced counterclockwise. Sometimes it will be desirable to change the parameter for a parametric curve to a different parameter that is better suited for the problem at hand. In this section we will investigate issues associated with changes of parameter, and we will show that arc length plays a special role in parametric representations of curves.

SMOOTH PARAMETRIZATIONS

Graphs of vector-valued functions range from continuous and smooth to discontinuous and wildly erratic. In this text we will not be concerned with graphs of the latter type, so we will need to impose restrictions to eliminate the unwanted behavior. We will say that $\mathbf{r}(t)$ is ***smoothly parametrized*** or that $\mathbf{r}(t)$ is a ***smooth function*** of t if $\mathbf{r}'(t)$ is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$ for any allowable value of t. Algebraically, smoothness implies that the components of $\mathbf{r}(t)$ have continuous derivatives that are not all zero for the same value of t, and geometrically, it implies that the tangent vector $\mathbf{r}'(t)$ varies continuously along the curve. For this reason a smoothly parametrized function is said to have a ***continuously turning tangent vector***.

Example 1 Determine whether the following vector-valued functions have continuously turning tangent vectors.

(a) $\mathbf{r}(t) = a\cos t\mathbf{i} + a\sin t\mathbf{j} + ct\mathbf{k} \quad (a > 0, c > 0)$
(b) $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}$

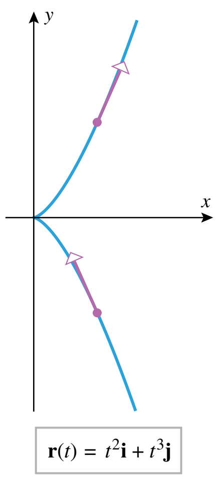


Figure 13.3.1

Solution (a). We have

$$\mathbf{r}'(t) = -a\sin t\mathbf{i} + a\cos t\mathbf{j} + c\mathbf{k}$$

The components are continuous functions, and there is no value of t for which all three of them are zero (verify), so $\mathbf{r}(t)$ has a continuously turning tangent vector. The graph of $\mathbf{r}(t)$ is the circular helix in Figure 13.1.2.

Solution (b). We have

$$\mathbf{r}'(t) = 2t\mathbf{i} + 3t^2\mathbf{j}$$

Although the components are continuous functions, they are both equal to zero if $t = 0$, so $\mathbf{r}(t)$ does not have a continuously turning tangent vector. The graph of $\mathbf{r}(t)$, which is shown in Figure 13.3.1, is a semicubical parabola traced in the upward direction (see Example 3 of Section 11.2). Observe that for values of t slightly less than zero the angle between $\mathbf{r}'(t)$ and $\mathbf{i}$ is near π, and for values of t slightly larger than zero the angle is near 0; hence there is a sudden reversal in the direction of the tangent vector as t increases through $t = 0$. ◀

ARC LENGTH FROM THE VECTOR VIEWPOINT

Recall from Theorem 6.4.3 that the arc length L of a parametric curve

$$x = x(t), \quad y = y(t) \qquad (a \le t \le b) \tag{1}$$

is given by the formula

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \tag{2}$$

Analogously, the arc length L of a parametric curve

$$x = x(t), \quad y = y(t), \quad z = z(t) \qquad (a \le t \le b) \tag{3}$$

in 3-space is given by the formula

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \tag{4}$$

Formulas (2) and (4) have vector forms that we can obtain by letting

$$\underbrace{\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}}_{\text{2-space}} \quad \text{or} \quad \underbrace{\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}}_{\text{3-space}}$$

It follows that

$$\underbrace{\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}}_{\text{2-space}} \quad \text{or} \quad \underbrace{\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}}_{\text{3-space}}$$

and hence

$$\left\|\frac{d\mathbf{r}}{dt}\right\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \quad \text{or} \quad \left\|\frac{d\mathbf{r}}{dt}\right\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

2-space 3-space

Substituting these expressions in (2) and (4) leads us to the following theorem.

13.3.1 THEOREM. *If C is the graph in 2-space or 3-space of a smooth vector-valued function $\mathbf{r}(t)$, then its arc length L from $t = a$ to $t = b$ is*

$$L = \int_a^b \left\|\frac{d\mathbf{r}}{dt}\right\| dt \tag{5}$$

Example 2 Find the arc length of that portion of the circular helix

$$x = \cos t, \quad y = \sin t, \quad z = t$$

from $t = 0$ to $t = \pi$.

Solution. Set $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k} = \langle \cos t, \sin t, t\rangle$. Then

$$\mathbf{r}'(t) = \langle -\sin t, \cos t, 1\rangle \quad \text{and} \quad \|\mathbf{r}'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1} = \sqrt{2}$$

From Theorem 13.3.1 the arc length of the helix is

$$L = \int_0^\pi \left\|\frac{d\mathbf{r}}{dt}\right\| dt = \int_0^\pi \sqrt{2}\, dt = \sqrt{2}\pi$$

◀

ARC LENGTH AS A PARAMETER

For many purposes the best parameter to use for representing a curve in 2-space or 3-space parametrically is the length of arc measured along the curve from some fixed reference point. This can be done as follows:

Step 1. Select an arbitrary point on the curve C to serve as a ***reference point***.

Step 2. Starting from the reference point, choose one direction along the curve to be the ***positive direction*** and the other to be the ***negative direction***.

Step 3. If P is a point on the curve, let s be the "signed" arc length along C from the reference point to P, where s is positive if P is in the positive direction from the reference point, and s is negative if P is in the negative direction. Figure 13.3.2 illustrates this idea.

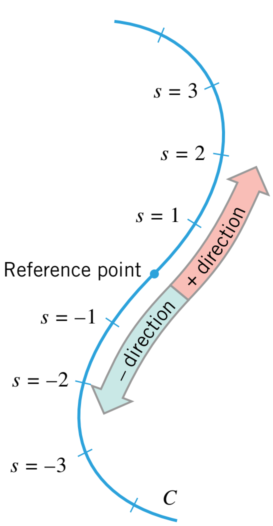


Figure 13.3.2

By this procedure, a unique point P on the curve is determined when a value for s is given. For example, $s = 2$ determines the point that is 2 units along the curve in the positive direction from the reference point, and $s = -\frac{3}{2}$ determines the point that is $\frac{3}{2}$ units along the curve in the negative direction from the reference point.

Let us now treat s as a variable. As the value of s changes, the corresponding point P moves along C and the coordinates of P become functions of s. Thus, in 2-space the coordinates of P are $(x(s), y(s))$, and in 3-space they are $(x(s), y(s), z(s))$. Therefore, in 2-space or 3-space the curve C is given by the parametric equations

$$x = x(s), \quad y = y(s) \quad \text{or} \quad x = x(s), \quad y = y(s), \quad z = z(s)$$

A parametric representation of a curve with arc length as the parameter is called an ***arc length parametrization*** of the curve. Note that a given curve will generally have infinitely many different arc length parametrizations, since the reference point and orientation can be chosen arbitrarily.

Example 3 Find the arc length parametrization of the circle $x^2 + y^2 = a^2$ with counterclockwise orientation and $(a, 0)$ as the reference point.

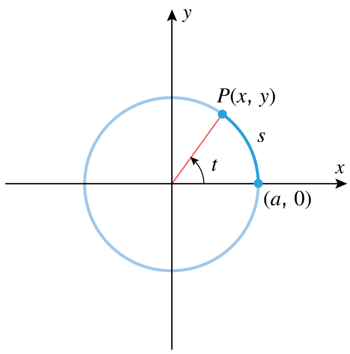


Figure 13.3.3

Solution. The circle with counterclockwise orientation can be represented by the parametric equations

$$x = a\cos t, \quad y = a\sin t \qquad (0 \le t \le 2\pi) \tag{6}$$

in which t can be interpreted as the angle in radian measure from the positive x-axis to the radius from the origin to the point $P(x, y)$ (Figure 13.3.3). If we take the positive direction for measuring the arc length to be counterclockwise, and we take $(a, 0)$ to be the reference point, then s and t are related by

$$s = at \quad \text{or} \quad t = s/a$$

Making this change of variable in (6) and noting that s increases from 0 to $2\pi a$ as t increases from 0 to 2π yields the following arc length parametrization of the circle:

$$x = a\cos(s/a), \quad y = a\sin(s/a) \qquad (0 \le s \le 2\pi a)$$ ◀

CHANGE OF PARAMETER

In many situations the solution of a problem can be simplified by choosing the parameter in a vector-valued function or a parametric curve in the right way. The two most common parameters for curves in 2-space or 3-space are time and arc length. However, there are other useful possibilities as well. For example, in analyzing the motion of a particle in 2-space, it is often desirable to parametrize its trajectory in terms of the angle ϕ between the tangent vector and the positive x-axis (Figure 13.3.4). Thus, our next objective is to develop methods for changing the parameter in a vector-valued function or parametric curve. This will allow us to move freely between different possible parametrizations.

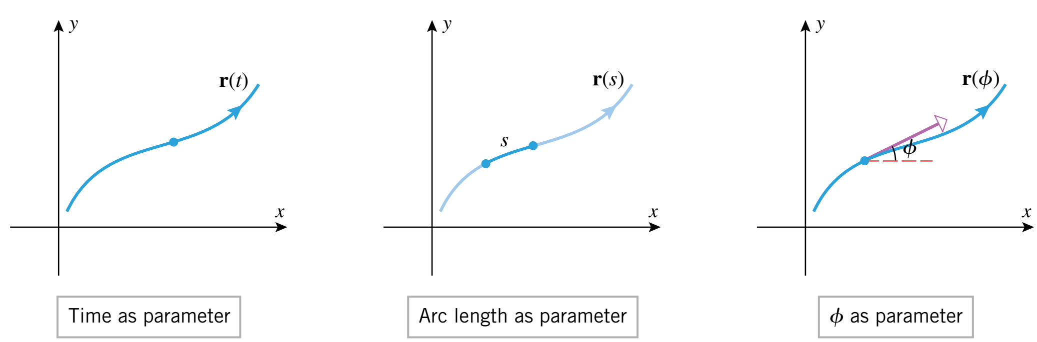


Figure 13.3.4

A ***change of parameter*** in a vector-valued function $\mathbf{r}(t)$ is a substitution $t = g(\tau)$ that produces a new vector-valued function $\mathbf{r}(g(\tau))$ having the same graph as $\mathbf{r}(t)$, but possibly traced differently as the parameter τ increases.

Example 4 Find a change of parameter $t = g(\tau)$ for the circle

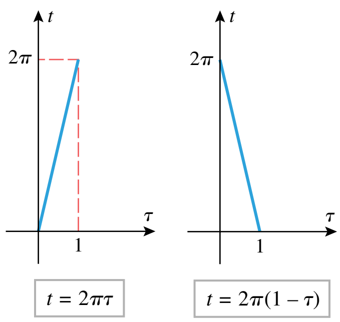
$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} \quad (0 \le t \le 2\pi)$$

such that

(a) the circle is traced counterclockwise as τ increases over the interval [0, 1];

(b) the circle is traced clockwise as τ increases over the interval [0, 1].

Solution (a). The given circle is traced counterclockwise as t increases. Thus, if we choose g to be an increasing function, then it will follow from the relationship $t = g(\tau)$ that



(*a*) (*b*)

Figure 13.3.5

t increases when τ increases, thereby ensuring that the circle will be traced counterclockwise as τ increases. We also want to choose g so that t increases from 0 to 2π as τ increases from 0 to 1. A simple choice of g that satisfies all of the required criteria is the linear function graphed in Figure 13.3.5*a*. The equation of this line is

$$t = g(\tau) = 2\pi\tau \tag{7}$$

which is the desired change of parameter. The resulting representation of the circle in terms of the parameter τ is

$$\mathbf{r}(g(\tau)) = \cos 2\pi\tau\,\mathbf{i} + \sin 2\pi\tau\,\mathbf{j} \quad (0 \le \tau \le 1)$$

Solution (b). To ensure that the circle is traced clockwise, we will choose g to be a decreasing function such that t decreases from 2π to 0 as τ increases from 0 to 1. A simple choice of g that achieves this is the linear function

$$t = g(\tau) = 2\pi(1-\tau) \tag{8}$$

graphed in Figure 13.3.5*b*. The resulting representation of the circle in terms of the parameter τ is

$$\mathbf{r}(g(\tau)) = \cos(2\pi(1-\tau))\mathbf{i} + \sin(2\pi(1-\tau))\mathbf{j} \quad (0 \le \tau \le 1)$$

which simplifies to (verify)

$$\mathbf{r}(g(\tau)) = \cos 2\pi\tau\,\mathbf{i} - \sin 2\pi\tau\,\mathbf{j} \quad (0 \le \tau \le 1)$$ ◀

When making a change of parameter $t = g(\tau)$ in a vector-valued function $\mathbf{r}(t)$, it will be important to ensure that the new vector-valued function $\mathbf{r}(g(\tau))$ is smooth if $\mathbf{r}(t)$ is smooth. To establish conditions under which this happens, we will need the following version of the chain rule for vector-valued functions. The proof is left as an exercise.

13.3.2 THEOREM (*Chain Rule*). *Let $\mathbf{r}(t)$ be a vector-valued function in 2-space or 3-space that is differentiable with respect to t. If $t = g(\tau)$ is a change of parameter in which g is differentiable with respect to τ, then $\mathbf{r}(g(\tau))$ is differentiable with respect to τ and*

$$\frac{d\mathbf{r}}{d\tau} = \frac{d\mathbf{r}}{dt}\frac{dt}{d\tau} \tag{9}$$

A change of parameter $t = g(\tau)$ in which $\mathbf{r}(g(\tau))$ is smooth if $\mathbf{r}(t)$ is smooth is called a ***smooth change of parameter***. It follows from (9) that $t = g(\tau)$ will be a smooth change of parameter if $dt/d\tau$ is continuous and $dt/d\tau \ne 0$ for all values of τ, since these conditions imply that $d\mathbf{r}/d\tau$ is continuous and nonzero if $d\mathbf{r}/dt$ is continuous and nonzero. Smooth changes of parameter fall into two categories—those for which $dt/d\tau > 0$ for all τ (called ***positive changes of parameter***) and those for which $dt/d\tau < 0$ for all τ (called ***negative changes of parameter***). A positive change of parameter preserves the orientation of a parametric curve, and a negative change of parameter reverses it.

Example 5 In Example 4 the change of parameter in (7) is positive since $dt/d\tau = 2\pi > 0$, and the change of parameter given by (8) is negative since $dt/d\tau = -2\pi < 0$. The positive change of parameter preserved the orientation of the circle, and the negative change of parameter reversed it. ◀

FINDING ARC LENGTH PARAMETRIZATIONS

Next we will consider the problem of finding an arc length parametrization of a vector-valued function that is expressed initially in terms of some other parameter t. The following theorem will provide a general method for doing this.

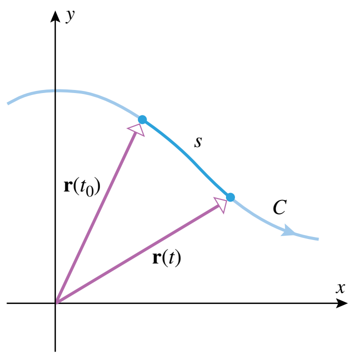


Figure 13.3.6

> **13.3.3 THEOREM.** *Let C be the graph of a smooth vector-valued function $\mathbf{r}(t)$ in 2-space or 3-space, and let $\mathbf{r}(t_0)$ be any point on C. Then the following formula defines a positive change of parameter from t to s, where s is an arc length parameter having $\mathbf{r}(t_0)$ as its reference point* (Figure 13.3.6):
>
> $$s = \int_{t_0}^{t} \left\| \frac{d\mathbf{r}}{du} \right\| du \tag{10}$$

Proof. From (5) with u as the variable of integration instead of t, the integral represents the arc length of that portion of C between $\mathbf{r}(t_0)$ and $\mathbf{r}(t)$ if $t > t_0$ and the negative of that arc length if $t < t_0$. Thus, s is the arc length parameter with $\mathbf{r}(t_0)$ as its reference point and its positive direction in the direction of increasing t. ▮

When needed, Formula (10) can be expressed in component form as

$$s = \int_{t_0}^{t} \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}\, du \qquad \boxed{\text{2-space}} \tag{11}$$

$$s = \int_{t_0}^{t} \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du \qquad \boxed{\text{3-space}} \tag{12}$$

Example 6 Find the arc length parametrization of the circular helix

$$\mathbf{r} = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k} \tag{13}$$

that has reference point $\mathbf{r}(0) = (1, 0, 0)$ and the same orientation as the given helix.

Solution. Replacing t by u in $\mathbf{r}$ for integration purposes and taking $t_0 = 0$ in Formula (10), we obtain

$$\mathbf{r} = \cos u\mathbf{i} + \sin u\mathbf{j} + u\mathbf{k}$$

$$\frac{d\mathbf{r}}{du} = (-\sin u)\mathbf{i} + \cos u\mathbf{j} + \mathbf{k}$$

$$\left\| \frac{d\mathbf{r}}{du} \right\| = \sqrt{(-\sin u)^2 + \cos^2 u + 1} = \sqrt{2}$$

$$s = \int_0^t \left\| \frac{d\mathbf{r}}{du} \right\| du = \int_0^t \sqrt{2}\, du = \sqrt{2}u \Big]_0^t = \sqrt{2}t$$

Thus, $t = s/\sqrt{2}$, so (13) can be reparametrized in terms of s as

$$\mathbf{r} = \cos\left(\frac{s}{\sqrt{2}}\right)\mathbf{i} + \sin\left(\frac{s}{\sqrt{2}}\right)\mathbf{j} + \frac{s}{\sqrt{2}}\mathbf{k}$$

We are guaranteed that this reparametrization preserves the orientation of the helix since Formula (10) produces a positive change of parameter. ◀

Example 7 A bug, starting at the reference point $(1, 0, 0)$ of the helix in Example 6, walks up the helix for a distance of 10 units. What are the bug's final coordinates?

Solution. From Example 6, the arc length parametrization of the helix relative to the reference point $(1, 0, 0)$ is

$$\mathbf{r} = \cos\left(\frac{s}{\sqrt{2}}\right)\mathbf{i} + \sin\left(\frac{s}{\sqrt{2}}\right)\mathbf{j} + \frac{s}{\sqrt{2}}\mathbf{k}$$

or, expressed parametrically,

$$x = \cos\left(\frac{s}{\sqrt{2}}\right), \quad y = \sin\left(\frac{s}{\sqrt{2}}\right), \quad z = \frac{s}{\sqrt{2}}$$

Thus, at $s = 10$ the coordinates are

$$\left(\cos\left(\frac{10}{\sqrt{2}}\right), \sin\left(\frac{10}{\sqrt{2}}\right), \frac{10}{\sqrt{2}}\right) \approx (0.705, 0.709, 7.07) \qquad \blacktriangleleft$$

Example 8 Recall from Formula (9) of Section 12.5 that the equation

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v} \tag{14}$$

is the vector form of the line that passes through the terminal point of $\mathbf{r}_0$ and is parallel to the vector $\mathbf{v}$. Find the arc length parametrization of the line that has reference point $\mathbf{r}_0$ and the same orientation as the given line.

Solution. Replacing t by u in $\mathbf{r}$ for integration purposes and taking $t_0 = 0$ in Formula (10), we obtain

$$\mathbf{r} = \mathbf{r}_0 + u\mathbf{v}$$

$$\frac{d\mathbf{r}}{du} = \mathbf{v} \qquad \boxed{\text{Since } \mathbf{r}_0 \text{ is constant}}$$

$$\left\|\frac{d\mathbf{r}}{du}\right\| = \|\mathbf{v}\|$$

$$s = \int_0^t \left\|\frac{d\mathbf{r}}{du}\right\| du = \int_0^t \|\mathbf{v}\|\, du = \|\mathbf{v}\|u\Big]_0^t = t\|\mathbf{v}\|$$

Thus, $t = s/\|\mathbf{v}\|$, so (14) can be reparametrized in terms of s as

$$\mathbf{r} = \mathbf{r}_0 + s\left(\frac{\mathbf{v}}{\|\mathbf{v}\|}\right) \tag{15}$$

REMARK. Comparing Formulas (14) and (15) shows that the vector equation of the line through the terminal point of $\mathbf{r}_0$ that is parallel to $\mathbf{v}$ can be reparametrized in terms of arc length with reference point $\mathbf{r}_0$ by normalizing $\mathbf{v}$ and then replacing t by s.

Example 9 Find the arc length parametrization of the line

$$x = 2t + 1, \quad y = 3t - 2$$

that has the same orientation as the given line and uses $(1, -2)$ as the reference point.

Solution. The line passes through the point $(1, -2)$ and is parallel to the vector $\mathbf{v} = 2\mathbf{i}+3\mathbf{j}$. To find the arc length parametrization of the line, we need only rewrite the given equations using $\mathbf{v}/\|\mathbf{v}\|$ rather than $\mathbf{v}$ to determine the direction and replace t by s. Since

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{2\mathbf{i} + 3\mathbf{j}}{\sqrt{13}} = \frac{2}{\sqrt{13}}\mathbf{i} + \frac{3}{\sqrt{13}}\mathbf{j}$$

it follows that the parametric equations for the line in terms of s are

$$x = \frac{2}{\sqrt{13}}s + 1, \quad y = \frac{3}{\sqrt{13}}s - 2 \qquad \blacktriangleleft$$

PROPERTIES OF ARC LENGTH PARAMETRIZATIONS

Because arc length parameters for a curve C are intimately related to the geometric characteristics of C, arc length parametrizations have properties that are not enjoyed by other parametrizations. For example, the following theorem shows that if a smooth curve is represented parametrically using an arc length parameter, then the tangent vectors all have length 1.

13.3.4 THEOREM.

(*a*) *If C is the graph of a smooth vector-valued function $\mathbf{r}(t)$ in 2-space or 3-space, where t is a general parameter, and if s is the arc length parameter for C defined by Formula* (10), *then for every value of t the tangent vector has length*

$$\left\|\frac{d\mathbf{r}}{dt}\right\| = \frac{ds}{dt} \tag{16}$$

(*b*) *If C is the graph of a smooth vector-valued function $\mathbf{r}(s)$ in 2-space or 3-space, where s is an arc length parameter, then for every value of s the tangent vector to C has length*

$$\left\|\frac{d\mathbf{r}}{ds}\right\| = 1 \tag{17}$$

(*c*) *If C is the graph of a smooth vector-valued function $\mathbf{r}(t)$ in 2-space or 3-space, and if*

$$\left\|\frac{d\mathbf{r}}{dt}\right\| = 1$$

for every value of t, then for any value of t_0 in the domain of $\mathbf{r}$, *the parameter $s = t - t_0$ is an arc length parameter that has its reference point at the point on C where $t = t_0$.*

Proof (a). This result follows by applying the Fundamental Theorem of Calculus (Theorem 5.6.3) to Formula (10).

Proof (b). Let $t = s$ in part (*a*).

Proof (c). It follows from Theorem 13.3.3 that the formula

$$s = \int_{t_0}^{t} \left\|\frac{d\mathbf{r}}{du}\right\| du$$

defines an arc length parameter for C with reference point $\mathbf{r}(0)$. However, $\|d\mathbf{r}/du\| = 1$ by hypothesis, so we can rewrite the formula for s as

$$s = \int_{t_0}^{t} du = u\Big]_{t_0}^{t} = t - t_0$$

The component forms of Formulas (16) and (17) will be of sufficient interest in later sections that we provide them here for reference:

$$\frac{ds}{dt} = \left\|\frac{d\mathbf{r}}{dt}\right\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \quad \text{2-space} \tag{18}$$

$$\frac{ds}{dt} = \left\|\frac{d\mathbf{r}}{dt}\right\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \quad \text{3-space} \tag{19}$$

$$\left\|\frac{d\mathbf{r}}{ds}\right\| = \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2} = 1 \quad \text{2-space} \tag{20}$$

$$\left\|\frac{d\mathbf{r}}{ds}\right\| = \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2} = 1 \qquad \boxed{\text{3-space}} \tag{21}$$

REMARK. Note that Formulas (18) and (19) do not involve t_0, and hence do not depend on where the reference point for s is chosen. This is to be expected, since changing the reference point shifts s by a constant (the arc length between the two reference points), and this constant drops out on differentiating.

EXERCISE SET 13.3

1. The accompanying figure shows the graph of the *four-cusped hypocycloid* $\mathbf{r}(t) = \cos^3 t\,\mathbf{i} + \sin^3 t\,\mathbf{j}$ $(0 \le t \le 2\pi)$.
(a) Give an informal explanation of why $\mathbf{r}(t)$ is not smooth.
(b) Confirm that $\mathbf{r}(t)$ is not smooth by examining $\mathbf{r}'(t)$.

2. The accompanying figure shows the graph of the vector-valued function $\mathbf{r}(t) = \sin t\,\mathbf{i} + \sin^2 t\,\mathbf{j}$ $(0 \le t \le 2\pi)$. Show that this parametric curve is not smooth, even though it has no corners. Give an informal explanation of what causes the lack of smoothness.

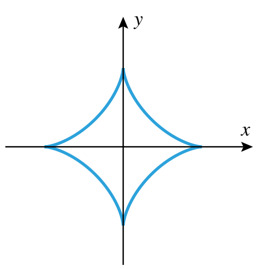


Figure Ex-1

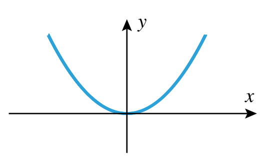


Figure Ex-2

In Exercises 3–6, determine whether $\mathbf{r}(t)$ is a smooth function of the parameter t.

3. $\mathbf{r}(t) = t^3\mathbf{i} + (3t^2 - 2t)\mathbf{j} + t^2\mathbf{k}$

4. $\mathbf{r}(t) = \cos t^2\mathbf{i} + \sin t^2\mathbf{j} + e^{-t}\mathbf{k}$

5. $\mathbf{r}(t) = te^{-t}\mathbf{i} + (t^2 - 2t)\mathbf{j} + \cos \pi t\mathbf{k}$

6. $\mathbf{r}(t) = \sin \pi t\mathbf{i} + (2t - \ln t)\mathbf{j} + (t^2 - t)\mathbf{k}$

In Exercises 7–10, find the arc length of the parametric curve.

7. $x = \cos^3 t$, $y = \sin^3 t$, $z = 2$; $0 \le t \le \pi/2$

8. $x = 3\cos t$, $y = 3\sin t$, $z = 4t$; $0 \le t \le \pi$

9. $x = e^t$, $y = e^{-t}$, $z = \sqrt{2}t$; $0 \le t \le 1$

10. $x = \frac{1}{2}t$, $y = \frac{1}{3}(1-t)^{3/2}$, $z = \frac{1}{3}(1+t)^{3/2}$; $-1 \le t \le 1$

In Exercises 11–14, find the arc length of the graph of $\mathbf{r}(t)$.

11. $\mathbf{r}(t) = t^3\mathbf{i} + t\mathbf{j} + \frac{1}{2}\sqrt{6}t^2\mathbf{k}$; $1 \le t \le 3$

12. $\mathbf{r}(t) = (4 + 3t)\mathbf{i} + (2 - 2t)\mathbf{j} + (5 + t)\mathbf{k}$; $3 \le t \le 4$

13. $\mathbf{r}(t) = 3\cos t\mathbf{i} + 3\sin t\mathbf{j} + t\mathbf{k}$; $0 \le t \le 2\pi$

14. $\mathbf{r}(t) = t^2\mathbf{i} + (\cos t + t\sin t)\mathbf{j} + (\sin t - t\cos t)\mathbf{k}$; $0 \le t \le \pi$

In Exercises 15–18, calculate $d\mathbf{r}/d\tau$ by the chain rule, and then check your result by expressing $\mathbf{r}$ in terms of τ and differentiating.

15. $\mathbf{r} = t\mathbf{i} + t^2\mathbf{j}$; $t = 4\tau + 1$

16. $\mathbf{r} = \langle 3\cos t, 3\sin t\rangle$; $t = \pi\tau$

17. $\mathbf{r} = e^t\mathbf{i} + 4e^{-t}\mathbf{j}$; $t = \tau^2$

18. $\mathbf{r} = \mathbf{i} + 3t^{3/2}\mathbf{j} + t\mathbf{k}$; $t = 1/\tau$

19. (a) Find the arc length parametrization of the line

$$x = t, \quad y = t$$

that has the same orientation as the given line and has reference point $(0, 0)$.

(b) Find the arc length parametrization of the line

$$x = t, \quad y = t, \quad z = t$$

that has the same orientation as the given line and has reference point $(0, 0, 0)$.

20. Find arc length parametrizations of the lines in Exercise 19 that have the stated reference points but are oriented opposite to the given lines.

21. (a) Find the arc length parametrization of the line

$$x = 1 + t, \quad y = 3 - 2t, \quad z = 4 + 2t$$

that has the same direction as the given line and has reference point $(1, 3, 4)$.

(b) Use the parametric equations obtained in part (a) to find the point on the line that is 25 units from the reference point in the direction of increasing parameter.

22. (a) Find the arc length parametrization of the line

$$x = -5 + 3t, \quad y = 2t, \quad z = 5 + t$$

that has the same direction as the given line and has reference point $(-5, 0, 5)$.

(b) Use the parametric equations obtained in part (a) to find the point on the line that is 10 units from the reference point in the direction of increasing parameter.

In Exercises 23–28, find an arc length parametrization of the curve that has the same orientation as the given curve and has $t = 0$ as the reference point.

23. $\mathbf{r}(t) = (3 + \cos t)\mathbf{i} + (2 + \sin t)\mathbf{j};\ \ 0 \le t \le 2\pi$

24. $\mathbf{r}(t) = \cos^3 t\mathbf{i} + \sin^3 t\mathbf{j};\ \ 0 \le t \le \pi/2$

25. $\mathbf{r}(t) = \frac{1}{3}t^3\mathbf{i} + \frac{1}{2}t^2\mathbf{j};\ \ t \ge 0$

26. $\mathbf{r}(t) = (1 + t)^2\mathbf{i} + (1 + t)^3\mathbf{j};\ \ 0 \le t \le 1$

27. $\mathbf{r}(t) = e^t \cos t\mathbf{i} + e^t \sin t\mathbf{j};\ \ 0 \le t \le \pi/2$

28. $\mathbf{r}(t) = \sin e^t\mathbf{i} + \cos e^t\mathbf{j} + \sqrt{3}e^t\mathbf{k};\ \ t \ge 0$

29. Show that the arc length of the circular helix $x = a\cos t$, $y = a\sin t$, $z = ct$ for $0 \le t \le t_0$ is $t_0\sqrt{a^2 + c^2}$.

30. Use the result in Exercise 29 to show the circular helix

$$\mathbf{r} = a\cos t\mathbf{i} + a\sin t\mathbf{j} + ct\mathbf{k}$$

can be expressed as

$$\mathbf{r} = \left(a\cos\frac{s}{w}\right)\mathbf{i} + \left(a\sin\frac{s}{w}\right)\mathbf{j} + \frac{cs}{w}\mathbf{k}$$

where $w = \sqrt{a^2 + c^2}$ and s is an arc length parameter with reference point at $(a, 0, 0)$.

31. Find an arc length parametrization of the cycloid

$$\begin{aligned} x &= at - a\sin t \\ y &= a - a\cos t \end{aligned} \qquad (0 \le t \le 2\pi)$$

with $(0, 0)$ as the reference point.

32. Show that in cylindrical coordinates a curve given by the parametric equations $r = r(t)$, $\theta = \theta(t)$, $z = z(t)$ for $a \le t \le b$ has arc length

$$L = \int_a^b \sqrt{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

[*Hint:* Use the relationships $x = r\cos\theta$, $y = r\sin\theta$.]

33. In each part, use the formula in Exercise 32 to find the arc length of the curve.

(a) $r = e^{2t}, \theta = t, z = e^{2t};\ \ 0 \le t \le \ln 2$

(b) $r = t^2, \theta = \ln t, z = \frac{1}{3}t^3;\ \ 1 \le t \le 2$

34. Show that in spherical coordinates a curve given by the parametric equations $\rho = \rho(t)$, $\theta = \theta(t)$, $\phi = \phi(t)$ for $a \le t \le b$ has arc length

$$L = \int_a^b \sqrt{\left(\frac{d\rho}{dt}\right)^2 + \rho^2\sin^2\phi\left(\frac{d\theta}{dt}\right)^2 + \rho^2\left(\frac{d\phi}{dt}\right)^2}\,dt$$

[*Hint:* $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$.]

35. In each part, use the formula in Exercise 34 to find the arc length of the curve.

(a) $\rho = e^{-t}, \theta = 2t, \phi = \pi/4;\ \ 0 \le t \le 2$

(b) $\rho = 2t, \theta = \ln t, \phi = \pi/6;\ \ 1 \le t \le 5$

36. (a) Show that $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$ $(-1 \le t \le 1)$ is a smooth vector-valued function, but the change of parameter $t = \tau^3$ produces a vector-valued function that is not smooth, yet has the same graph as $\mathbf{r}(t)$.

(b) Examine how the two vector-valued functions are traced and see if you can explain what causes the problem.

37. Find a change of parameter $t = g(\tau)$ for the semicircle

$$\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} \quad (0 \le t \le \pi)$$

such that

(a) the semicircle is traced counterclockwise as τ varies over the interval $[0, 1]$

(b) the semicircle is traced clockwise as τ varies over the interval $[0, 1]$.

38. What change of parameter $t = g(\tau)$ would you make if you wanted to trace the graph of $\mathbf{r}(t)$ $(0 \le t \le 1)$ in the opposite direction with τ varying from 0 to 1?

39. As illustrated in the accompanying figure, copper cable with a diameter of $\frac{1}{2}$ inch is to be wrapped in a circular helix around a cylinder that has a 12-inch diameter. What length of cable (measured along its centerline) will make one complete turn around the cylinder in a distance of 20 inches (between centerlines) measured parallel to the axis of the cylinder?

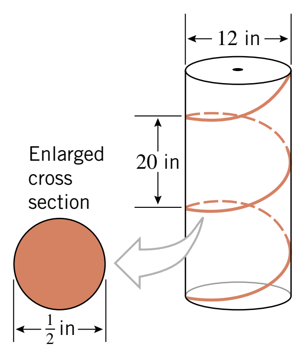


Figure Ex-39

40. Let $x = \cos t$, $y = \sin t$, $z = t^{3/2}$. Find

(a) $\|\mathbf{r}'(t)\|$ (b) $\dfrac{ds}{dt}$ (c) $\displaystyle\int_0^2 \|\mathbf{r}'(t)\|\,dt$.

41. Let $\mathbf{r}(t) = \ln t\mathbf{i} + 2t\mathbf{j} + t^2\mathbf{k}$. Find

(a) $\|\mathbf{r}'(t)\|$ (b) $\dfrac{ds}{dt}$ (c) $\displaystyle\int_1^3 \|\mathbf{r}'(t)\|\,dt$.

42. Prove: If $\mathbf{r}(t)$ is a smoothly parametrized function, then the angles between $\mathbf{r}'(t)$ and the vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are continuous functions of t.

43. Prove the vector form of the chain rule for 2-space (Theorem 13.3.2) by expressing $\mathbf{r}(t)$ in terms of components.

13.4 UNIT TANGENT, NORMAL, AND BINORMAL VECTORS

In this section we will discuss some of the fundamental geometric properties of vector-valued functions. Our work here will have important applications to the study of motion along a curved path in 2-space or 3-space and to the study of the geometric properties of curves and surfaces.

UNIT TANGENT VECTORS

Recall that if C is the graph of a *smooth* vector-valued function $\mathbf{r}(t)$ in 2-space or 3-space, then the vector $\mathbf{r}'(t)$ is nonzero, tangent to C, and points in the direction of increasing parameter. Thus, by normalizing $\mathbf{r}'(t)$ we obtain a unit vector

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \tag{1}$$

that is tangent to C and points in the direction of increasing parameter. We call $\mathbf{T}(t)$ the ***unit tangent vector*** to C at t.

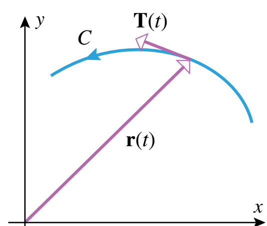


Figure 13.4.1

REMARK. Unless stated otherwise, we will assume that $\mathbf{T}(t)$ is positioned with its initial point at the terminal point of $\mathbf{r}(t)$ as in Figure 13.4.1. This will ensure that $\mathbf{T}(t)$ is actually tangent to the graph of $\mathbf{r}(t)$ and not simply parallel to the tangent line.

Example 1 Find the unit tangent vector to the graph of $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}$ at the point where $t = 2$.

Solution. Since

$$\mathbf{r}'(t) = 2t\mathbf{i} + 3t^2\mathbf{j}$$

we obtain

$$\mathbf{T}(2) = \frac{\mathbf{r}'(2)}{\|\mathbf{r}'(2)\|} = \frac{4\mathbf{i} + 12\mathbf{j}}{\sqrt{160}} = \frac{4\mathbf{i} + 12\mathbf{j}}{4\sqrt{10}} = \frac{1}{\sqrt{10}}\mathbf{i} + \frac{3}{\sqrt{10}}\mathbf{j}$$

The graph of $\mathbf{r}(t)$ and the vector $\mathbf{T}(2)$ are shown in Figure 13.4.2. ◀

UNIT NORMAL VECTORS

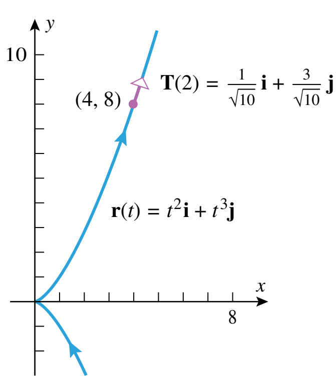


Figure 13.4.2

Recall from Theorem 13.2.9 that if a vector-valued function $\mathbf{r}(t)$ has constant norm, then $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are orthogonal vectors. In particular, $\mathbf{T}(t)$ has constant norm 1, so $\mathbf{T}(t)$ and $\mathbf{T}'(t)$ are orthogonal vectors. This implies that $\mathbf{T}'(t)$ is perpendicular to the tangent line to C at t, so we say that $\mathbf{T}'(t)$ is ***normal*** to C at t. It follows that if $\mathbf{T}'(t) \neq \mathbf{0}$, and if we normalize $\mathbf{T}'(t)$, then we obtain a unit vector

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} \tag{2}$$

that is normal to C and points in the same direction as $\mathbf{T}'(t)$. We call $\mathbf{N}(t)$ the ***principal unit normal vector*** to C at t or more simply the ***unit normal vector***. Observe that the unit normal vector is only defined at points where $\mathbf{T}'(t) \neq \mathbf{0}$. Unless stated otherwise, we will assume that this condition is satisfied. In particular, this *excludes* straight lines.

REMARK. In 2-space there are two unit vectors that are orthogonal to $\mathbf{T}(t)$, and in 3-space there are infinitely many such vectors (Figure 13.4.3). In both cases the principal unit normal is that particular normal that points in the direction of $\mathbf{T}'(t)$. After the next example we will show that for a nonlinear parametric curve in 2-space the principal unit normal is the one that points "inward" toward the concave side of the curve.

Example 2 Find $\mathbf{T}(t)$ and $\mathbf{N}(t)$ for the circular helix

$$x = a\cos t, \quad y = a\sin t, \quad z = ct$$

where $a > 0$.

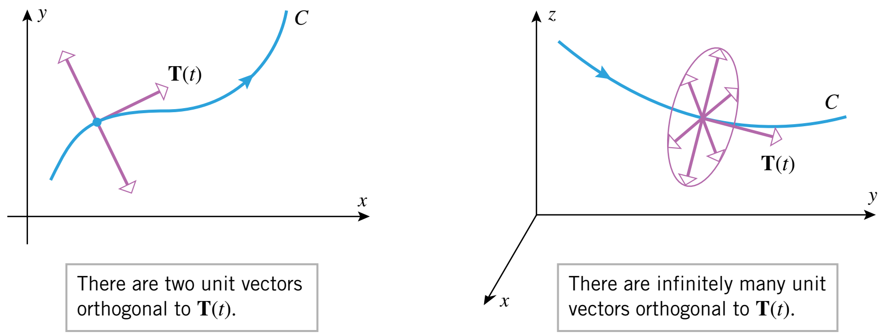


Figure 13.4.3

Solution. The radius vector for the helix is

$$\mathbf{r}(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} + ct\mathbf{k}$$

Thus,

$$\mathbf{r}'(t) = (-a\sin t)\mathbf{i} + a\cos t\,\mathbf{j} + c\mathbf{k}$$

$$\|\mathbf{r}'(t)\| = \sqrt{(-a\sin t)^2 + (a\cos t)^2 + c^2} = \sqrt{a^2+c^2}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = -\frac{a\sin t}{\sqrt{a^2+c^2}}\mathbf{i} + \frac{a\cos t}{\sqrt{a^2+c^2}}\mathbf{j} + \frac{c}{\sqrt{a^2+c^2}}\mathbf{k}$$

$$\mathbf{T}'(t) = -\frac{a\cos t}{\sqrt{a^2+c^2}}\mathbf{i} - \frac{a\sin t}{\sqrt{a^2+c^2}}\mathbf{j}$$

$$\|\mathbf{T}'(t)\| = \sqrt{\left(-\frac{a\cos t}{\sqrt{a^2+c^2}}\right)^2 + \left(-\frac{a\sin t}{\sqrt{a^2+c^2}}\right)^2} = \sqrt{\frac{a^2}{a^2+c^2}} = \frac{a}{\sqrt{a^2+c^2}}$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = (-\cos t)\mathbf{i} - (\sin t)\mathbf{j}$$ ◀

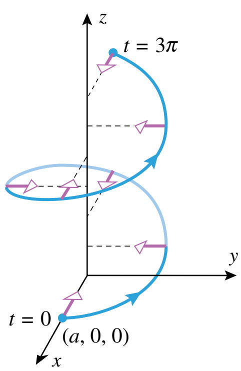


Figure 13.4.4

FOR THE READER. Because the **k** component of $\mathbf{N}(t)$ is zero, this vector lies in a horizontal plane for every value of t. Show that $\mathbf{N}(t)$ actually points directly toward the z-axis for all t (Figure 13.4.4).

INWARD UNIT NORMAL VECTORS IN 2-SPACE

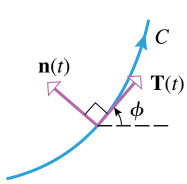


Figure 13.4.5

Our next objective is to show that for a nonlinear parametric curve C in 2-space the unit normal vector always points toward the concave side of C. For this purpose, let $\phi(t)$ be the angle from the positive x-axis to $\mathbf{T}(t)$, and let $\mathbf{n}(t)$ be the unit vector that results when $\mathbf{T}(t)$ is rotated counterclockwise through an angle of $\pi/2$ (Figure 13.4.5). Since $\mathbf{T}(t)$ and $\mathbf{n}(t)$ are unit vectors, it follows from Formula (12) of Section 12.2 that these vectors can be expressed as

$$\mathbf{T}(t) = \cos\phi(t)\mathbf{i} + \sin\phi(t)\mathbf{j} \tag{3}$$

and

$$\mathbf{n}(t) = \cos[\phi(t)+\pi/2]\mathbf{i} + \sin[\phi(t)+\pi/2]\mathbf{j} = -\sin\phi(t)\mathbf{i} + \cos\phi(t)\mathbf{j} \tag{4}$$

Observe that on intervals where $\phi(t)$ is increasing the vector $\mathbf{n}(t)$ points *toward* the concave side of C, and on intervals where $\phi(t)$ is decreasing it points *away* from the concave side (Figure 13.4.6).

Now let us differentiate $\mathbf{T}(t)$ by using Formula (3) and applying the chain rule. This yields

$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{d\phi}\frac{d\phi}{dt} = [(-\sin\phi)\mathbf{i} + (\cos\phi)\mathbf{j}]\frac{d\phi}{dt}$$

and thus from (4)

$$\frac{d\mathbf{T}}{dt} = \mathbf{n}(t)\frac{d\phi}{dt} \tag{5}$$

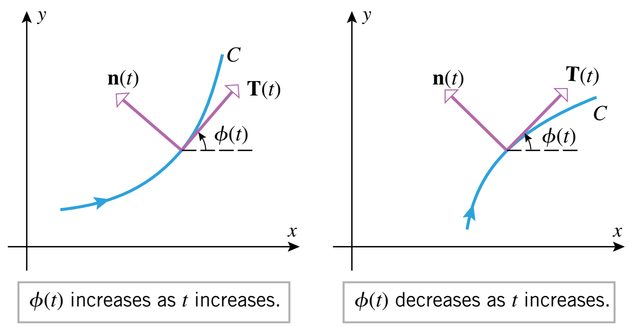


Figure 13.4.6

But $d\phi/dt > 0$ on intervals where $\phi(t)$ is increasing and $d\phi/dt < 0$ on intervals where $\phi(t)$ is decreasing. Thus, it follows from (5) that $d\mathbf{T}/dt$ has the same direction as $\mathbf{n}(t)$ on intervals where $\phi(t)$ is increasing and the opposite direction on intervals where $\phi(t)$ is decreasing. Therefore, $\mathbf{T}'(t) = d\mathbf{T}/dt$ points "inward" toward the concave side of the curve in all cases, and hence so does $\mathbf{N}(t)$. For this reason, $\mathbf{N}(t)$ is also called the ***inward unit normal*** when applied to curves in 2-space.

COMPUTING T AND N FOR CURVES PARAMETRIZED BY ARC LENGTH

In the case where $\mathbf{r}(s)$ is parametrized by arc length, the procedures for computing the unit tangent vector $\mathbf{T}(s)$ and the unit normal vector $\mathbf{N}(s)$ are simpler than in the general case. For example, we showed in Theorem 13.3.4 that if s is an arc length parameter, then $\|\mathbf{r}'(s)\| = 1$. Thus, Formula (1) for the unit tangent vector simplifies to

$$\mathbf{T}(s) = \mathbf{r}'(s) \tag{6}$$

and consequently Formula (2) for the unit normal vector simplifies to

$$\mathbf{N}(s) = \frac{\mathbf{r}''(s)}{\|\mathbf{r}''(s)\|} \tag{7}$$

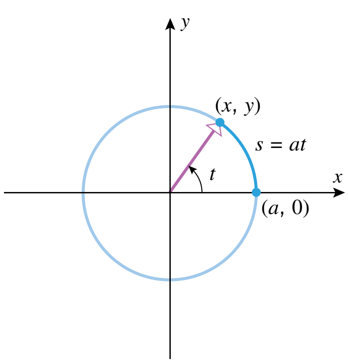


Figure 13.4.7

Example 3 The circle of radius a with counterclockwise orientation and centered at the origin can be represented by the vector-valued function

$$\mathbf{r} = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} \quad (0 \le t \le 2\pi) \tag{8}$$

In this representation we can interpret t as the angle in radian measure from the positive x-axis to the radius vector (Figure 13.4.7). This angle subtends an arc of length $s = at$ on the circle, so we can reparametrize the circle in terms of s by substituting s/a for t in (8). This yields

$$\mathbf{r}(s) = a\cos(s/a)\mathbf{i} + a\sin(s/a)\mathbf{j} \quad (0 \le s \le 2\pi a)$$

To find $\mathbf{T}(s)$ and $\mathbf{N}(s)$ from Formulas (6) and (7), we must compute $\mathbf{r}'(s)$, $\mathbf{r}''(s)$, and $\|\mathbf{r}''(s)\|$. Doing so, we obtain

$$\mathbf{r}'(s) = -\sin(s/a)\mathbf{i} + \cos(s/a)\mathbf{j}$$

$$\mathbf{r}''(s) = -(1/a)\cos(s/a)\mathbf{i} - (1/a)\sin(s/a)\mathbf{j}$$

$$\|\mathbf{r}''(s)\| = \sqrt{(-1/a)^2\cos^2(s/a) + (-1/a)^2\sin^2(s/a)} = 1/a$$

Thus,

$$\mathbf{T}(s) = \mathbf{r}'(s) = -\sin(s/a)\mathbf{i} + \cos(s/a)\mathbf{j}$$

$$\mathbf{N}(s) = \mathbf{r}''(s)/\|\mathbf{r}''(s)\| = -\cos(s/a)\mathbf{i} - \sin(s/a)\mathbf{j}$$

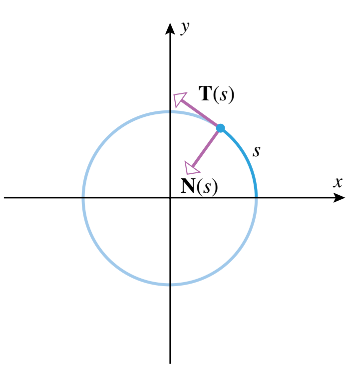


Figure 13.4.8

so $\mathbf{N}(s)$ points toward the center of the circle for all s (Figure 13.4.8). This makes sense geometrically and is also consistent with our earlier observation that in 2-space the unit normal vector is the inward normal. ◀

BINORMAL VECTORS IN 3-SPACE

If C is the graph of a vector-valued function $\mathbf{r}(t)$ in 3-space, then we define the ***binormal vector*** to C at t to be

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) \tag{9}$$

It follows from properties of the cross product that $\mathbf{B}(t)$ is orthogonal to both $\mathbf{T}(t)$ and $\mathbf{N}(t)$ and is oriented relative to $\mathbf{T}(t)$ and $\mathbf{N}(t)$ by the right-hand rule. Moreover, $\mathbf{T}(t) \times \mathbf{N}(t)$ is a unit vector since

$$\|\mathbf{T}(t) \times \mathbf{N}(t)\| = \|\mathbf{T}(t)\|\|\mathbf{N}(t)\| \sin(\pi/2) = 1$$

Thus, $\{\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}(t)\}$ is a set of three mutually orthogonal unit vectors.

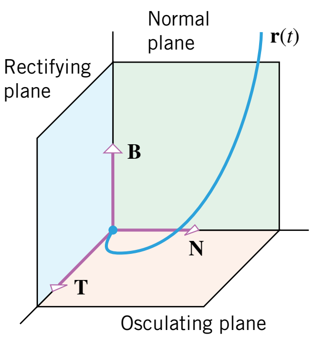
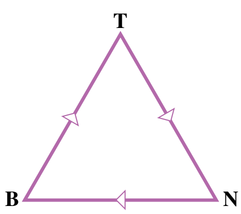


Figure 13.4.9

Just as the vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ determine a right-handed coordinate system in 3-space, so do the vectors $\mathbf{T}(t)$, $\mathbf{N}(t)$, and $\mathbf{B}(t)$. At each point on a smooth parametric curve C in 3-space, these vectors determine three mutually perpendicular planes that pass through the point—the **TB**-plane (called the ***rectifying plane***), the **TN**-plane (called the ***osculating plane***), and the **NB**-plane (called the ***normal plane***) (Figure 13.4.9). Moreover, one can show that a coordinate system determined by $\mathbf{T}(t)$, $\mathbf{N}(t)$, and $\mathbf{B}(t)$ is right-handed in the sense that each of these vectors is related to the other two by the right-hand rule (Figure 13.4.10):



Each vector is the cross product of the other two taken in clockwise order.

Figure 13.4.10

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t), \quad \mathbf{N}(t) = \mathbf{B}(t) \times \mathbf{T}(t), \quad \mathbf{T}(t) = \mathbf{N}(t) \times \mathbf{B}(t) \tag{10}$$

The coordinate system determined by $\mathbf{T}(t)$, $\mathbf{N}(t)$, and $\mathbf{B}(t)$ is called the **TNB-*frame*** or sometimes the ***Frenet frame*** in honor of the French mathematician Jean Frédéric Frenet (1816–1900) who pioneered its application to the study of space curves. Typically, the xyz-coordinate system determined by the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ remains fixed, whereas the **TNB**-frame changes as its origin moves along the curve C (Figure 13.4.11).

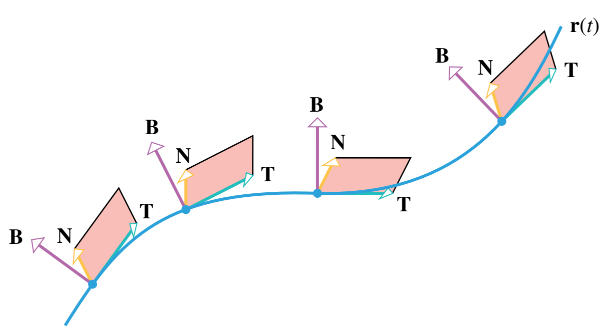


Figure 13.4.11

Formula (9) expresses $\mathbf{B}(t)$ in terms of $\mathbf{T}(t)$ and $\mathbf{N}(t)$. Alternatively, the binormal $\mathbf{B}(t)$ can be expressed directly in terms of $\mathbf{r}(t)$ as

$$\mathbf{B}(t) = \frac{\mathbf{r}'(t) \times \mathbf{r}''(t)}{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|} \tag{11}$$

and in the case where the parameter is arc length it can be expressed in terms of $\mathbf{r}(s)$ as

$$\mathbf{B}(s) = \frac{\mathbf{r}'(s) \times \mathbf{r}''(s)}{\|\mathbf{r}''(s)\|} \tag{12}$$

We omit the proof.

EXERCISE SET 13.4

1. In each part, sketch the unit tangent and normal vectors at the points P, Q, and R, taking into account the orientation of the curve C.

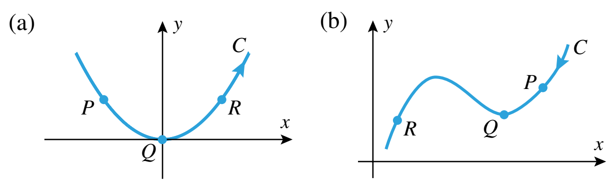


Figure Ex-1

2. Make a rough sketch that shows the ellipse

$$\mathbf{r}(t) = 3\cos t\mathbf{i} + 2\sin t\mathbf{j}$$

for $0 \le t \le 2\pi$ and the unit tangent and normal vectors at the points $t = 0$, $t = \pi/4$, $t = \pi/2$, and $t = \pi$.

In Exercises 3–10, find $\mathbf{T}(t)$ and $\mathbf{N}(t)$ at the given point.

3. $\mathbf{r}(t) = (t^2 - 1)\mathbf{i} + t\mathbf{j};\ \ t = 1$

4. $\mathbf{r}(t) = \frac{1}{2}t^2\mathbf{i} + \frac{1}{3}t^3\mathbf{j};\ \ t = 1$

5. $\mathbf{r}(t) = 5\cos t\mathbf{i} + 5\sin t\mathbf{j};\ \ t = \pi/3$

6. $\mathbf{r}(t) = \ln t\mathbf{i} + t\mathbf{j};\ \ t = e$

7. $\mathbf{r}(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j} + t\mathbf{k};\ \ t = \pi/2$

8. $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{2}t^2\mathbf{j} + \frac{1}{3}t^3\mathbf{k};\ \ t = 0$

9. $x = e^t\cos t,\ y = e^t\sin t,\ z = e^t;\ \ t = 0$

10. $x = \cosh t,\ y = \sinh t,\ z = t;\ \ t = \ln 2$

11. In the remark following Example 8 of Section 13.3, we observed that a line $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$ can be parametrized in terms of an arc length parameter s with reference point $\mathbf{r}_0$ by normalizing $\mathbf{v}$. Use this result to show that the tangent line to the graph of $\mathbf{r}(t)$ at the point t_0 can be expressed as

$$\mathbf{r} = \mathbf{r}(t_0) + s\mathbf{T}(t_0)$$

where s is an arc length parameter with reference point $\mathbf{r}(t_0)$.

12. Use the result in Exercise 11 to show that the tangent line to the parabola

$$x = t, \quad y = t^2$$

at the point $(1, 1)$ can be expressed parametrically as

$$x = 1 + \frac{s}{\sqrt{5}}, \quad y = 1 + \frac{2s}{\sqrt{5}}$$

In Exercises 13 and 14, use the result in Exercise 11 to find parametric equations for the tangent line to the graph of $\mathbf{r}(t)$ at t_0 in terms of an arc length parameter s.

13. $\mathbf{r}(t) = \sin t\mathbf{i} + \cos t\mathbf{j} + \frac{1}{2}t^2\mathbf{k};\ \ t_0 = 0$

14. $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + \sqrt{9 - t^2}\mathbf{k};\ \ t_0 = 1$

In Exercises 15–18, use the formula $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$ to find $\mathbf{B}(t)$, and then check your answer by using Formula (11) to find $\mathbf{B}(t)$ directly from $\mathbf{r}(t)$.

15. $\mathbf{r}(t) = 3\sin t\mathbf{i} + 3\cos t\mathbf{j} + 4t\mathbf{k}$

16. $\mathbf{r}(t) = e^t\sin t\mathbf{i} + e^t\cos t\mathbf{j} + 3\mathbf{k}$

17. $\mathbf{r}(t) = (\sin t - t\cos t)\mathbf{i} + (\cos t + t\sin t)\mathbf{j} + \mathbf{k}$

18. $\mathbf{r}(t) = a\cos t\mathbf{i} + a\sin t\mathbf{j} + ct\mathbf{k} \quad (a \neq 0, c \neq 0)$

In Exercises 19 and 20, find $\mathbf{T}(t)$, $\mathbf{N}(t)$, and $\mathbf{B}(t)$ for the given value of t. Then find equations for the osculating, normal, and rectifying planes at the point that corresponds to that value of t.

19. $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + \mathbf{k};\ \ t = \pi/4$

20. $\mathbf{r}(t) = e^t\mathbf{i} + e^t\cos t\mathbf{j} + e^t\sin t\mathbf{k};\ \ t = 0$

21. (a) Use the formula $\mathbf{N}(t) = \mathbf{B}(t) \times \mathbf{T}(t)$ and Formulas (1) and (11) to show that $\mathbf{N}(t)$ can be expressed in terms of $\mathbf{r}(t)$ as

$$\mathbf{N}(t) = \frac{\mathbf{r}'(t) \times \mathbf{r}''(t)}{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|} \times \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$$

(b) Use properties of cross products to show that the formula in part (a) can be expressed as

$$\mathbf{N}(t) = \frac{(\mathbf{r}'(t) \times \mathbf{r}''(t)) \times \mathbf{r}'(t)}{\|(\mathbf{r}'(t) \times \mathbf{r}''(t)) \times \mathbf{r}'(t)\|}$$

(c) Use the result in part (b) and Exercise 39 of Section 12.4 to show that $\mathbf{N}(t)$ can be expressed directly in terms of $\mathbf{r}(t)$ as

$$\mathbf{N}(t) = \frac{\mathbf{u}(t)}{\|\mathbf{u}(t)\|}$$

where

$$\mathbf{u}(t) = \|\mathbf{r}'(t)\|^2\mathbf{r}''(t) - (\mathbf{r}'(t) \cdot \mathbf{r}''(t))\mathbf{r}'(t)$$

22. Use the result in part (b) of Exercise 21 to find the unit normal vector requested in
(a) Exercise 3 (b) Exercise 7.

In Exercises 23 and 24, use the result in part (c) of Exercise 21 to find $\mathbf{N}(t)$.

23. $\mathbf{r}(t) = \sin t\mathbf{i} + \cos t\mathbf{j} + t\mathbf{k}$ **24.** $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$

13.5 CURVATURE

In this section we will consider the problem of obtaining a numerical measure of how sharply a curve in 2-space or 3-space bends. Our results will have applications in geometry and in the study of motion along a curved path.

DEFINITION OF CURVATURE

Suppose that C is the graph of a smooth vector-valued function in 2-space or 3-space that is parametrized in terms of arc length. Figure 13.5.1 suggests that for a curve in 2-space the "sharpness" of the bend in C is closely related to $d\mathbf{T}/ds$, which is the rate of change of the unit tangent vector $\mathbf{T}$ with respect to s. (Keep in mind that $\mathbf{T}$ has constant length, so only its direction changes.) If C is a straight line (no bend), then the direction of $\mathbf{T}$ remains constant (Figure 13.5.1a); if C bends slightly, then $\mathbf{T}$ undergoes a gradual change of direction (Figure 13.5.1b); and if C bends sharply, then $\mathbf{T}$ undergoes a rapid change of direction (Figure 13.5.1c).

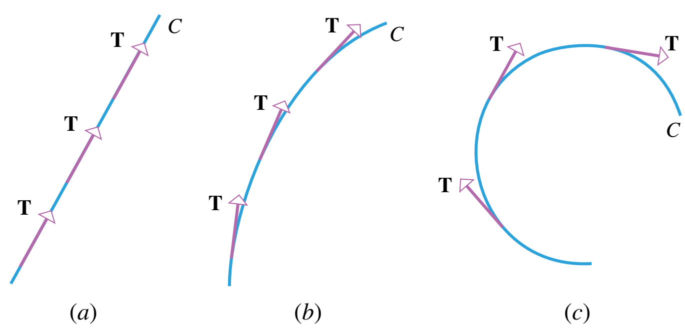


Figure 13.5.1

The situation in 3-space is more complicated because bends in a curve are not limited to a single plane—they can occur in all directions, as illustrated by the complicated tube plot in Figure 13.1.3. To describe the bending characteristics of a curve in 3-space completely, one must take into account $d\mathbf{T}/ds$, $d\mathbf{N}/ds$, and $d\mathbf{B}/ds$. A complete study of this topic would take us too far afield, so we will limit our discussion to $d\mathbf{T}/ds$, which is the most important of these derivatives in applications.

13.5.1 DEFINITION. If C is a smooth curve in 2-space or 3-space that is parametrized by arc length, then the ***curvature*** of C, denoted by $\kappa = \kappa(s)$ (κ = Greek "kappa"), is defined by

$$\kappa(s) = \left\| \frac{d\mathbf{T}}{ds} \right\| = \|\mathbf{r}''(s)\| \tag{1}$$

Observe that $\kappa(s)$ is a real-valued function of s, since it is the *length* of $d\mathbf{T}/ds$ that measures the curvature. In general, the curvature will vary from point to point along a curve; however, the following example shows that the curvature is constant for circles in 2-space, as you might expect.

Example 1 In Example 3 of Section 13.4 we showed that the circle of radius a, centered at the origin, can be parametrized in terms of arc length as

$$\mathbf{r}(s) = a\cos\left(\frac{s}{a}\right)\mathbf{i} + a\sin\left(\frac{s}{a}\right)\mathbf{j} \quad (0 \le s \le 2\pi a)$$

Thus,

$$\mathbf{r}''(s) = -\frac{1}{a}\cos\left(\frac{s}{a}\right)\mathbf{i} - \frac{1}{a}\sin\left(\frac{s}{a}\right)\mathbf{j}$$

and hence from (1)

$$\kappa(s) = \|\mathbf{r}''(s)\| = \sqrt{\left[-\frac{1}{a}\cos\left(\frac{s}{a}\right)\right]^2 + \left[-\frac{1}{a}\sin\left(\frac{s}{a}\right)\right]^2} = \frac{1}{a}$$

so the circle has constant curvature $1/a$. ◀

The next example shows that lines have zero curvature, which is consistent with the fact that they do not bend.

Example 2 Recall from the remark following Example 8 of Section 13.3 that a line in 2-space or 3-space can be parametrized in terms of arc length as

$$\mathbf{r} = \mathbf{r}_0 + s\mathbf{u}$$

where the terminal point of $\mathbf{r}_0$ is a point on the line and $\mathbf{u}$ is a unit vector parallel to the line. Thus,

$$\mathbf{r}'(s) = \frac{d\mathbf{r}}{ds} = \underbrace{\frac{d}{ds}[\mathbf{r}_0 + s\mathbf{u}]}_{\mathbf{r}_0 \text{ is constant}} = \mathbf{0} + \mathbf{u} = \mathbf{u}$$

and hence

$$\mathbf{r}''(s) = \frac{d\mathbf{r}'}{ds} = \underbrace{\frac{d}{ds}[\mathbf{u}]}_{\mathbf{u} \text{ is constant}} = \mathbf{0}$$

Thus,

$$\kappa(s) = \|\mathbf{r}''(s)\| = 0$$ ◀

FORMULAS FOR CURVATURE

Formula (1) is only applicable if the curve is parametrized in terms of arc length. The following theorem provides two formulas for curvature in terms of a general parameter t.

13.5.2 THEOREM. *If $\mathbf{r}(t)$ is a smooth vector-valued function in 2-space or 3-space, then for each value of t at which $\mathbf{T}'(t)$ and $\mathbf{r}''(t)$ exist, the curvature κ can be expressed as*

(a) $$\kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} \qquad (2)$$

(b) $$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} \qquad (3)$$

Proof (a). It follows from Formula (1) and Formulas (16) and (17) of Section 13.3 that

$$\kappa(t) = \left\|\frac{d\mathbf{T}}{ds}\right\| = \left\|\frac{d\mathbf{T}/dt}{ds/dt}\right\| = \left\|\frac{d\mathbf{T}/dt}{\|d\mathbf{r}/dt\|}\right\| = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$$

Proof (b). It follows from Formula (1) of Section 13.4 that

$$\mathbf{r}'(t) = \|\mathbf{r}'(t)\|\mathbf{T}(t) \qquad (4)$$

so

$$\mathbf{r}''(t) = \|\mathbf{r}'(t)\|'\mathbf{T}(t) + \|\mathbf{r}'(t)\|\mathbf{T}'(t) \qquad (5)$$

But from Formula (2) of Section 13.4 and part (*a*) of this theorem we have

$$\mathbf{T}'(t) = \|\mathbf{T}'(t)\|\mathbf{N}(t) \quad \text{and} \quad \|\mathbf{T}'(t)\| = \kappa(t)\|\mathbf{r}'(t)\|$$

so

$$\mathbf{T}'(t) = \kappa(t)\|\mathbf{r}'(t)\|\mathbf{N}(t)$$

Substituting this into (5) yields

$$\mathbf{r}''(t) = \|\mathbf{r}'(t)\|'\mathbf{T}(t) + \kappa(t)\|\mathbf{r}'(t)\|^2\mathbf{N}(t) \tag{6}$$

Thus, from (4) and (6)

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \|\mathbf{r}'(t)\|\|\mathbf{r}'(t)\|'(\mathbf{T}(t) \times \mathbf{T}(t)) + \kappa(t)\|\mathbf{r}'(t)\|^3(\mathbf{T}(t) \times \mathbf{N}(t))$$

But the cross product of a vector with itself is zero, so this equation simplifies to

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \kappa(t)\|\mathbf{r}'(t)\|^3(\mathbf{T}(t) \times \mathbf{N}(t)) = \kappa(t)\|\mathbf{r}'(t)\|^3\mathbf{B}(t)$$

It follows from this equation and the fact that $\mathbf{B}(t)$ is a unit vector that

$$\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = \kappa(t)\|\mathbf{r}'(t)\|^3$$

Formula (3) now follows. ∎

REMARKS. Formula (2) is useful if $\mathbf{T}(t)$ is known or is easy to obtain; however, Formula (3) will usually be easier to apply, since it involves only $\mathbf{r}(t)$ and its derivatives. We also note that cross products were defined only for vectors in 3-space, so to use Formula (3) in 2-space we must first write the 2-space function $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ as the 3-space function $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + 0\mathbf{k}$ with a zero $\mathbf{k}$ component.

Example 3 Find $\kappa(t)$ for the circular helix

$$x = a\cos t, \quad y = a\sin t, \quad z = ct$$

where $a > 0$.

Solution. The radius vector for the helix is

$$\mathbf{r}(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} + ct\mathbf{k}$$

Thus,

$$\mathbf{r}'(t) = (-a\sin t)\mathbf{i} + a\cos t\,\mathbf{j} + c\mathbf{k}$$

$$\mathbf{r}''(t) = (-a\cos t)\mathbf{i} + (-a\sin t)\mathbf{j}$$

so

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin t & a\cos t & c \\ -a\cos t & -a\sin t & 0 \end{vmatrix} = (ac\sin t)\mathbf{i} - (ac\cos t)\mathbf{j} + a^2\mathbf{k}$$

Therefore,

$$\|\mathbf{r}'(t)\| = \sqrt{(-a\sin t)^2 + (a\cos t)^2 + c^2} = \sqrt{a^2 + c^2}$$

and

$$\begin{aligned}\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| &= \sqrt{(ac\sin t)^2 + (-ac\cos t)^2 + a^4} \\ &= \sqrt{a^2c^2 + a^4} = a\sqrt{a^2 + c^2}\end{aligned}$$

so

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{a\sqrt{a^2 + c^2}}{(\sqrt{a^2 + c^2})^3} = \frac{a}{a^2 + c^2}$$

Note that κ does not depend on t, which tells us that the helix has constant curvature. ◀

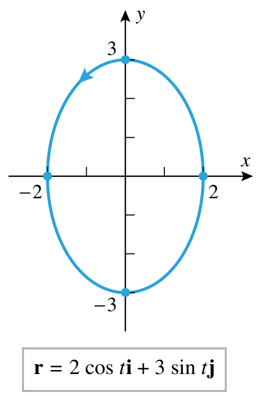


Figure 13.5.2

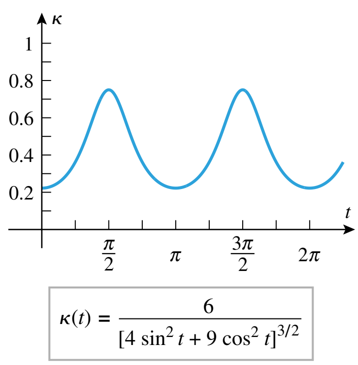


Figure 13.5.3

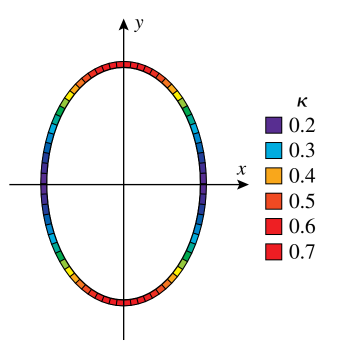


Figure 13.5.4

Example 4 The graph of the vector equation

$$\mathbf{r} = 2\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j} \quad (0 \le t \le 2\pi)$$

is the ellipse in Figure 13.5.2. Find the curvature of the ellipse at the endpoints of the major and minor axes, and use a graphing utility to generate the graph of $\kappa(t)$.

Solution. To apply Formula (3), we must treat the ellipse as a curve in the xy-plane of an xyz-coordinate system by adding a zero $\mathbf{k}$ component and writing its equation as

$$\mathbf{r} = 2\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j} + 0\mathbf{k}$$

It is not essential to write the zero $\mathbf{k}$ component explicitly as long as you assume it to be there when you calculate a cross product. Thus,

$$\mathbf{r}'(t) = (-2\sin t)\mathbf{i} + 3\cos t\,\mathbf{j}$$
$$\mathbf{r}''(t) = (-2\cos t)\mathbf{i} + (-3\sin t)\mathbf{j}$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2\sin t & 3\cos t & 0 \\ -2\cos t & -3\sin t & 0 \end{vmatrix} = [(6\sin^2 t) + (6\cos^2 t)]\mathbf{k} = 6\mathbf{k}$$

Therefore,

$$\|\mathbf{r}'(t)\| = \sqrt{(-2\sin t)^2 + (3\cos t)^2} = \sqrt{4\sin^2 t + 9\cos^2 t}$$
$$\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = 6$$

so

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{6}{[4\sin^2 t + 9\cos^2 t]^{3/2}} \tag{7}$$

The endpoints of the minor axis are $(2, 0)$ and $(-2, 0)$, which correspond to $t = 0$ and $t = \pi$, respectively. Substituting these values in (7) yields the same curvature at both points, namely

$$\kappa = \kappa(0) = \kappa(\pi) = \frac{6}{9^{3/2}} = \frac{6}{27} = \frac{2}{9}$$

The endpoints of the major axis are $(0, 3)$ and $(0, -3)$, which correspond to $t = \pi/2$ and $t = 3\pi/2$, respectively; from (7) the curvature at these points is

$$\kappa = \kappa\left(\frac{\pi}{2}\right) = \kappa\left(\frac{3\pi}{2}\right) = \frac{6}{4^{3/2}} = \frac{3}{4}$$

Observe that the curvature is greater at the ends of the major axis than at the ends of the minor axis, as you might expect. Figure 13.5.3 shows the graph κ versus t. This graph illustrates clearly that the curvature is minimum at $t = 0$ (the right end of the minor axis), increases to a maximum at $t = \pi/2$ (the top of the major axis), decreases to a minimum again at $t = \pi$ (the left end of the minor axis), and continues cyclically in this manner. Figure 13.5.4 provides another way of picturing the curvature. ◀

RADIUS OF CURVATURE

In the last example we found the curvature at the ends of the minor axis to be $\frac{2}{9}$ and the curvature at the ends of the major axis to be $\frac{3}{4}$. To obtain a better understanding of the meaning of these numbers, recall from Example 1 that a circle of radius a has a constant curvature of $1/a$; thus, the curvature of the ellipse at the ends of the minor axis is the same as that of a circle of radius $\frac{9}{2}$, and the curvature at the ends of the major axis is the same as that of a circle of radius $\frac{4}{3}$ (Figure 13.5.5).

In general, if a curve C in 2-space has nonzero curvature κ at a point P, then the circle of radius $\rho = 1/\kappa$ sharing a common tangent with C at P, and centered on the concave side of the curve at P, is called the ***circle of curvature*** or ***osculating circle*** at P (Figure 13.5.6). The osculating circle and the curve C not only touch at P but they have equal curvatures at that point. In this sense, the osculating circle is the circle that best approximates the curve

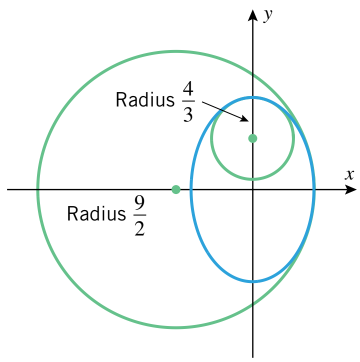


Figure 13.5.5

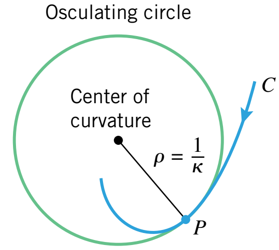


Figure 13.5.6

C near P. The radius ρ of the osculating circle at P is called the ***radius of curvature*** at P, and the center of the circle is called the ***center of curvature*** at P (Figure 13.5.6).

AN INTERPRETATION OF CURVATURE IN 2-SPACE

A useful geometric interpretation of curvature in 2-space can be obtained by considering the angle ϕ measured counterclockwise from the direction of the positive x-axis to the unit tangent vector $\mathbf{T}$ (Figure 13.5.7). By Formula (12) of Section 12.2, we can express $\mathbf{T}$ in terms of ϕ as

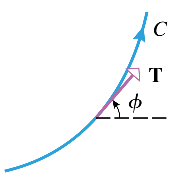


Figure 13.5.7

$$\mathbf{T}(\phi) = \cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}$$

Thus,

$$\frac{d\mathbf{T}}{d\phi} = (-\sin\phi)\mathbf{i} + \cos\phi\,\mathbf{j}$$

$$\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}}{d\phi}\frac{d\phi}{ds}$$

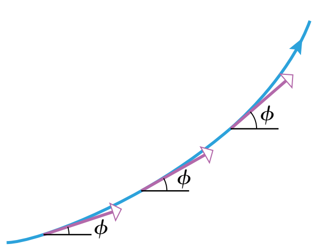
from which we obtain

$$\kappa(s) = \left\|\frac{d\mathbf{T}}{ds}\right\| = \left|\frac{d\phi}{ds}\right|\left\|\frac{d\mathbf{T}}{d\phi}\right\| = \left|\frac{d\phi}{ds}\right|\sqrt{(-\sin\phi)^2 + \cos^2\phi} = \left|\frac{d\phi}{ds}\right|$$

In summary, we have shown that

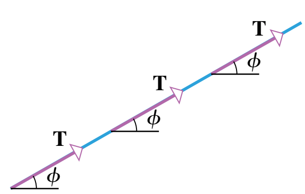
$$\boxed{\kappa(s) = \left|\frac{d\phi}{ds}\right|} \tag{8}$$

which tells us that curvature in 2-space can be interpreted as the magnitude of the rate of change of ϕ with respect to s—the greater the curvature, the more rapidly ϕ changes with s (Figure 13.5.8). In the case of a straight line, the angle ϕ is constant (Figure 13.5.9) and consequently $\kappa(s) = |d\phi/ds| = 0$, which is consistent with the fact that a straight line has zero curvature at every point.



In 2-space, $\kappa(s)$ is the magnitude of the rate of change of ϕ with respect to s.

Figure 13.5.8



ϕ is constant, so the line has zero curvature.

Figure 13.5.9

FORMULA SUMMARY

We conclude this section with a summary of formulas for **T**, **N**, and **B**. These formulas have either been derived in the text or are easily derivable from formulas we have already established.

$$\mathbf{T}(s) = \mathbf{r}'(s) \tag{9}$$

$$\mathbf{N}(s) = \frac{1}{\kappa(s)}\frac{d\mathbf{T}}{ds} = \frac{\mathbf{r}''(s)}{\|\mathbf{r}''(s)\|} = \frac{\mathbf{r}''(s)}{\kappa(s)} \tag{10}$$

$$\mathbf{B}(s) = \frac{\mathbf{r}'(s) \times \mathbf{r}''(s)}{\|\mathbf{r}''(s)\|} = \frac{\mathbf{r}'(s) \times \mathbf{r}''(s)}{\kappa(s)} \tag{11}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \tag{12}$$

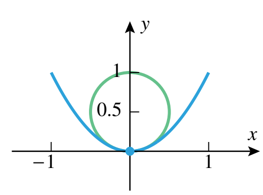
$$\mathbf{B}(t) = \frac{\mathbf{r}'(t) \times \mathbf{r}''(t)}{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|} \tag{13}$$

$$\mathbf{N}(t) = \mathbf{B}(t) \times \mathbf{T}(t) \tag{14}$$

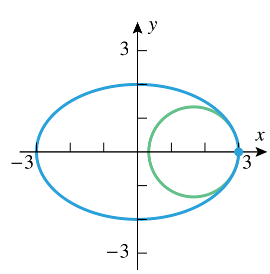
EXERCISE SET 13.5 Graphing Utility C CAS

In Exercises 1 and 2, use the osculating circle shown in the figure to estimate the curvature at the indicated point.

1.



2.



In Exercises 3–10, use Formula (3) to find $\kappa(t)$.

3. $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}$

4. $\mathbf{r}(t) = 4\cos t\,\mathbf{i} + \sin t\,\mathbf{j}$

5. $\mathbf{r}(t) = e^{3t}\mathbf{i} + e^{-t}\mathbf{j}$

6. $x = 1 - t^3, \quad y = t - t^2$

7. $\mathbf{r}(t) = 4\cos t\,\mathbf{i} + 4\sin t\,\mathbf{j} + t\mathbf{k}$

8. $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{2}t^2\mathbf{j} + \frac{1}{3}t^3\mathbf{k}$

9. $x = \cosh t, \quad y = \sinh t, \quad z = t$

10. $\mathbf{r}(t) = \mathbf{i} + t\mathbf{j} + t^2\mathbf{k}$

In Exercises 11–14, find the curvature and the radius of curvature at the stated point.

11. $\mathbf{r}(t) = 3\cos t\,\mathbf{i} + 4\sin t\,\mathbf{j} + t\mathbf{k}; \quad t = \pi/2$

12. $\mathbf{r}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j} + t\mathbf{k}; \quad t = 0$

13. $x = e^t\cos t,\ y = e^t\sin t,\ z = e^t; \quad t = 0$

14. $x = \sin t,\ y = \cos t,\ z = \frac{1}{2}t^2; \quad t = 0$

In Exercises 15 and 16, confirm that s is an arc length parameter by showing that $\|d\mathbf{r}/ds\| = 1$, and then apply Formula (1) to find $\kappa(s)$.

15. $\mathbf{r} = \sin\left(1 + \dfrac{s}{2}\right)\mathbf{i} + \cos\left(1 + \dfrac{s}{2}\right)\mathbf{j} + \sqrt{3}\left(1 + \dfrac{s}{2}\right)\mathbf{k}$

16. $\mathbf{r} = \left(1 - \frac{2}{3}s\right)^{3/2}\mathbf{i} + \left(\frac{2}{3}s\right)^{3/2}\mathbf{j} \quad \left(0 \le s \le \frac{3}{2}\right)$

17. (a) Use Formula (3) to show that in 2-space the curvature of a smooth parametric curve

$$x = x(t), \qquad y = y(t)$$

is

$$\kappa(t) = \frac{|x'y'' - y'x''|}{(x'^2 + y'^2)^{3/2}}$$

where primes denote differentiation with respect to t.

(b) Use the result in part (a) to show that in 2-space the curvature of the plane curve given by $y = f(x)$ is

$$\kappa(x) = \frac{|d^2y/dx^2|}{[1 + (dy/dx)^2]^{3/2}}$$

[*Hint:* Express $y = f(x)$ parametrically with $x = t$ as the parameter.]

18. Use part (b) of Exercise 17 to show that the curvature of $y = f(x)$ can be expressed in terms of the angle of inclination of the tangent line as

$$\kappa(\phi) = \left|\frac{d^2y}{dx^2}\cos^3\phi\right|$$

[*Hint:* $\tan\phi = dy/dx$.]

> In Exercises 19–24, use the result in Exercise 17(b) to find the curvature at the stated point.

19. $y = \sin x;\ x = \pi/2$

20. $y = x^3/3;\ x = 0$

21. $y = 1/x;\ x = 1$

22. $y = e^{-x};\ x = 1$

23. $y = \tan x;\ x = \pi/4$

24. $y^2 - 4x^2 = 9;\ (2, 5)$

> In Exercises 25–30, use the result in Exercise 17(a) to find the curvature at the stated point.

25. $x = t^2,\ y = t^3;\ t = \frac{1}{2}$

26. $x = 4\cos t,\ y = \sin t;\ t = \pi/2$

27. $x = e^{3t},\ y = e^{-t};\ t = 0$

28. $x = 1 - t^3,\ y = t - t^2;\ t = 1$

29. $x = t,\ y = 1/t;\ t = 1$

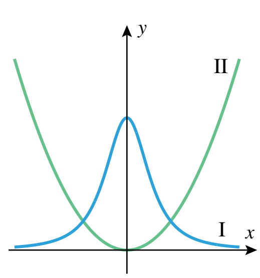
30. $x = 2\sin 2t,\ y = 3\sin t;\ t = \pi/2$

31. In each part, use the formulas in Exercise 17 to help find the radius of curvature at the stated points. Then sketch the graph together with the osculating circles at those points.
(a) $y = \cos x$ at $x = 0$ and $x = \pi$
(b) $x = 2\cos t,\ y = \sin t\ \ (0 \le t \le 2\pi)$ at $t = 0$ and $t = \pi/2$

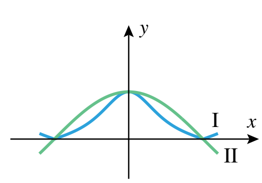
32. Use the formula in Exercise 17(a) to find $\kappa(t)$ for the curve $x = e^{-t}\cos t,\ y = e^{-t}\sin t$. Then sketch the graph of $\kappa(t)$.

> In each part of Exercises 33 and 34, the graphs of $f(x)$ and the associated curvature function $\kappa(x)$ are shown. Determine which is which, and explain your reasoning.

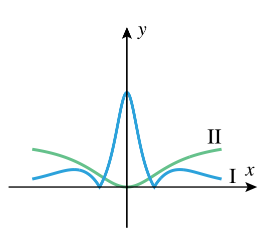
33. (a)



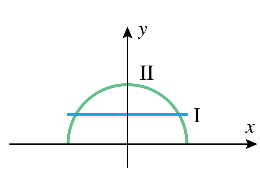
(b)



34. (a)



(b)



> In Exercises 35 and 36, use a graphing utility to generate the graph of $y = f(x)$, and then make a conjecture about the shape of the graph of $y = \kappa(x)$. Check your conjecture by generating the graph of $y = \kappa(x)$.

35. $f(x) = xe^{-x}$ for $0 \le x \le 5$

36. $f(x) = x^3 - x$ for $-1 \le x \le 1$

37. (a) If you have a CAS, read the documentation on calculating higher-order derivatives. Then use the CAS and part (b) of Exercise 17 to find $\kappa(x)$ for $f(x) = x^4 - 2x^2$.
(b) Use the CAS to generate the graphs of $f(x) = x^4 - 2x^2$ and $\kappa(x)$ on the same screen for $-2 \le x \le 2$.
(c) Find the radius of curvature at each relative extremum.
(d) Make a reasonably accurate hand-drawn sketch that shows the graph of $f(x) = x^4 - 2x^2$ and the osculating circles in their correct proportions at the relative extrema.

38. (a) Use a CAS to graph the parametric curve $x = t\cos t$, $y = t\sin t$ for $t \ge 0$.
(b) Make a conjecture about the behavior of $\kappa(t)$ as $t \to +\infty$.
(c) Use the CAS and part (a) of Exercise 17 to find $\kappa(t)$.
(d) Check your conjecture by finding the limit of $\kappa(t)$ as $t \to +\infty$.

39. Use the formula in Exercise 17(a) to show that for a curve in polar coordinates described by $r = f(\theta)$ the curvature is

$$\kappa(\theta) = \frac{\left|r^2 + 2\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}\right|}{\left[r^2 + \left(\dfrac{dr}{d\theta}\right)^2\right]^{3/2}}$$

[*Hint:* Let θ be the parameter and use the relationships $x = r\cos\theta,\ y = r\sin\theta$.]

40. Use the result in Exercise 39 to show that a circle has constant curvature.

> In Exercises 41–44, use the formula of Exercise 39 to find the curvature at the indicated point.

41. $r = 1 + \cos\theta;\ \theta = \pi/2$

42. $r = e^{2\theta};\ \theta = 1$

43. $r = \sin 3\theta;\ \theta = 0$

44. $r = \theta;\ \theta = 1$

45. The accompanying figure is the graph of the radius of curvature versus θ in rectangular coordinates for the cardioid $r = 1 + \cos\theta$. In words, explain what the graph tells you about the cardioid.

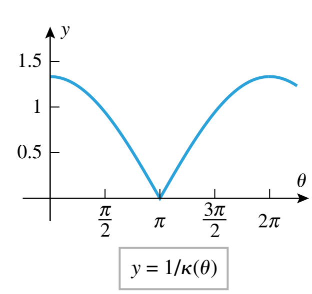


Figure Ex-45

46. Use the formula in Exercise 39 and a graphing utility to generate the graph in Exercise 45.

47. Find the radius of curvature of the parabola $y^2 = 4px$ at $(0, 0)$.

48. At what point(s) does $y = e^x$ have maximum curvature?

49. At what point(s) does $4x^2+9y^2 = 36$ have minimum radius of curvature?

50. Find the value of $x, x > 0$, where $y = x^3$ has maximum curvature.

51. Find the maximum and minimum values of the radius of curvature for the curve $x = \cos t$, $y = \sin t$, $z = \cos t$.

52. Find the minimum value of the radius of curvature for the curve $x = e^t$, $y = e^{-t}$, $z = \sqrt{2}t$.

53. Use the formula in Exercise 39 to show that the curvature of the polar curve $r = e^{a\theta}$ is inversely proportional to r.

54. Use the formula in Exercise 39 and a CAS to show that the curvature of the lemniscate $r = \sqrt{a \cos 2\theta}$ is directly proportional to r.

55. (a) Use the result in Exercise 18 to show that for the parabola $y = x^2$ the curvature $\kappa(\phi)$ at points where the tangent line has an angle of inclination of ϕ is

$$\kappa(\phi) = |2\cos^3\phi|$$

(b) Use the result in part (a) to find the radius of curvature of the parabola at the point on the parabola where the tangent line has slope 1.

(c) Make a sketch with reasonably accurate proportions that shows the osculating circle at the point on the parabola where the tangent line has slope 1.

56. The ***evolute*** of a smooth parametric curve C in 2-space is the curve formed from the centers of curvature of C. The accompanying figure shows the ellipse $x = 3\cos t$, $y = 2\sin t\,(0 \le t \le 2\pi)$ and its evolute graphed together.

(a) Which points on the evolute correspond to $t = 0$ and $t = \pi/2$?

(b) In what direction is the evolute traced as t increases from 0 to 2π?

(c) What does the evolute of a circle look like? Explain your reasoning.

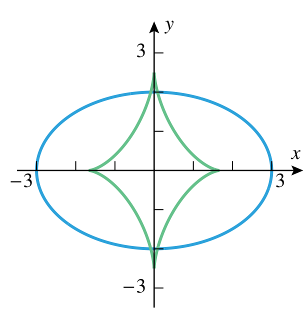


Figure Ex-56

In Exercises 57–60, we will be concerned with the problem of creating a single smooth curve by piecing together two separate smooth curves. If two smooth curves C_1 and C_2 are joined at a point P to form a curve C, then we will say that C_1 and C_2 make a ***smooth transition*** at P if the curvature of C is continuous at P.

57. Show that the transition at $x = 0$ from the horizontal line $y = 0$ for $x \le 0$ to the parabola $y = x^2$ for $x > 0$ is not smooth, whereas the transition to $y = x^3$ for $x > 0$ is smooth.

58. (a) Sketch the graph of the curve defined piecewise by $y = x^2$ for $x < 0$, $y = x^4$ for $x \ge 0$.

(b) Show that for the curve in part (a) the transition at $x = 0$ is not smooth.

59. The accompanying figure shows the arc of a circle of radius r with center at $(0, r)$. Find the value of a so that there is a smooth transition from the circle to the parabola $y = ax^2$ at the point where $x = 0$.

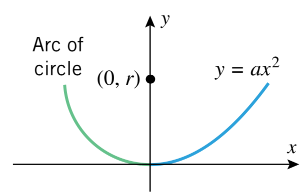


Figure Ex-59

60. Find $a, b,$ and c so that there is a smooth transition at $x = 0$ from the curve $y = e^x$ for $x \le 0$ to the parabola $y = ax^2 + bx + c$ for $x > 0$. [*Hint:* The curvature is continuous at those points where y'' is continuous.]

In Exercises 61–64, we assume that s is an arc length parameter for a smooth vector-valued function $\mathbf{r}(s)$ in 3-space and that $d\mathbf{T}/ds$ and $d\mathbf{N}/ds$ exist at each point on the curve. This implies that $d\mathbf{B}/ds$ exists as well, since $\mathbf{B} = \mathbf{T} \times \mathbf{N}$.

61. Show that

$$\frac{d\mathbf{T}}{ds} = \kappa(s)\mathbf{N}(s)$$

and use this result to obtain the formulas in (10).

62. (a) Show that $d\mathbf{B}/ds$ is perpendicular to $\mathbf{B}(s)$.

(b) Show that $d\mathbf{B}/ds$ is perpendicular to $\mathbf{T}(s)$. [*Hint:* Use the fact that $\mathbf{B}(s)$ is perpendicular to both $\mathbf{T}(s)$ and $\mathbf{N}(s)$, and differentiate $\mathbf{B}\cdot\mathbf{T}$ with respect to s.]

(c) Use the results in parts (a) and (b) to show that $d\mathbf{B}/ds$ is a scalar multiple of $\mathbf{N}(s)$. The *negative* of this scalar is called the ***torsion*** of $\mathbf{r}(s)$ and is denoted by $\tau(s)$. Thus,

$$\frac{d\mathbf{B}}{ds} = -\tau(s)\mathbf{N}(s)$$

(d) Show that $\tau(s) = 0$ for all s if the graph of $\mathbf{r}(s)$ lies in a plane. [*Note:* For reasons that we cannot discuss here, the torsion is related to the "twisting" properties of the curve, and $\tau(s)$ is regarded as a numerical measure of the tendency for the curve to twist out of the osculating plane.]

63. Let κ be the curvature of C and τ the torsion (defined in Exercise 62). By differentiating $\mathbf{N} = \mathbf{B} \times \mathbf{T}$ with respect to s, show that $d\mathbf{N}/ds = -\kappa\mathbf{T} + \tau\mathbf{B}$.

64. The following derivatives, known as the ***Frenet–Serret formulas***, are fundamental in the theory of curves in 3-space:

$$d\mathbf{T}/ds = \kappa\mathbf{N} \qquad \text{[Exercise 61]}$$
$$d\mathbf{N}/ds = -\kappa\mathbf{T} + \tau\mathbf{B} \qquad \text{[Exercise 63]}$$
$$d\mathbf{B}/ds = -\tau\mathbf{N} \qquad \text{[Exercise 62(c)]}$$

Use the first two Frenet–Serret formulas and the fact that $\mathbf{r}'(s) = \mathbf{T}$ if $\mathbf{r} = \mathbf{r}(s)$ to show that

$$\tau = \frac{[\mathbf{r}'(s) \times \mathbf{r}''(s)] \cdot \mathbf{r}'''(s)}{\|\mathbf{r}''(s)\|^2} \quad \text{and} \quad \mathbf{B} = \frac{\mathbf{r}'(s) \times \mathbf{r}''(s)}{\|\mathbf{r}''(s)\|}$$

65. Use the results in Exercise 64 and the results in Exercise 30 of Section 13.3 to show that for the circular helix

$$\mathbf{r} = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} + ct\mathbf{k}$$

with $a > 0$ the torsion and the binormal vector are

$$\tau = \frac{c}{w^2}$$

and

$$\mathbf{B} = \left(\frac{c}{w}\sin\frac{s}{w}\right)\mathbf{i} - \left(\frac{c}{w}\cos\frac{s}{w}\right)\mathbf{j} + \left(\frac{a}{w}\right)\mathbf{k}$$

where $w = \sqrt{a^2 + c^2}$ and s has reference point $(a, 0, 0)$.

66. (a) Use the chain rule and the first two Frenet–Serret formulas in Exercise 64 to show that

$$\mathbf{T}' = \kappa s'\mathbf{N} \quad \text{and} \quad \mathbf{N}' = -\kappa s'\mathbf{T} + \tau s'\mathbf{B}$$

where primes denote differentiation with respect to t.

(b) Show that Formulas (4) and (6) can be written in the form

$$\mathbf{r}'(t) = s'\mathbf{T} \quad \text{and} \quad \mathbf{r}''(t) = s''\mathbf{T} + \kappa(s')^2\mathbf{N}$$

(c) Use the results in parts (a) and (b) to show that

$$\mathbf{r}'''(t) = [s''' - \kappa^2(s')^3]\mathbf{T} + [3\kappa s's'' + \kappa'(s')^2]\mathbf{N} + \kappa\tau(s')^3\mathbf{B}$$

(d) Use the results in parts (b) and (c) to show that

$$\tau(t) = \frac{[\mathbf{r}'(t) \times \mathbf{r}''(t)] \cdot \mathbf{r}'''(t)}{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|^2}$$

In Exercises 67–70, use the formula in Exercise 66(d) to find the torsion $\tau = \tau(t)$.

67. The twisted cubic $\mathbf{r}(t) = 2t\mathbf{i} + t^2\mathbf{j} + \frac{1}{3}t^3\mathbf{k}$

68. The circular helix $\mathbf{r}(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} + ct\mathbf{k}$

69. $\mathbf{r}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j} + \sqrt{2}t\mathbf{k}$

70. $\mathbf{r}(t) = (t - \sin t)\mathbf{i} + (1 - \cos t)\mathbf{j} + t\mathbf{k}$

13.6 MOTION ALONG A CURVE

In earlier sections we considered the motion of a particle along a line. In that situation there are only two directions in which the particle can move—the positive direction or the negative direction. Motion in 2-space or 3-space is more complicated because there are infinitely many directions in which a particle can move. In this section we will show how vectors can be used to analyze motion along curves in 2-space or 3-space.

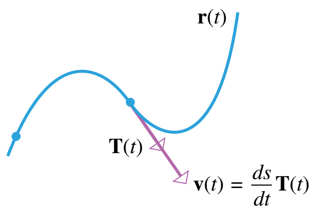
VELOCITY, ACCELERATION, AND SPEED

Let us assume that the motion of a particle in 2-space or 3-space is described by a smooth vector-valued function $\mathbf{r}(t)$ in which the parameter t denotes time; we will call this the ***position function*** or ***trajectory*** of the particle. As the particle moves along its trajectory, its direction of motion and its speed can vary from instant to instant. Thus, before we can undertake any analysis of such motion, we must have clear answers to the following questions:

- What is the direction of motion of the particle at an instant of time?
- What is the speed of the particle at an instant of time?

We will define the direction of motion at time t to be the direction of the unit tangent vector $\mathbf{T}(t)$, and we will define the speed to be ds/dt—the instantaneous rate of change of the arc length traveled by the particle from an arbitrary reference point. Taking this a step further, we will combine the speed and the direction of motion to form the vector

$$\mathbf{v}(t) = \frac{ds}{dt}\mathbf{T}(t) \tag{1}$$



The length of the velocity vector is the speed of the particle, and the direction of the velocity vector is the direction of motion.

Figure 13.6.1

which we call the ***velocity*** of the particle at time t. Thus, at each instant of time the velocity vector $\mathbf{v}(t)$ points in the direction of motion and has a magnitude that is equal to the speed of the particle (Figure 13.6.1).

Recall that for motion along a coordinate line the velocity function is the derivative of the position function. The same is true for motion along a curve, since

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = \frac{ds}{dt}\mathbf{T}(t) = \mathbf{v}(t)$$

For motion along a coordinate line, the acceleration function was defined to be the derivative of the velocity function. The definition is the same for motion along a curve.

13.6.1 DEFINITION. If $\mathbf{r}(t)$ is the position function of a particle moving along a curve in 2-space or 3-space, then the ***instantaneous velocity***, ***instantaneous acceleration***, and ***instantaneous speed*** of the particle at time t are defined by

$$\text{velocity} = \mathbf{v}(t) = \frac{d\mathbf{r}}{dt} \tag{2}$$

$$\text{acceleration} = \mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} \tag{3}$$

$$\text{speed} = \|\mathbf{v}(t)\| = \frac{ds}{dt} \tag{4}$$

As shown in Table 13.6.1, the position, velocity, acceleration, and speed can also be expressed in component form:

Table 13.6.1

	2-SPACE	3-SPACE
POSITION	$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$	$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$
VELOCITY	$\mathbf{v}(t) = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}$	$\mathbf{v}(t) = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$
ACCELERATION	$\mathbf{a}(t) = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j}$	$\mathbf{a}(t) = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k}$
SPEED	$\|\mathbf{v}(t)\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$	$\|\mathbf{v}(t)\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$

Example 1 A particle moves along a circular path in such a way that its x- and y-coordinates at time t are

$$x = 2\cos t, \quad y = 2\sin t$$

(a) Find the instantaneous velocity and speed of the particle at time t.

(b) Sketch the path of the particle, and show the position and velocity vectors at time $t = \pi/4$ with the velocity vector drawn so that its initial point is at the tip of the position vector.

(c) Show that at each instant the acceleration vector is perpendicular to the velocity vector.

Solution (a). At time t, the position vector is

$$\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}$$

so the instantaneous velocity and speed are

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}$$
$$\|\mathbf{v}(t)\| = \sqrt{(-2\sin t)^2 + (2\cos t)^2} = 2$$

Solution (b). The graph of the parametric equations is a circle of radius 2 centered at the origin. At time $t = \pi/4$ the position and velocity vectors of the particles are

$$\mathbf{r}(\pi/4) = 2\cos(\pi/4)\mathbf{i} + 2\sin(\pi/4)\mathbf{j} = \sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}$$
$$\mathbf{v}(\pi/4) = -2\sin(\pi/4)\mathbf{i} + 2\cos(\pi/4)\mathbf{j} = -\sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}$$

These vectors and the circle are shown in Figure 13.6.2.

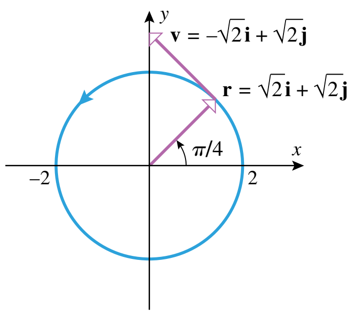


Figure 13.6.2

Solution (c). At time t, the acceleration vector is

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = -2\cos t\,\mathbf{i} - 2\sin t\,\mathbf{j}$$

One way of showing that $\mathbf{v}(t)$ and $\mathbf{a}(t)$ are perpendicular is to show that their dot product is zero (try it). However, it is easier to observe that $\mathbf{a}(t)$ is the negative of $\mathbf{r}(t)$, which implies that $\mathbf{v}(t)$ and $\mathbf{a}(t)$ are perpendicular, since at each point on a circle the radius and tangent line are perpendicular. ◀

Since $\mathbf{v}(t)$ can be obtained by differentiating $\mathbf{r}(t)$, and since $\mathbf{a}(t)$ can be obtained by differentiating $\mathbf{v}(t)$, it follows that $\mathbf{r}(t)$ can be obtained by integrating $\mathbf{v}(t)$, and $\mathbf{v}(t)$ can be obtained by integrating $\mathbf{a}(t)$. However, such integrations do not produce unique functions because constants of integration occur. Typically, initial conditions are required to determine these constants.

Example 2 A particle moves through 3-space in such a way that its velocity is

$$\mathbf{v}(t) = \mathbf{i} + t\mathbf{j} + t^2\mathbf{k}$$

Find the coordinates of the particle at time $t = 1$ given that the particle is at the point $(-1, 2, 4)$ at time $t = 0$.

Solution. Integrating the velocity function to obtain the position function yields

$$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int (\mathbf{i} + t\mathbf{j} + t^2\mathbf{k})\,dt = t\mathbf{i} + \frac{t^2}{2}\mathbf{j} + \frac{t^3}{3}\mathbf{k} + \mathbf{C} \tag{5}$$

where $\mathbf{C}$ is a vector constant of integration. Since the coordinates of the particle at time $t = 0$ are $(-1, 2, 4)$, the position vector at time $t = 0$ is

$$\mathbf{r}(0) = -\mathbf{i} + 2\mathbf{j} + 4\mathbf{k} \tag{6}$$

It follows on substituting $t = 0$ in (5) and equating the result with (6) that

$$\mathbf{C} = -\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$$

Substituting this value of $\mathbf{C}$ in (5) and simplifying yields

$$\mathbf{r}(t) = (t-1)\mathbf{i} + \left(\frac{t^2}{2} + 2\right)\mathbf{j} + \left(\frac{t^3}{3} + 4\right)\mathbf{k}$$

Thus, at time $t = 1$ the position vector of the particle is

$$\mathbf{r}(1) = 0\mathbf{i} + \frac{5}{2}\mathbf{j} + \frac{13}{3}\mathbf{k}$$

so its coordinates at that instant are $\left(0, \frac{5}{2}, \frac{13}{3}\right)$. ◀

DISPLACEMENT AND DISTANCE TRAVELED

If a particle travels along a curve C in 2-space or 3-space, the ***displacement*** of the particle over the time interval $t_1 \leq t \leq t_2$ is commonly denoted by $\Delta\mathbf{r}$ and is defined as

$$\Delta\mathbf{r} = \mathbf{r}(t_2) - \mathbf{r}(t_1) \tag{7}$$

(Figure 13.6.3). The displacement vector, which describes the change in position of the particle during the time interval, can be obtained by integrating the velocity function from t_1 to t_2:

$$\Delta\mathbf{r} = \int_{t_1}^{t_2} \mathbf{v}(t)\,dt = \int_{t_1}^{t_2} \frac{d\mathbf{r}}{dt}\,dt = \mathbf{r}(t)\Big]_{t_1}^{t_2} = \mathbf{r}(t_2) - \mathbf{r}(t_1) \qquad \boxed{\text{Displacement}} \tag{8}$$

It follows from Theorem 13.3.1 that we can find the distance s traveled by a particle over a time interval $t_1 \leq t \leq t_2$ by integrating the speed over that interval, since

$$s = \int_{t_1}^{t_2} \left\| \frac{d\mathbf{r}}{dt} \right\| dt = \int_{t_1}^{t_2} \|\mathbf{v}(t)\|\,dt \qquad \boxed{\text{Distance traveled}} \tag{9}$$

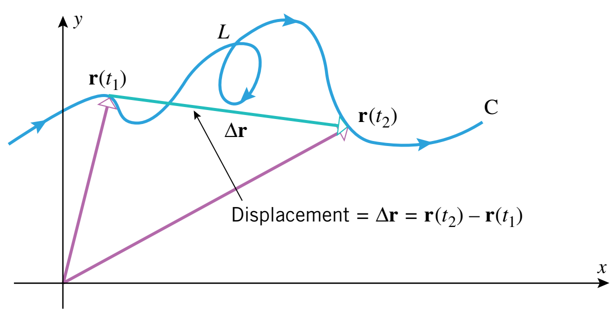


Figure 13.6.3

Example 3 Suppose that a particle moves along a circular helix in 3-space so that its position vector at time t is

$$\mathbf{r}(t) = (4\cos\pi t)\mathbf{i} + (4\sin\pi t)\mathbf{j} + t\mathbf{k}$$

Find the distance traveled and the displacement of the particle during the time interval $1 \leq t \leq 5$.

Solution. We have

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = (-4\pi\sin\pi t)\mathbf{i} + (4\pi\cos\pi t)\mathbf{j} + \mathbf{k}$$

$$\|\mathbf{v}(t)\| = \sqrt{(-4\pi\sin\pi t)^2 + (4\pi\cos\pi t)^2 + 1} = \sqrt{16\pi^2 + 1}$$

Thus, it follows from (9) that the distance traveled by the particle from time $t = 1$ to $t = 5$ is

$$s = \int_1^5 \sqrt{16\pi^2 + 1}\,dt = 4\sqrt{16\pi^2 + 1}$$

Moreover, it follows from (8) that the displacement over the time interval is

$$\begin{aligned}
\Delta\mathbf{r} &= \mathbf{r}(5) - \mathbf{r}(1) \\
&= (4\cos 5\pi\mathbf{i} + 4\sin 5\pi\mathbf{j} + 5\mathbf{k}) - (4\cos\pi\mathbf{i} + 4\sin\pi\mathbf{j} + \mathbf{k}) \\
&= (-4\mathbf{i} + 5\mathbf{k}) - (-4\mathbf{i} + \mathbf{k}) = 4\mathbf{k}
\end{aligned}$$

which tells us that the change in the position of the particle over the time interval was 4 units straight up. ◀

NORMAL AND TANGENTIAL COMPONENTS OF ACCELERATION

You know from your experience as an automobile passenger that if a car speeds up rapidly, then your body is thrown back against the backrest of the seat. You also know that if the car rounds a turn in the road, then your body is thrown toward the outside of the curve—the greater the curvature in the road, the greater the force with which you are thrown. The explanation of these effects can be understood by resolving the velocity and acceleration components of the motion into vector components that are parallel to the unit tangent and unit normal vectors. The following theorem explains how this can be done.

13.6.2 THEOREM. *If a particle moves along a smooth curve C in 2-space or 3-space, then at each point on the curve velocity and acceleration vectors can be written as*

$$\mathbf{v} = \frac{ds}{dt}\mathbf{T} \tag{10}$$

$$\mathbf{a} = \frac{d^2s}{dt^2}\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N} \tag{11}$$

where s is an arc length parameter for the curve, and $\mathbf{T}$, $\mathbf{N}$, and κ denote the unit tangent vector, unit normal vector, and curvature at the point (Figure 13.6.4).

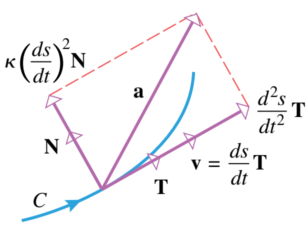


Figure 13.6.4

Proof. Formula (10) is just a restatement of (1). To obtain (11), we differentiate both sides of (10) with respect to t; this yields

$$\begin{aligned}
\mathbf{a} = \frac{d}{dt}\left(\frac{ds}{dt}\mathbf{T}\right) &= \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\frac{d\mathbf{T}}{dt} \\
&= \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\frac{d\mathbf{T}}{ds}\frac{ds}{dt} \\
&= \frac{d^2s}{dt^2}\mathbf{T} + \left(\frac{ds}{dt}\right)^2\frac{d\mathbf{T}}{ds} \\
&= \frac{d^2s}{dt^2}\mathbf{T} + \left(\frac{ds}{dt}\right)^2\kappa\mathbf{N} \qquad \boxed{\text{Formula (10) of Section 13.5}}
\end{aligned}$$

from which (11) follows. ■

The coefficients of $\mathbf{T}$ and $\mathbf{N}$ in (11) are commonly denoted by

$$a_T = \frac{d^2s}{dt^2} \qquad a_N = \kappa\left(\frac{ds}{dt}\right)^2 \tag{12–13}$$

in which case Formula (11) is expressed as

$$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N} \tag{14}$$

In this formula the scalars a_T and a_N are called the ***tangential scalar component of acceleration*** and the ***normal scalar component of acceleration***, and the vectors $a_T\mathbf{T}$ and $a_N\mathbf{N}$ are called the ***tangential vector component of acceleration*** and the ***normal vector component of acceleration***.

The scalar components of acceleration explain the effect that you experience when a car speeds up rapidly or rounds a turn. The rapid increase in speed produces a large value for d^2s/dt^2, which results in a large tangential scalar component of acceleration; and by Newton's second law this corresponds to a large tangential force on the car in the direction

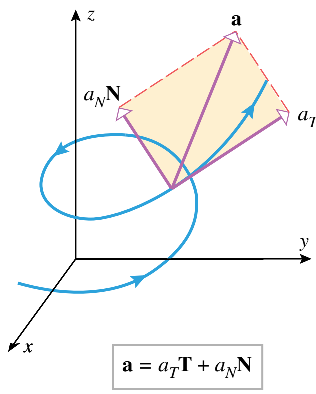


Figure 13.6.5

of motion. To understand the effect of rounding a turn, observe that the normal scalar component of acceleration has the curvature κ and the square of speed ds/dt as factors. Thus, sharp turns or turns taken at high speed both produce large normal forces on the car.

REMARK. Formula (14) applies to motion in both 2-space and 3-space. What is interesting is that the 3-space formula does not involve the binormal vector **B**, so the acceleration vector always lies in the plane of **T** and **N** (the osculating plane), even for the most twisting paths of motion (Figure 13.6.5).

Although Formulas (12) and (13) provide useful insight into the behavior of particles moving along curved paths, they are not always the best formulas for computations. The following theorem provides some more useful formulas that relate a_T, a_N, and κ to the velocity **v** and acceleration **a**.

13.6.3 THEOREM. *If a particle moves along a smooth curve C in 2-space or 3-space, then at each point on the curve the velocity* **v** *and the acceleration* **a** *are related to* a_T, a_N, *and* κ *by the formulas*

$$a_T = \frac{\mathbf{v}\cdot\mathbf{a}}{\|\mathbf{v}\|} \qquad a_N = \frac{\|\mathbf{v}\times\mathbf{a}\|}{\|\mathbf{v}\|} \qquad \kappa = \frac{\|\mathbf{v}\times\mathbf{a}\|}{\|\mathbf{v}\|^3} \tag{15–17}$$

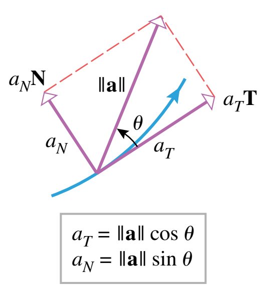


Figure 13.6.6

Proof. As illustrated in Figure 13.6.6, let θ be the angle between the vector **a** and the vector a_T**T**. Thus,

$$a_T = \|\mathbf{a}\|\cos\theta \quad \text{and} \quad a_N = \|\mathbf{a}\|\sin\theta$$

from which we obtain

$$a_T = \|\mathbf{a}\|\cos\theta = \frac{\|\mathbf{v}\|\,\|\mathbf{a}\|\cos\theta}{\|\mathbf{v}\|} = \frac{\mathbf{v}\cdot\mathbf{a}}{\|\mathbf{v}\|}$$

$$a_N = \|\mathbf{a}\|\sin\theta = \frac{\|\mathbf{v}\|\,\|\mathbf{a}\|\sin\theta}{\|\mathbf{v}\|} = \frac{\|\mathbf{v}\times\mathbf{a}\|}{\|\mathbf{v}\|}$$

$$\kappa = \frac{a_N}{(ds/dt)^2} = \frac{a_N}{\|\mathbf{v}\|^2} = \frac{1}{\|\mathbf{v}\|^2}\,\frac{\|\mathbf{v}\times\mathbf{a}\|}{\|\mathbf{v}\|} = \frac{\|\mathbf{v}\times\mathbf{a}\|}{\|\mathbf{v}\|^3}$$

REMARK. Theorem 13.6.3 applies to motion in 2-space and 3-space, but for motion in 2-space you will have to add a zero **k** component to **v** and **a** to calculate the cross product. Also, recall that for nonlinear smooth curves in 2-space the unit normal vector **N** is the inward normal; that is, it points toward the concave side of the curve. Thus, the same is true for a_N**N**, since a_N is a nonnegative scalar.

Example 4 Suppose that a particle moves through 3-space so that its position vector at time t is

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$$

(The path is the twisted cubic shown in Figure 13.1.5.)

(a) Find the scalar tangential and normal components of acceleration at time t.

(b) Find the scalar tangential and normal components of acceleration at time $t = 1$.

(c) Find the vector tangential and normal components of acceleration at time $t = 1$.

(d) Find the curvature of the path at the point where the particle is located at time $t = 1$.

Solution (a). We have

$$\mathbf{v}(t) = \mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}$$
$$\mathbf{a}(t) = \mathbf{v}'(t) = 2\mathbf{j} + 6t\mathbf{k}$$
$$\|\mathbf{v}(t)\| = \sqrt{1 + 4t^2 + 9t^4}$$
$$\mathbf{v}(t) \cdot \mathbf{a}(t) = 4t + 18t^3$$
$$\mathbf{v}(t) \times \mathbf{a}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = 6t^2\mathbf{i} - 6t\mathbf{j} + 2\mathbf{k}$$

Thus, from (15) and (16)

$$a_T = \frac{\mathbf{v} \cdot \mathbf{a}}{\|\mathbf{v}\|} = \frac{4t + 18t^3}{\sqrt{1 + 4t^2 + 9t^4}}$$
$$a_N = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|} = \frac{\sqrt{36t^4 + 36t^2 + 4}}{\sqrt{1 + 4t^2 + 9t^4}} = 2\sqrt{\frac{9t^4 + 9t^2 + 1}{9t^4 + 4t^2 + 1}}$$

Solution (b). At time $t = 1$, the components a_T and a_N in part (a) are

$$a_T = \frac{22}{\sqrt{14}} \approx 5.88 \quad \text{and} \quad a_N = 2\sqrt{\frac{19}{14}} \approx 2.33$$

Solution (c). Since $\mathbf{T}$ and $\mathbf{v}$ have the same direction, $\mathbf{T}$ can be obtained by normalizing $\mathbf{v}$, that is,

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|}$$

At time $t = 1$ we have

$$\mathbf{T}(1) = \frac{\mathbf{v}(1)}{\|\mathbf{v}(1)\|} = \frac{\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}}{\|\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}\|} = \frac{1}{\sqrt{14}}(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$$

From this and part (b) we obtain the vector tangential component of acceleration:

$$a_T(1)\mathbf{T}(1) = \frac{22}{\sqrt{14}}\mathbf{T}(1) = \frac{11}{7}(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) = \frac{11}{7}\mathbf{i} + \frac{22}{7}\mathbf{j} + \frac{33}{7}\mathbf{k}$$

To find the normal vector component of acceleration, we rewrite $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$ as

$$a_N\mathbf{N} = \mathbf{a} - a_T\mathbf{T}$$

Thus, at time $t = 1$ the normal vector component of acceleration is

$$\begin{aligned} a_N(1)\mathbf{N}(1) &= \mathbf{a}(1) - a_T(1)\mathbf{T}(1) \\ &= (2\mathbf{j} + 6\mathbf{k}) - \left(\frac{11}{7}\mathbf{i} + \frac{22}{7}\mathbf{j} + \frac{33}{7}\mathbf{k}\right) \\ &= -\frac{11}{7}\mathbf{i} - \frac{8}{7}\mathbf{j} + \frac{9}{7}\mathbf{k} \end{aligned}$$

Solution (d). We will apply Formula (17) with $t = 1$. From part (a)

$$\|\mathbf{v}(1)\| = \sqrt{14} \quad \text{and} \quad \mathbf{v}(1) \times \mathbf{a}(1) = 6\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$$

Thus, at time $t = 1$

$$\kappa = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|^3} = \frac{\sqrt{76}}{(\sqrt{14})^3} = \frac{1}{14}\sqrt{\frac{38}{7}} \approx 0.17$$

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FOR THE READER. It follows from Figure 13.6.6 and the Theorem of Pythagoras that a_N can be expressed in terms of $\|\mathbf{v}\|$ and a_T as

$$a_N = \sqrt{\|\mathbf{a}\|^2 - a_T^2} \tag{18}$$

Confirm that this is so in Example 4.

A MODEL OF PROJECTILE MOTION

Earlier in this text we examined various problems concerned with objects moving *vertically* in the Earth's gravitational field (see Free-Fall Model 4.4.4, Example 4 of Section 5.7, and the subsection of Section 9.1 entitled A Model of Free-Fall Motion Retarded by Air Resistance). Now we will consider the motion of a projectile launched along a *curved* path in the Earth's gravitational field. For this purpose we will need the vector version of Newton's Second Law of Motion (9.1.1)

$$\mathbf{F} = m\mathbf{a} \tag{19}$$

and we will need to make three modeling assumptions:

- The mass m of the object is constant.
- The only force acting on the object after it is launched is the force of the Earth's gravity. (Thus, air resistance and the gravitational effect of other planets and celestial objects are ignored.)
- The object remains sufficiently close to the Earth that we can assume the force of gravity to be constant.

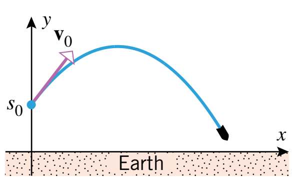

Figure 13.6.7

Let us assume that at time $t = 0$ an object of mass m is launched from a height of s_0 above the Earth with an initial velocity vector of $\mathbf{v}_0$. Furthermore, let us introduce an xy-coordinate system as shown in Figure 13.6.7. In this coordinate system the positive y-direction is up, the origin is at the surface of the Earth, and the initial coordinate of the object is $(0, s_0)$. Our objective is to use basic principles of physics to derive the velocity function $\mathbf{v}(t)$ and the position function $\mathbf{r}(t)$ from the acceleration function $\mathbf{a}(t)$ of the object. Our starting point is the physical observation that the downward force $\mathbf{F}$ of the Earth's gravity on an object of mass m is

$$\mathbf{F} = -mg\mathbf{j}$$

where g is the acceleration due to gravity (see 9.4.3). It follows from this fact and Newton's second law (19) that

$$m\mathbf{a} = -mg\mathbf{j}$$

or on canceling m from both sides

$$\mathbf{a} = -g\mathbf{j} \tag{20}$$

Observe that this acceleration function does not involve t and hence is constant. We can now obtain the velocity function $\mathbf{v}(t)$ by integrating this acceleration function and using the initial condition $\mathbf{v}(0) = \mathbf{v}_0$ to find the constant of integration. Integrating (20) with respect to t and keeping in mind that $-g\mathbf{j}$ is constant yields

$$\mathbf{v}(t) = \int -g\mathbf{j}\,dt = -gt\mathbf{j} + \mathbf{c}_1$$

where $\mathbf{c}_1$ is a vector constant of integration. Substituting $t = 0$ in this equation and using the initial condition $\mathbf{v}(0) = \mathbf{v}_0$ yields

$$\mathbf{v}_0 = \mathbf{c}_1$$

Thus, the velocity function of the object is

$$\mathbf{v}(t) = -gt\mathbf{j} + \mathbf{v}_0 \tag{21}$$

To obtain the position function $\mathbf{r}(t)$ of the object, we will integrate the velocity function and use the known initial position of the object to find the constant of integration. For this purpose observe that the object has coordinates $(0, s_0)$ at time $t = 0$, so the position vector at that time is

$$\mathbf{r}(0) = 0\mathbf{i} + s_0\mathbf{j} = s_0\mathbf{j} \tag{22}$$

This is the initial condition that we will need to find the constant of integration. Integrating (21) with respect to t yields

$$\mathbf{r}(t) = \int (-gt\mathbf{j} + \mathbf{v}_0)\, dt = -\tfrac{1}{2}gt^2\mathbf{j} + t\mathbf{v}_0 + \mathbf{c}_2 \tag{23}$$

where $\mathbf{c}_2$ is another vector constant of integration. Substituting $t = 0$ in (23) and using initial condition (22) yields

$$s_0\mathbf{j} = \mathbf{c}_2$$

so that (23) can be written as

$$\mathbf{r}(t) = \left(-\tfrac{1}{2}gt^2 + s_0\right)\mathbf{j} + t\mathbf{v}_0 \tag{24}$$

This formula expresses the position function of the object in terms of its known initial position and velocity.

REMARK. Observe that the mass of the object does not enter into the final formulas for velocity and position. Physically, this means that the mass has no influence on the trajectory or the velocity of the object—these are completely determined by the initial position and velocity. This explains the famous observation of Galileo that two objects of different mass, released from the same height, will reach the ground at the same time if air resistance is neglected.

PARAMETRIC EQUATIONS OF PROJECTILE MOTION

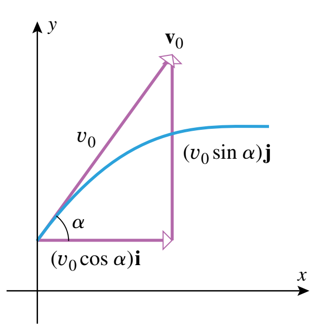

Figure 13.6.8

Formulas (21) and (24) can be used to obtain parametric equations for the position and velocity in terms of the initial speed of the object and the angle that the initial velocity vector makes with the positive x-axis. For this purpose, let $v_0 = \|\mathbf{v}_0\|$ be the initial speed, let α be the angle that the initial velocity vector $\mathbf{v}_0$ makes with the positive x-axis, let v_x and v_y be the horizontal and vertical scalar components of $\mathbf{v}(t)$ at time t, and let x and y be the horizontal and vertical components of $\mathbf{r}(t)$ at time t. As illustrated in Figure 13.6.8, the initial velocity vector can be expressed as

$$\mathbf{v}_0 = (v_0\cos\alpha)\mathbf{i} + (v_0\sin\alpha)\mathbf{j} \tag{25}$$

Substituting this expression in (24) and combining like components yields (verify)

$$\mathbf{r}(t) = (v_0\cos\alpha)t\mathbf{i} + \left(s_0 + (v_0\sin\alpha)t - \tfrac{1}{2}gt^2\right)\mathbf{j} \tag{26}$$

which is equivalent to the parametric equations

$$x = (v_0\cos\alpha)t, \qquad y = s_0 + (v_0\sin\alpha)t - \tfrac{1}{2}gt^2 \tag{27}$$

Similarly, substituting (25) in (21) and combining like components yields

$$\mathbf{v}(t) = (v_0\cos\alpha)\mathbf{i} + (v_0\sin\alpha - gt)\mathbf{j}$$

which is equivalent to the parametric equations

$$v_x = v_0\cos\alpha, \qquad v_y = v_0\sin\alpha - gt \tag{28}$$

The parameter t can be eliminated in (27) by solving the first equation for t and substituting in the second equation. We leave it for you to show that this yields

$$y = s_0 + (\tan\alpha)x - \left(\frac{g}{2v_0^2\cos^2\alpha}\right)x^2 \tag{29}$$

which is the equation of a parabola, since the right side is a quadratic polynomial in x. Thus, we have shown that the trajectory of the projectile is a parabolic arc.

Example 5 A shell, fired from a cannon, has a muzzle speed (the speed as it leaves the barrel) of 800 ft/s. The barrel makes an angle of 45° with the horizontal and, for simplicity, the barrel opening is assumed to be at ground level.

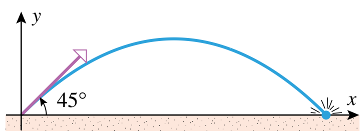

Figure 13.6.9

(a) Find parametric equations for the shell's trajectory relative to the coordinate system in Figure 13.6.9.

(b) How high does the shell rise?

(c) How far does the shell travel horizontally?

(d) What is the speed of the shell at its point of impact with the ground?

Solution (a). From (27) with $v_0 = 800$ ft/s, $\alpha = 45°$, $s_0 = 0$ ft (since the shell starts at ground level), and $g = 32$ ft/s^2, we obtain the parametric equations

$$x = (800\cos 45°)t, \quad y = (800\sin 45°)t - 16t^2 \qquad (t \geq 0)$$

which simplify to

$$x = 400\sqrt{2}t, \quad y = 400\sqrt{2}t - 16t^2 \qquad (t \geq 0) \tag{30}$$

Solution (b). The maximum height of the shell is the maximum value of y in (30), which occurs when $dy/dt = 0$, that is, when

$$400\sqrt{2} - 32t = 0 \quad \text{or} \quad t = \frac{25\sqrt{2}}{2}$$

Substituting this value of t in (30) yields

$$y = 5000 \text{ ft}$$

as the maximum height of the shell.

Solution (c). The shell will hit the ground when $y = 0$. From (30), this occurs when

$$400\sqrt{2}t - 16t^2 = 0 \quad \text{or} \quad t(400\sqrt{2} - 16t) = 0$$

The solution $t = 0$ corresponds to the initial position of the shell and the solution $t = 25\sqrt{2}$ to the time of impact. Substituting the latter value in the equation for x in (30) yields

$$x = 20{,}000 \text{ ft}$$

as the horizontal distance traveled by the shell.

Solution (d). From (30), the position function of the shell is

$$\mathbf{r}(t) = 400\sqrt{2}t\mathbf{i} + (400\sqrt{2}t - 16t^2)\mathbf{j}$$

so that the velocity function is

$$\mathbf{v}(t) = \mathbf{r}'(t) = 400\sqrt{2}\mathbf{i} + (400\sqrt{2} - 32t)\mathbf{j}$$

From part (c), impact occurs when $t = 25\sqrt{2}$, so the velocity vector at this point is

$$\mathbf{v}(25\sqrt{2}) = 400\sqrt{2}\mathbf{i} + [400\sqrt{2} - 32(25\sqrt{2})]\mathbf{j} = 400\sqrt{2}\mathbf{i} - 400\sqrt{2}\mathbf{j}$$

Thus, the speed at impact is

$$\|\mathbf{v}(25\sqrt{2})\| = \sqrt{(400\sqrt{2})^2 + (-400\sqrt{2})^2} = 800 \text{ ft/s}$$

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EXERCISE SET 13.6 Graphing Utility C CAS

In Exercises 1–4, $\mathbf{r}(t)$ is the position vector of a particle moving in the plane. Find the velocity, acceleration, and speed at an arbitrary time t. Then sketch the path of the particle together with the velocity and acceleration vectors at the indicated time t.

1. $\mathbf{r}(t) = 3\cos t\mathbf{i} + 3\sin t\mathbf{j};\ t = \pi/3$
2. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j};\ t = 2$
3. $\mathbf{r}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j};\ t = 0$
4. $\mathbf{r}(t) = (2+4t)\mathbf{i} + (1-t)\mathbf{j};\ t = 1$

In Exercises 5–8, find the velocity, speed, and acceleration at the given time t of a particle moving along the given curve.

5. $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{2}t^2\mathbf{j} + \frac{1}{3}t^3\mathbf{k};\ t = 1$
6. $x = 1+3t,\ y = 2-4t,\ z = 7+t;\ t = 2$
7. $x = 2\cos t,\ y = 2\sin t,\ z = t;\ t = \pi/4$
8. $\mathbf{r}(t) = e^t\sin t\mathbf{i} + e^t\cos t\mathbf{j} + t\mathbf{k};\ t = \pi/2$

9. As illustrated in the accompanying figure, suppose that the equations of motion of a particle moving along an elliptic path are $x = a\cos\omega t$, $y = b\sin\omega t$.
(a) Show that the acceleration is directed toward the origin.
(b) Show that the magnitude of the acceleration is proportional to the distance from the particle to the origin.

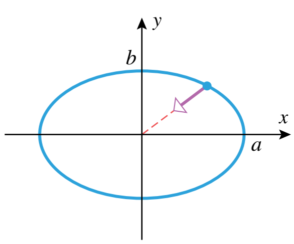


Figure Ex-9

10. Suppose that a particle vibrates in such a way that its position function is $\mathbf{r}(t) = 16\sin\pi t\mathbf{i} + 4\cos 2\pi t\mathbf{j}$, where distance is in millimeters and t is in seconds.
(a) Find the velocity and acceleration at time $t = 1$ s.
(b) Show that the particle moves along a parabolic curve.
(c) Show that the particle moves back and forth along the curve.

11. Suppose that the position vector of a particle moving in the plane is $\mathbf{r} = 12\sqrt{t}\mathbf{i} + t^{3/2}\mathbf{j}$, $t > 0$. Find the minimum speed of the particle and its location when it has this speed.

12. Suppose that the motion of a particle is described by the position vector $\mathbf{r} = (t - t^2)\mathbf{i} - t^2\mathbf{j}$. Find the minimum speed of the particle and its location when it has this speed.

13. Suppose that the position function of a particle moving in 2-space is $\mathbf{r} = \sin 3t\mathbf{i} - 2\cos 3t\mathbf{j}$.
(a) Use a graphing utility to graph the speed of the particle versus time from $t = 0$ to $t = 2\pi/3$.
(b) What are the maximum and minimum speeds of the particle?
(c) Use the graph to estimate the time at which the maximum speed first occurs.
(d) Find the exact time at which the maximum speed first occurs.

14. Suppose that the position function of a particle moving in 3-space is $\mathbf{r} = 3\cos 2t\mathbf{i} + \sin 2t\mathbf{j} + 4t\mathbf{k}$.
(a) Use a graphing utility to graph the speed of the particle versus time from $t = 0$ to $t = \pi$.
(b) Use the graph to estimate the maximum and minimum speeds of the particle.
(c) Use the graph to estimate the time at which the maximum speed first occurs.
(d) Find the exact values of the maximum and minimum speeds and the exact time at which the maximum speed first occurs.

In Exercises 15–18, use the given information to find the position and velocity vectors of the particle.

15. $\mathbf{a}(t) = -\cos t\mathbf{i} - \sin t\mathbf{j};\ \mathbf{v}(0) = \mathbf{i};\ \mathbf{r}(0) = \mathbf{j}$
16. $\mathbf{a}(t) = \mathbf{i} + e^{-t}\mathbf{j};\ \mathbf{v}(0) = 2\mathbf{i} + \mathbf{j};\ \mathbf{r}(0) = \mathbf{i} - \mathbf{j}$
17. $\mathbf{a}(t) = \sin t\mathbf{i} + \cos t\mathbf{j} + e^t\mathbf{k};\ \mathbf{v}(0) = \mathbf{k};\ \mathbf{r}(0) = -\mathbf{i} + \mathbf{k}$
18. $\mathbf{a}(t) = (t+1)^{-2}\mathbf{j} - e^{-2t}\mathbf{k};\ \mathbf{v}(0) = 3\mathbf{i} - \mathbf{j};\ \mathbf{r}(0) = 2\mathbf{k}$

19. What can you say about the trajectory of a particle that moves in 2-space or 3-space with zero acceleration? Justify your answer.

20. Recall from Theorem 13.2.9 that if $\mathbf{r}(t)$ is a vector-valued function in 2-space or 3-space, and if $\|\mathbf{r}(t)\|$ is constant for all t, then $\mathbf{r}(t)\cdot\mathbf{r}'(t) = 0$.
(a) Translate this theorem into a statement about the motion of a particle in 2-space or 3-space.
(b) Replace $\mathbf{r}(t)$ by $\mathbf{r}'(t)$ in the theorem, and translate the result into a statement about the motion of a particle in 2-space or 3-space.

21. Find, to the nearest degree, the angle between $\mathbf{v}$ and $\mathbf{a}$ for $\mathbf{r} = t^3\mathbf{i} + t^2\mathbf{j}$ when $t = 1$.

22. Show that the angle between $\mathbf{v}$ and $\mathbf{a}$ is constant for the position vector $\mathbf{r} = e^t\cos t\mathbf{i} + e^t\sin t\mathbf{j}$. Find the angle.

23. (a) Suppose that at time $t = t_0$ an electron has a position vector of $\mathbf{r} = 3.5\mathbf{i} - 1.7\mathbf{j} + \mathbf{k}$, and at a later time $t = t_1$ it has a position vector of $\mathbf{r} = 4.2\mathbf{i} + \mathbf{j} - 2.4\mathbf{k}$. What is the displacement of the electron during the time interval from t_0 to t_1?
(b) Suppose that during a certain time interval a proton has a displacement of $\Delta\mathbf{r} = 0.7\mathbf{i} + 2.9\mathbf{j} - 1.2\mathbf{k}$ and its final position vector is known to be $\mathbf{r} = 3.6\mathbf{k}$. What was the initial position vector of the proton?

24. Suppose that the position function of a particle moving along a circle in the xy-plane is $\mathbf{r} = 5\cos 2\pi t\mathbf{i} + 5\sin 2\pi t\mathbf{j}$.

(a) Sketch some typical displacement vectors over the time interval from $t = 0$ to $t = 1$.
(b) What is the distance traveled by the particle during the time interval?

In Exercises 25–28, find the displacement and the distance traveled over the indicated time interval.

25. $\mathbf{r} = t^2\mathbf{i} + \frac{1}{3}t^3\mathbf{j};\ 1 \le t \le 3$

26. $\mathbf{r} = (1 - 3\sin t)\mathbf{i} + 3\cos t\mathbf{j};\ 0 \le t \le 3\pi/2$

27. $\mathbf{r} = e^t\mathbf{i} + e^{-t}\mathbf{j} + \sqrt{2}t\mathbf{k};\ 0 \le t \le \ln 3$

28. $\mathbf{r} = \cos 2t\mathbf{i} + (1 - \cos 2t)\mathbf{j} + \left(3 + \frac{1}{2}\cos 2t\right)\mathbf{k};\ 0 \le t \le \pi$

In Exercises 29 and 30, the position vectors of two particles are given. Show that the particles move along the same path but the speed of the first is constant and the speed of the second is not.

29. $\mathbf{r}_1 = 2\cos 3t\mathbf{i} + 2\sin 3t\mathbf{j}$
$\mathbf{r}_2 = 2\cos(t^2)\mathbf{i} + 2\sin(t^2)\mathbf{j} \quad (t \ge 0)$

30. $\mathbf{r}_1 = (3 + 2t)\mathbf{i} + t\mathbf{j} + (1 - t)\mathbf{k}$
$\mathbf{r}_2 = (5 - 2t^3)\mathbf{i} + (1 - t^3)\mathbf{j} + t^3\mathbf{k}$

In Exercises 31–38, the position function of a particle is given. Use Theorem 13.6.3 to find
(a) the scalar tangential and normal components of acceleration at the stated time t;
(b) the vector tangential and normal components of acceleration at the stated time t;
(c) the curvature of the path at the point where the particle is located at the stated time t.

31. $\mathbf{r} = e^{-t}\mathbf{i} + e^t\mathbf{j};\ t = 0$

32. $\mathbf{r} = \cos(t^2)\mathbf{i} + \sin(t^2)\mathbf{j};\ t = \sqrt{\pi}/2$

33. $\mathbf{r} = (t^3 - 2t)\mathbf{i} + (t^2 - 4)\mathbf{j};\ t = 1$

34. $\mathbf{r} = e^t\cos t\mathbf{i} + e^t\sin t\mathbf{j};\ t = \pi/4$

35. $\mathbf{r} = (1/t)\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k};\ t = 1$

36. $\mathbf{r} = e^t\mathbf{i} + e^{-2t}\mathbf{j} + t\mathbf{k};\ t = 0$

37. $\mathbf{r} = 3\sin t\mathbf{i} + 2\cos t\mathbf{j} - \sin 2t\mathbf{k};\ t = \pi/2$

38. $\mathbf{r} = 2\mathbf{i} + t^3\mathbf{j} - 16\ln t\mathbf{k};\ t = 1$

In Exercises 39–42, $\mathbf{v}$ and $\mathbf{a}$ are given at a certain instant of time. Find a_T, a_N, $\mathbf{T}$, and $\mathbf{N}$ at this instant.

39. $\mathbf{v} = -4\mathbf{j},\ \mathbf{a} = 2\mathbf{i} + 3\mathbf{j}$ **40.** $\mathbf{v} = \mathbf{i} + 2\mathbf{j},\ \mathbf{a} = 3\mathbf{i}$

41. $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k},\ \mathbf{a} = \mathbf{i} + 2\mathbf{k}$

42. $\mathbf{v} = 3\mathbf{i} - 4\mathbf{k},\ \mathbf{a} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$

In Exercises 43–46, the speed $\|\mathbf{v}\|$ of a particle at an arbitrary time t is given. Find the scalar tangential component of acceleration at the indicated time.

43. $\|\mathbf{v}\| = \sqrt{3t^2 + 4};\ t = 2$

44. $\|\mathbf{v}\| = \sqrt{t^2 + e^{-3t}};\ t = 0$

45. $\|\mathbf{v}\| = \sqrt{(4t - 1)^2 + \cos^2\pi t};\ t = \frac{1}{4}$

46. $\|\mathbf{v}\| = \sqrt{t^4 + 5t^2 + 3};\ t = 1$

47. The nuclear accelerator at the Enrico Fermi Laboratory is circular with a radius of 1 km. Find the scalar normal component of acceleration of a proton moving around the accelerator with a constant speed of 2.9×10^5 km/s.

48. Suppose that a particle moves with nonzero acceleration along the curve $y = f(x)$. Use part (b) of Exercise 17 in Section 13.5 to show that the acceleration vector is tangent to the curve at each point where $f''(x) = 0$.

In Exercises 49 and 50, use the given information and Exercise 17 of Section 13.5 to find the normal scalar component of acceleration as a function of x.

49. A particle moves along the parabola $y = x^2$ with a constant speed of 3 units per second.

50. A particle moves along the curve $x = \ln y$ with a constant speed of 2 units per second.

In Exercises 51 and 52, use the given information to find the normal scalar component of acceleration at time $t = 1$.

51. $\mathbf{a}(1) = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k};\ a_T(1) = 3$

52. $\|\mathbf{a}(1)\| = 9;\ a_T(1)\mathbf{T}(1) = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

53. An automobile travels at a constant speed around a curve whose radius of curvature is 1000 m. What is the maximum allowable speed if the maximum acceptable value for the normal scalar component of acceleration is 1.5 m/s^2?

54. If an automobile of mass m rounds a curve, then its inward vector component of acceleration $a_N\mathbf{N}$ is caused by the frictional force $\mathbf{F}$ of the road. Thus, it follows from the vector form of Newton's second law [Equation (19)] that the frictional force and the normal scalar component of acceleration are related by the equation $\mathbf{F} = ma_N\mathbf{N}$. Thus,

$$\|\mathbf{F}\| = m\kappa\left(\frac{ds}{dt}\right)^2$$

Use this result to find the magnitude of the frictional force in newtons exerted by the road on a 500-kg go-cart driven at a speed of 10 km/h around a circular track of radius 15 m. [*Note:* 1 N = 1 kg·m/s^2]

55. A shell is fired from ground level with a muzzle speed of 320 ft/s and elevation angle of 60°. Find
(a) parametric equations for the shell's trajectory
(b) the maximum height reached by the shell
(c) the horizontal distance traveled by the shell
(d) the speed of the shell at impact.

56. Solve Exercise 55 assuming that the muzzle speed is 980 m/s and the elevation angle is 45°.

57. A rock is thrown downward from the top of a building, 168 ft high, at an angle of 60° with the horizontal. How far from the base of the building will the rock land if its initial speed is 80 ft/s?

58. Solve Exercise 57 assuming that the rock is thrown horizontally at a speed of 80 ft/s.

59. A shell is to be fired from ground level at an elevation angle of 30°. What should the muzzle speed be in order for the maximum height of the shell to be 2500 ft?

60. A shell, fired from ground level at an elevation angle of 45°, hits the ground 24,500 m away. Calculate the muzzle speed of the shell.

61. Find two elevation angles that will enable a shell, fired from ground level with a muzzle speed of 800 ft/s, to hit a ground-level target 10,000 ft away.

62. A ball rolls off a table 4 ft high while moving at a constant speed of 5 ft/s.
(a) How long does it take for the ball to hit the floor after it leaves the table?
(b) At what speed does the ball hit the floor?
(c) If a ball were dropped from rest at table height just as the rolling ball leaves the table, which ball would hit the ground first? Justify your answer.

63. As illustrated in the accompanying figure, a fire hose sprays water with an initial velocity of 40 ft/s at an angle of 60° with the horizontal.
(a) Confirm that the water will clear corner point A.
(b) Confirm that the water will hit the roof.
(c) How far from corner point A will the water hit the roof?

64. What is the minimum initial velocity that will allow the water in Exercise 63 to hit the roof?

65. As shown in the accompanying figure, water is sprayed from a hose with an initial velocity of 35 m/s at an angle of 45° with the horizontal.
(a) What is the radius of curvature of the stream at the point where it leaves the hose?
(b) What is the maximum height of the stream above the nozzle of the hose?

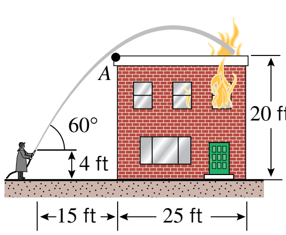


Figure Ex-63

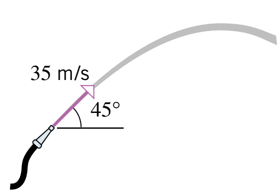


Figure Ex-65

66. As illustrated in the accompanying figure, a train is traveling on a curved track. At a point where the train is traveling at a speed of 132 ft/s and the radius of curvature of the track is 3000 ft, the engineer hits the brakes to make the train slow down at a constant rate of 7.5 ft/s^2.
(a) Find the magnitude of the acceleration vector at the instant the engineer hits the brakes.
(b) Approximate the angle between the acceleration vector and the unit tangent vector $\mathbf{T}$ at the instant the engineer hits the brakes.

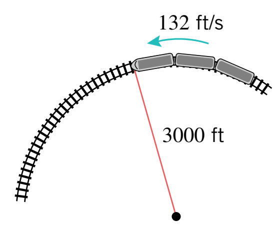


Figure Ex-66

67. A shell is fired from ground level at an elevation angle of α and a muzzle speed of v_0.
(a) Show that the maximum height reached by the shell is

$$\text{maximum height} = \frac{(v_0 \sin \alpha)^2}{2g}$$

(b) The ***horizontal range*** R of the shell is the horizontal distance traveled when the shell returns to ground level. Show that $R = (v_0^2 \sin 2\alpha)/g$. For what elevation angle will the range be maximum? What is the maximum range?

68. A shell is fired from ground level with an elevation angle α and a muzzle speed of v_0. Find two angles that can be used to hit a target at ground level that is a distance of three-fourths the maximum range of the shell. Express your answer to the nearest tenth of a degree. [*Hint:* See Exercise 67(b).]

69. At time $t = 0$ a baseball that is 5 ft above the ground is hit with a bat. The ball leaves the bat with a speed of 80 ft/s at an angle of 30° above the horizontal.
(a) How long will it take for the baseball to hit the ground? Express your answer to the nearest hundredth of a second.
(b) Use the result in part (a) to find the horizontal distance traveled by the ball. Express your answer to the nearest tenth of a foot.

70. At time $t = 0$ a projectile is fired from a height h above level ground at an elevation angle of α with a speed v. Let R be the horizontal distance to the point where the projectile hits the ground.
(a) Show that α and R must satisfy the equation

$$g(\sec^2 \alpha)R^2 - 2v^2(\tan \alpha)R - 2v^2 h = 0$$

(b) If g, h, and v are constant, then the equation in part (a) defines R implicitly as a function of α. Let R_0 be the maximum value of R and α_0 the value of α when $R = R_0$. Use implicit differentiation to find $dR/d\alpha$ and show that

$$\tan \alpha_0 = \frac{v^2}{gR_0}$$

[*Hint:* Assume that $dR/d\alpha = 0$ when R is maximum.]
(c) Use the results in parts (a) and (b) to show that

$$R_0 = \frac{v}{g}\sqrt{v^2 + 2gh}$$

and

$$\alpha_0 = \tan^{-1} \frac{v}{\sqrt{v^2 + 2gh}}$$

C **71.** At time $t = 0$ a skier leaves the end of a ski jump with a speed of v_0 ft/s at an angle α with the horizontal (see the accompanying figure). The skier lands 259 ft down the incline 2.9 s later.
(a) Approximate v_0 to the nearest ft/s and α to the nearest degree.
(b) Use a CAS or a calculating utility with a numerical integration capability to approximate the distance traveled by the skier.
(Use $g = 32$ ft/s^2 as the acceleration due to gravity.)

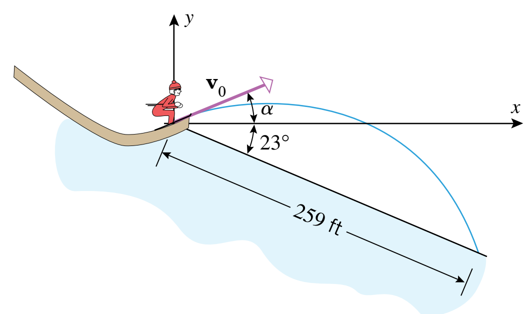


Figure Ex-71

13.7 KEPLER'S LAWS OF PLANETARY MOTION

One of the great advances in the history of astronomy occurred in the early 1600s when Johannes Kepler deduced from empirical data that all planets in our solar system move in elliptical orbits with the Sun at a focus. Subsequently, Isaac Newton showed mathematically that such planetary motion is the consequence of an inverse-square law of gravitational attraction. In this section we will use the concepts developed in the preceding sections of this chapter to derive three basic laws of planetary motion, known as* ***Kepler's laws***.

KEPLER'S LAWS

In Section 11.6 we stated the following laws of planetary motion that were published by Johannes Kepler in 1609 in his book known as *Astronomia Nova*.

13.7.1 KEPLER'S LAWS.

- First law (***Law of Orbits***). Each planet moves in an elliptical orbit with the Sun at a focus.
- Second law (***Law of Areas***). Equal areas are swept out in equal times by the line from the Sun to a planet.
- Third law (***Law of Periods***). The square of a planet's period (the time it takes the planet to complete one orbit about the Sun) is proportional to the cube of the semimajor axis of its orbit.

CENTRAL FORCES

If a particle moves under the influence of a *single* force that is always directed toward a fixed point O, then the particle is said to be moving in a ***central force field***. The force is called a ***central force***, and the point O is called the ***center of force***. For example, in the simplest model of planetary motion, it is assumed that the only force acting on a planet is the force of the Sun's gravity, directed toward the center of the Sun. This model, which produces Kepler's laws, ignores the forces that other celestial objects exert on the planet as well as the minor effect that the planet's gravity has on the Sun. Central force models are also used to study the motion of comets, asteroids, planetary moons, and artificial satellites. They also have important applications in electromagnetics. Our objective in this section is to develop some basic principles about central force fields and then use those results to derive Kepler's laws.

Suppose that a particle P of mass m moves in a central force field due to a force $\mathbf{F}$ that is directed toward a fixed point O, and let $\mathbf{r} = \mathbf{r}(t)$ be the position vector from O to P

*See biography on p. 779.

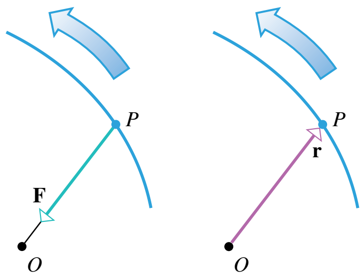


Figure 13.7.1

(Figure 13.7.1). Let $\mathbf{v} = \mathbf{v}(t)$ and $\mathbf{a} = \mathbf{a}(t)$ be the velocity and acceleration functions of the particle, and assume that $\mathbf{F}$ and $\mathbf{a}$ are related by Newton's second law ($\mathbf{F} = m\mathbf{a}$).

Our first objective is to show that the particle P moves in a plane containing the point O. For this purpose observe that $\mathbf{a}$ has the same direction as $\mathbf{F}$ by Newton's second law, and this implies that $\mathbf{a}$ and $\mathbf{r}$ are oppositely directed vectors. Thus, it follows from part (c) of Theorem 12.4.5 that

$$\mathbf{r} \times \mathbf{a} = \mathbf{0}$$

Since the velocity and acceleration of the particle are given by $\mathbf{v} = d\mathbf{r}/dt$ and $\mathbf{a} = d\mathbf{v}/dt$, respectively, we have

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{v}) = \mathbf{r} \times \frac{d\mathbf{v}}{dt} + \frac{d\mathbf{r}}{dt} \times \mathbf{v} = (\mathbf{r} \times \mathbf{a}) + (\mathbf{v} \times \mathbf{v}) = \mathbf{0} + \mathbf{0} = \mathbf{0} \tag{1}$$

Integrating the left and right sides of this equation with respect to t yields

$$\mathbf{r} \times \mathbf{v} = \mathbf{b} \tag{2}$$

where $\mathbf{b}$ is a constant (independent of t). However, $\mathbf{b}$ is orthogonal to both $\mathbf{r}$ and $\mathbf{v}$, so we can conclude that $\mathbf{r} = \mathbf{r}(t)$ and $\mathbf{v} = \mathbf{v}(t)$ lie in a fixed plane containing the point O.

REMARK. The preceding discussion shows that each planet moves in a plane through the center of the Sun. Astronomers call this plane the ***ecliptic*** of the planet.

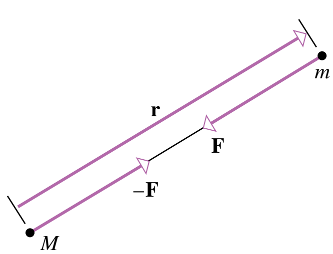
NEWTON'S LAW OF UNIVERSAL GRAVITATION

Our next objective is to derive the position function of a particle moving under a central force in a polar coordinate system. For this purpose we will need the following result, known as ***Newton's Law of Universal Gravitation***.

13.7.2 NEWTON'S LAW OF UNIVERSAL GRAVITATION. Every particle of matter in the Universe attracts every other particle of matter in the Universe with a force that is proportional to the product of their masses and inversely proportional to the square of the distance between them. Specifically, if a particle of mass M and a particle of mass m are at a distance r from one another, then they attract each other with equal and opposite forces, $\mathbf{F}$ and $-\mathbf{F}$, of magnitude

$$\|\mathbf{F}\| = \frac{GMm}{r^2} \tag{3}$$

where G is a constant called the ***universal gravitational constant***.



M exerts force $\mathbf{F}$ on m, and m exerts force $-\mathbf{F}$ on M.

Figure 13.7.2

To obtain a formula for the vector force $\mathbf{F}$ that mass M exerts on mass m, we will let $\mathbf{r}$ be the radius vector from mass M to mass m (Figure 13.7.2). Thus, the distance r between the masses is $\|\mathbf{r}\|$, and the force $\mathbf{F}$ can be expressed in terms of $\mathbf{r}$ as

$$\mathbf{F} = \|\mathbf{F}\|\left(-\frac{\mathbf{r}}{\|\mathbf{r}\|}\right) = \|\mathbf{F}\|\left(-\frac{\mathbf{r}}{r}\right)$$

which from (3) can be expressed as

$$\mathbf{F} = -\frac{GMm}{r^3}\mathbf{r} \tag{4}$$

We start by finding a formula for the acceleration function. To do this we use Formula (4) and Newton's second law to obtain

$$m\mathbf{a} = -\frac{GMm}{r^3}\mathbf{r}$$

from which we obtain

$$\mathbf{a} = -\frac{GM}{r^3}\mathbf{r} \tag{5}$$

REMARK. Observe that the acceleration **a** depends on the mass M but not on the mass m. Thus, for example, the acceleration of a planet is affected by the mass of the Sun but not by its own mass.

To obtain a formula for the position function of the mass m, we will need to introduce a coordinate system and make some assumptions about the initial conditions. Let us assume:

- The distance r from m to M is minimum at time $t = 0$.
- The mass m has nonzero position and velocity vectors $\mathbf{r}_0$ and $\mathbf{v}_0$ at time $t = 0$.
- A polar coordinate system is introduced with its pole at mass M and oriented so $\theta = 0$ at time $t = 0$.
- The vector $\mathbf{v}_0$ is perpendicular to the polar axis at time $t = 0$.

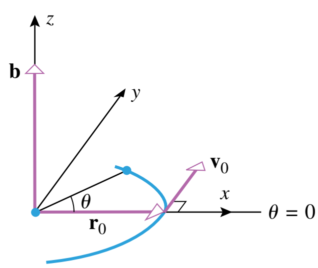


Figure 13.7.3

Moreover, to ensure that the polar angle θ increases with t, let us agree to observe this polar coordinate system looking toward the pole from the terminal point of the vector $\mathbf{b} = \mathbf{r}_0 \times \mathbf{v}_0$. We will also find it useful to superimpose an xyz-coordinate system on the polar coordinate system with the positive z-axis in the direction of **b** (Figure 13.7.3).

For computational purposes, it will be helpful to denote $\|\mathbf{r}_0\|$ by r_0 and $\|\mathbf{v}_0\|$ by v_0, in which case we can express the vectors $\mathbf{r}_0$ and $\mathbf{v}_0$ in xyz-coordinates as

$$\mathbf{r}_0 = r_0\mathbf{i} \quad \text{and} \quad \mathbf{v}_0 = v_0\mathbf{j}$$

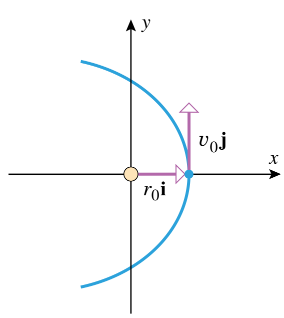


Figure 13.7.4

and the vector **b** as

$$\mathbf{b} = \mathbf{r}_0 \times \mathbf{v}_0 = r_0\mathbf{i} \times v_0\mathbf{j} = r_0 v_0\mathbf{k} \tag{6}$$

(Figure 13.7.4). It will also be useful to introduce the unit vector

$$\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j} \tag{7}$$

which will allow us to express the polar form of the position vector **r** as

$$\mathbf{r} = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j} = r(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}) = r\mathbf{u} \tag{8}$$

and to express the acceleration vector **a** in terms of **u** by rewriting (5) as

$$\mathbf{a} = -\frac{GM}{r^2}\mathbf{u} \tag{9}$$

We are now ready to derive the position function of the mass m in polar coordinates. For this purpose, recall from (2) that the vector $\mathbf{b} = \mathbf{r} \times \mathbf{v}$ is constant, so it follows from (6) that the relationship

$$\mathbf{b} = \mathbf{r} \times \mathbf{v} = r_0 v_0\mathbf{k} \tag{10}$$

holds for *all* values of t. Now let us examine **b** from another point of view. It follows from (8) that

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d}{dt}(r\mathbf{u}) = r\frac{d\mathbf{u}}{dt} + \frac{dr}{dt}\mathbf{u}$$

and hence

$$\mathbf{b} = \mathbf{r} \times \mathbf{v} = (r\mathbf{u}) \times \left(r\frac{d\mathbf{u}}{dt} + \frac{dr}{dt}\mathbf{u}\right) = r^2\mathbf{u} \times \frac{d\mathbf{u}}{dt} + r\frac{dr}{dt}\mathbf{u} \times \mathbf{u} = r^2\mathbf{u} \times \frac{d\mathbf{u}}{dt} \tag{11}$$

But (7) implies that

$$\frac{d\mathbf{u}}{dt} = \frac{d\mathbf{u}}{d\theta}\frac{d\theta}{dt} = (-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j})\frac{d\theta}{dt}$$

so

$$\mathbf{u} \times \frac{d\mathbf{u}}{dt} = \frac{d\theta}{dt}\mathbf{k} \tag{12}$$

Substituting (12) in (11) yields

$$\mathbf{b} = r^2\frac{d\theta}{dt}\mathbf{k} \tag{13}$$

Thus, it follows from (7), (9), and (13) that

$$\mathbf{a} \times \mathbf{b} = -\frac{GM}{r^2}(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}) \times \left(r^2\frac{d\theta}{dt}\mathbf{k}\right)$$

$$= GM(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j})\frac{d\theta}{dt} = GM\frac{d\mathbf{u}}{dt} \tag{14}$$

From this formula and the fact that $d\mathbf{b}/dt = \mathbf{0}$ (since $\mathbf{b}$ is constant), we obtain

$$\frac{d}{dt}(\mathbf{v} \times \mathbf{b}) = \mathbf{v} \times \frac{d\mathbf{b}}{dt} + \frac{d\mathbf{v}}{dt} \times \mathbf{b} = \mathbf{a} \times \mathbf{b} = GM\frac{d\mathbf{u}}{dt}$$

Integrating both sides of this equation with respect to t yields

$$\mathbf{v} \times \mathbf{b} = GM\mathbf{u} + \mathbf{C} \tag{15}$$

where $\mathbf{C}$ is a vector constant of integration. This constant can be obtained by evaluating both sides of the equation at $t = 0$. We leave it as an exercise to show that

$$\mathbf{C} = (r_0 v_0^2 - GM)\mathbf{i} \tag{16}$$

from which it follows that

$$\mathbf{v} \times \mathbf{b} = GM\mathbf{u} + (r_0 v_0^2 - GM)\mathbf{i} \tag{17}$$

We can now obtain the position function by computing the scalar triple product $\mathbf{r} \cdot (\mathbf{v} \times \mathbf{b})$ in two ways. First we use (10) and property (11) of Section 12.4 to obtain

$$\mathbf{r} \cdot (\mathbf{v} \times \mathbf{b}) = (\mathbf{r} \times \mathbf{v}) \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{b} = r_0^2 v_0^2 \tag{18}$$

and next we use (17) to obtain

$$\mathbf{r} \cdot (\mathbf{v} \times \mathbf{b}) = \mathbf{r} \cdot (GM\mathbf{u}) + \mathbf{r} \cdot (r_0 v_0^2 - GM)\mathbf{i}$$

$$= \mathbf{r} \cdot \left(GM\frac{\mathbf{r}}{r}\right) + r\mathbf{u} \cdot (r_0 v_0^2 - GM)\mathbf{i}$$

$$= GMr + r(r_0 v_0^2 - GM)\cos\theta$$

If we now equate this to (18), we obtain

$$r_0^2 v_0^2 = GMr + r(r_0 v_0^2 - GM)\cos\theta$$

which when solved for r gives

$$r = \frac{r_0^2 v_0^2}{GM + (r_0 v_0^2 - GM)\cos\theta} = \frac{\dfrac{r_0^2 v_0^2}{GM}}{1 + \left(\dfrac{r_0 v_0^2}{GM} - 1\right)\cos\theta} \tag{19}$$

or more simply

$$r = \frac{k}{1 + e\cos\theta} \tag{20}$$

where

$$k = \frac{r_0^2 v_0^2}{GM} \quad \text{and} \quad e = \frac{r_0 v_0^2}{GM} - 1 \tag{21–22}$$

We will leave it as an exercise to show that $e \geq 0$. Accepting this to be so, it follows by comparing (20) to Formula (3) of Section 11.6 that the trajectory is a conic section with eccentricity e, the focus at the pole, and $d = k/e$. Thus, depending on whether $e < 1$, $e = 1$, or $e > 1$, the trajectory will be, respectively, an ellipse, a parabola, or a hyperbola (Figure 13.7.5).

Note from Formula (22) that e depends on r_0 and v_0, so the exact form of the trajectory is determined by the mass M and the initial conditions. If the initial conditions are such that $e < 1$, then the mass m becomes trapped in an elliptical orbit; otherwise the mass m "escapes" and never returns to its initial position. Accordingly, the initial velocity that

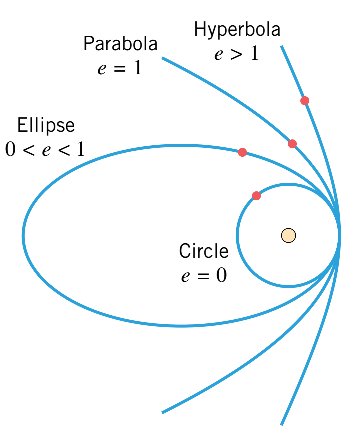


Figure 13.7.5

produces an eccentricity of $e = 1$ is called the ***escape speed*** and is denoted by v_{esc}. Thus, it follows from (22) that

$$v_{\text{esc}} = \sqrt{\frac{2GM}{r_0}} \tag{23}$$

(verify).

KEPLER'S FIRST AND SECOND LAWS

It follows from our general discussion of central force fields that the planets have elliptical orbits with the Sun at the focus, which is Kepler's first law. To derive Kepler's second law, we begin by equating (10) and (13) to obtain

$$r^2\frac{d\theta}{dt} = r_0v_0 \tag{24}$$

To prove that the radial line from the center of the Sun to the center of a planet sweeps out equal areas in equal times, let $r = f(\theta)$ denote the polar equation of the planet, and let A denote the area swept out by the radial line as it varies from any fixed angle θ_0 to an angle θ. It follows from Theorem 11.3.2 that A can be expressed as

$$A = \int_{\theta_0}^{\theta} \frac{1}{2}[f(\phi)]^2\,d\phi$$

where the dummy variable ϕ is introduced for the integration to reserve θ for the upper limit. It now follows from Part 2 of the Fundamental Theorem of Calculus and the chain rule that

$$\frac{dA}{dt} = \frac{dA}{d\theta}\frac{d\theta}{dt} = \frac{1}{2}[f(\theta)]^2\frac{d\theta}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt}$$

Thus, it follows from (24) that

$$\frac{dA}{dt} = \frac{1}{2}r_0v_0 \tag{25}$$

which shows that A changes at a constant rate. This implies that equal areas are swept out in equal times.

KEPLER'S THIRD LAW

To derive Kepler's third law, we let a and b be the semimajor and semiminor axes of the elliptical orbit, and we recall that the area of this ellipse is πab. It follows by integrating (25) that in t units of time the radial line will sweep out an area of $A = \frac{1}{2}r_0v_0t$. Thus, if T denotes the time required for the planet to make one revolution around the Sun (the period), then the radial line will sweep out the area of the entire ellipse during that time and hence

$$\pi ab = \tfrac{1}{2}r_0v_0T$$

from which we obtain

$$T^2 = \frac{4\pi^2a^2b^2}{r_0^2v_0^2} \tag{26}$$

However, it follows from Formula (1) of Section 11.6 and the relationship $c^2 = a^2 - b^2$ for an ellipse that

$$e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a}$$

Thus, $b^2 = a^2(1 - e^2)$ and hence (26) can be written as

$$T^2 = \frac{4\pi^2a^4(1 - e^2)}{r_0^2v_0^2} \tag{27}$$

But comparing Equation (20) to Equation (17) of Section 11.6 shows that

$$k = a(1 - e^2)$$

Finally, substituting this expression and (21) in (27) yields

$$T^2 = \frac{4\pi^2 a^3}{r_0^2 v_0^2} k = \frac{4\pi^2 a^3}{r_0^2 v_0^2} \frac{r_0^2 v_0^2}{GM} = \frac{4\pi^2}{GM} a^3 \tag{28}$$

Thus, we have proved that T^2 is proportional to a^3, which is Kepler's third law. When convenient, Formula (28) can also be expressed as

$$\boxed{T = \frac{2\pi}{\sqrt{GM}} a^{3/2}} \tag{29}$$

ARTIFICIAL SATELLITES

Kepler's second and third laws and Formula (23) also apply to satellites that orbit a celestial body; we need only interpret M to be the mass of the body exerting the force and m to be the mass of the satellite. Values of GM that are required in many of the formulas in this section have been determined experimentally for various attracting bodies (Table 13.7.1).

Table 13.7.1

ATTRACTING BODY	INTERNATIONAL SYSTEM	BRITISH ENGINEERING SYSTEM
Earth	$GM = 3.99 \times 10^{14}$ m^3/s^2 $GM = 3.99 \times 10^{5}$ km^3/s^2	$GM = 1.41 \times 10^{16}$ ft^3/s^2 $GM = 1.24 \times 10^{12}$ mi^3/h^2
Sun	$GM = 1.33 \times 10^{20}$ m^3/s^2 $GM = 1.33 \times 10^{11}$ km^3/s^2	$GM = 4.69 \times 10^{21}$ ft^3/s^2 $GM = 4.13 \times 10^{17}$ mi^3/h^2
Moon	$GM = 4.90 \times 10^{12}$ m^3/s^2 $GM = 4.90 \times 10^{3}$ km^3/s^2	$GM = 1.73 \times 10^{14}$ ft^3/s^2 $GM = 1.53 \times 10^{10}$ mi^3/h^2

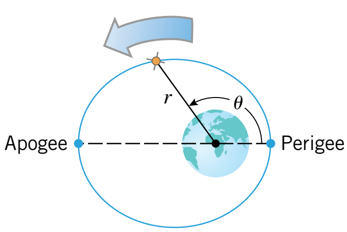


Figure 13.7.6

Recall that for orbits of planets around the Sun, the point at which the distance between the center of the planet and the center of the Sun is maximum is called the *aphelion* and the point at which it is minimum the *perihelion*. For satellites around the Earth the point at which the maximum distance occurs is called the ***apogee*** and the point at which the minimum distance occurs is called the ***perigee*** (Figure 13.7.6). The actual distances between the centers at apogee and perigee are called the ***apogee distance*** and the ***perigee distance***.

Example 1 A geosynchronous orbit for a satellite is a circular orbit about the equator of the Earth in which the satellite stays fixed over a point on the equator. Use the fact that the Earth makes one revolution about its axis every 24 hours to find the altitude in miles of a communications satellite in geosynchronous orbit. Assume the Earth to be a sphere of radius 4000 mi.

Solution. To remain fixed over a point on the equator, the satellite must have a period of $T = 24$ h. It follows from (28) or (29) and the Earth value of $GM = 1.24 \times 10^{12}$ mi^3/h^2 from Table 13.7.1 that

$$a = \sqrt[3]{\frac{GMT^2}{4\pi^2}} = \sqrt[3]{\frac{(1.24 \times 10^{12})(24)^2}{4\pi^2}} \approx 26{,}250 \text{ mi}$$

and hence the altitude h of the satellite is

$$h = 26{,}250 - 4000 = 22{,}250 \text{ mi} \qquad \blacktriangleleft$$

EXERCISE SET 13.7

In Exercises that require numerical values, use Table 13.7.1 and the following values, where needed:

radius of Earth = 4000 mi = 6440 km
radius of Moon = 1080 mi = 1740 km
1 year (Earth year) = 365 days

1. Suppose that a particle is in an elliptical orbit in a central force field in which the center of force is at a focus, and let $r_{\min}$ and $r_{\max}$ denote the minimum and maximum distances from the particle to the center of force. Review the discussion of ellipses in polar coordinates in Section 11.6, and show that if the ellipse has eccentricity e and semimajor axis a, then $r_{\min} = a(1-e)$ and $r_{\max} = a(1+e)$.

2. (a) Use the results in Exercise 1 to show that
$$e = \frac{r_{\max} - r_{\min}}{r_{\max} + r_{\min}}$$
(b) Show that
$$r_{\max} = r_{\min}\frac{1+e}{1-e}$$

3. (a) Obtain the value of $\mathbf{C}$ given in Formula (16) by setting $t = 0$ in (15).
(b) Use Formulas (7), (17), and (22) to show that
$$\mathbf{v} \times \mathbf{b} = GM[(e + \cos\theta)\mathbf{i} + \sin\theta\,\mathbf{j}]$$
(c) Show that $\|\mathbf{v} \times \mathbf{b}\| = \|\mathbf{v}\|\|\mathbf{b}\|$.
(d) Use the results in parts (b) and (c) to show that the speed of a particle in an elliptical orbit is
$$v = \frac{v_0}{1+e}\sqrt{e^2 + 2e\cos\theta + 1}$$

4. Use the result in Exercise 3(d) to show that when a particle in an elliptical orbit with eccentricity e reaches an end of the minor axis, its speed is
$$v = v_0\sqrt{\frac{1-e}{1+e}}$$

5. Use the result in Exercise 3(d) to show that for a particle in an elliptical orbit with eccentricity e, the maximum and minimum speeds are related by
$$v_{\max} = v_{\min}\frac{1+e}{1-e}$$

6. Use Formula (22) and the result in part (d) of Exercise 3 to show that the speed v of a particle in a circular orbit of radius r_0 is constant and is given by
$$v = \sqrt{\frac{GM}{r_0}}$$

7. Use the result in Exercise 6 to find the speed in km/s of a satellite in a circular orbit that is 200 km above the surface of the Earth.

8. Use the result in Exercise 6 to find the speed in mi/h of a communications satellite that is in geosynchronous orbit around the Earth. [See Example 1.]

9. Find the escape speed in km/s for a space probe in a circular orbit that is 300 km above the surface of the Earth.

10. The universal gravitational constant is approximately
$$G = 6.67 \times 10^{-11}\ \text{m}^3/\text{kg}\cdot\text{s}^2$$
and the semimajor axis of the Earth's orbit is approximately
$$a = 149.6 \times 10^6\ \text{km}$$
Estimate the mass of the Sun in kg.

11. (a) The eccentricity of the Moon's orbit around the Earth is 0.055, and its semimajor axis is $a = 238{,}900$ mi. Find the maximum and minimum distances between the surface of the Earth and the surface of the Moon.
(b) Find the period of the Moon's orbit in days.

12. (a) *Vanguard 1* was launched in March 1958 with perigee and apogee altitudes above the Earth of 649 km and 4340 km, respectively. Find the length of the semimajor axis of its orbit.
(b) Use the result in part (a) of Exercise 2 to find the eccentricity of its orbit.
(c) Find the period of *Vanguard I* in minutes.

13. (a) Suppose that a space probe is in a circular orbit at an altitude of 180 mi above the surface of the Earth. Use the result in Exercise 6 to find its speed.
(b) During a very short period of time, a thruster rocket on the space probe is fired to increase the speed of the probe by 600 mi/h in its direction of motion. Find the eccentricity of the resulting elliptical orbit, and use the result in part (b) of Exercise 2 to find the apogee altitude.

14. Show that the quantity e defined by Formula (22) is nonnegative. [*Hint:* The polar axis was chosen so that r is minimum when $\theta = 0$.]

SUPPLEMENTARY EXERCISES

1. In words, what is meant by the graph of a vector-valued function $\mathbf{r}(t)$?

2. Describe the graph of the vector-valued function.
(a) $\mathbf{r} = \mathbf{r}_0 + t(\mathbf{r}_1 - \mathbf{r}_0)$
(b) $\mathbf{r} = \mathbf{r}_0 + t(\mathbf{r}_1 - \mathbf{r}_0) \quad (0 \le t \le 1)$
(c) $\mathbf{r} = \mathbf{r}_0 + t\mathbf{r}'(t_0)$

3. In words, describe what happens geometrically to $\mathbf{r}(t)$ if $\lim_{t \to a} \mathbf{r}(t) = \mathbf{L}$.

4. Suppose that $\mathbf{r}(t)$ is the position function of a particle moving in 2-space or 3-space. In each part, explain what the given quantity represents physically.

(a) $\left\| \dfrac{d\mathbf{r}}{dt} \right\|$ (b) $\displaystyle\int_{t_0}^{t_1} \left\| \frac{d\mathbf{r}}{dt} \right\| dt$ (c) $\|\mathbf{r}(t)\|$

5. Suppose that $\mathbf{r}(t)$ is a smooth vector-valued function. State the definitions of $\mathbf{T}(t)$, $\mathbf{N}(t)$, and $\mathbf{B}(t)$.

6. State the definition of "curvature" and explain what it means geometrically.

7. In Supplementary Exercise 34 of Chapter 11, we defined the Cornu spiral parametrically as

$$x = \int_0^t \cos\left(\frac{\pi u^2}{2}\right) du, \quad y = \int_0^t \sin\left(\frac{\pi u^2}{2}\right) du$$

This curve, which is graphed in the accompanying figure, is used in highway design to create a gradual transition from a straight road (zero curvature) to an exit ramp with positive curvature.
(a) Express the Cornu spiral as a vector-valued function $\mathbf{r}(t)$, and then use Theorem 13.3.4 to show that $s = t$ is the arc length parameter with reference point $(0, 0)$.
(b) Replace t by s and use Formula (1) of Section 13.5 to show that $\kappa(s) = \pi|s|$. [*Note:* If $s \ge 0$, then the curvature $\kappa(s) = \pi s$ increases from 0 at a constant rate with respect to s. This makes the spiral ideal for joining a curved road to a straight road.]
(c) What happens to the curvature of the Cornu spiral as $s \to +\infty$? In words, explain why this is consistent with the graph.

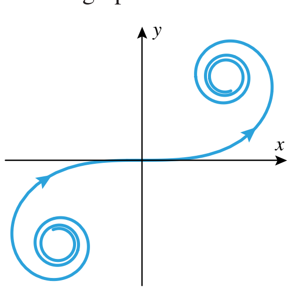


Figure Ex-7

8. (a) What does Theorem 13.2.9 tell you about the velocity vector of a particle that moves over a sphere?
(b) What does Theorem 13.2.9 tell you about the acceleration vector of a particle that moves with constant speed?
(c) Show that the particle with position function

$$\mathbf{r}(t) = \sqrt{1 - \tfrac{1}{4}\cos^2 t}\cos t\,\mathbf{i} + \sqrt{1 - \tfrac{1}{4}\cos^2 t}\sin t\,\mathbf{j} + \tfrac{1}{2}\cos t\,\mathbf{k}$$

moves over a sphere.

9. As illustrated in the accompanying figure, suppose that a particle moves counterclockwise around a circle of radius R centered at the origin at a constant rate of ω radians per second. This is called ***uniform circular motion***. If we assume that the particle is at the point $(R, 0)$ at time $t = 0$, then its position function will be

$$\mathbf{r}(t) = R\cos\omega t\,\mathbf{i} + R\sin\omega t\,\mathbf{j}$$

(a) Show that the velocity vector $\mathbf{v}(t)$ is always tangent to the circle and that the particle has constant speed v given by

$$v = R\omega$$

(b) Show that the acceleration vector $\mathbf{a}(t)$ is always directed toward the center of the circle and has constant magnitude a given by

$$a = R\omega^2$$

(c) Show that the time T required for the particle to make one complete revolution is

$$T = \frac{2\pi}{\omega} = \frac{2\pi R}{v}$$

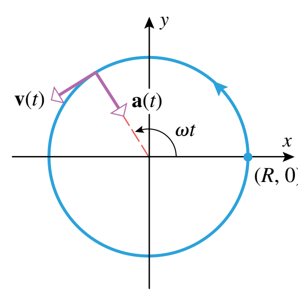


Figure Ex-9

10. If a particle of mass m has uniform circular motion (see Exercise 9), then the acceleration vector $\mathbf{a}(t)$ is called the ***centripetal acceleration***. According to Newton's second law, this acceleration must be produced by some force $\mathbf{F}(t)$, called the ***centripetal force***, that is related to $\mathbf{a}(t)$ by the equation $\mathbf{F}(t) = m\mathbf{a}(t)$. If this force is not present, then the particle cannot undergo uniform circular motion.
(a) Show that the direction of the centripetal force varies with time but that it has constant magnitude F given by

$$F = \frac{mv^2}{R}$$

(b) An astronaut with a mass of $m = 70$ kg orbits the Earth at an altitude of $h = 3200$ km with a constant speed of $v = 6.43$ km/s. Find her centripetal acceleration assuming that the radius of the Earth is 6440 km.

(c) What centripetal gravitational force in newtons does the Earth exert on the astronaut?

11. (a) Show that the graph of the vector-valued function $\mathbf{r}(t) = t\sin\pi t\mathbf{i} + t\mathbf{j} + t\cos\pi t\mathbf{k}$ lies on the surface of a cone, and sketch the cone.

(b) Find parametric equations for the intersection of the surfaces

$$y = x^2 \quad \text{and} \quad 2x^2 + y^2 + 6z^2 = 24$$

and sketch the intersection.

12. Sketch the graph of the vector-valued function that is defined piecewise by

$$\mathbf{r}(t) = \begin{cases} 3t\mathbf{i}, & 0 \le t \le \frac{1}{3} \\ (2-3t)\mathbf{i} + (3t-1)\mathbf{j}, & \frac{1}{3} \le t \le \frac{2}{3} \\ 3(1-t)\mathbf{j}, & \frac{2}{3} \le t \le 1 \end{cases}$$

13. Suppose that the position function of a point moving in the xy-plane is

$$\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j}$$

This equation can be expressed in polar coordinates by making the substitution

$$x(t) = r(t)\cos\theta(t), \quad y(t) = r(t)\sin\theta(t)$$

This yields

$$\mathbf{r} = r(t)\cos\theta(t)\mathbf{i} + r(t)\sin\theta(t)\mathbf{j}$$

which can be expressed as

$$\mathbf{r} = r(t)\mathbf{e}_r(t)$$

where $\mathbf{e}_r(t) = \cos\theta(t)\mathbf{i} + \sin\theta(t)\mathbf{j}$.

(a) Show that $\mathbf{e}_r(t)$ is a unit vector that has the same direction as the radius vector $\mathbf{r}$ if $r(t) > 0$ and that $\mathbf{e}_\theta(t) = -\sin\theta(t)\mathbf{i} + \cos\theta(t)\mathbf{j}$ is the unit vector that results when $\mathbf{e}_r(t)$ is rotated counterclockwise through an angle of $\pi/2$. The vector $\mathbf{e}_r(t)$ is called the ***radial unit vector***, and the vector $\mathbf{e}_\theta(t)$ is called the ***transverse unit vector*** (see the accompanying figure).

(b) Show that the velocity function $\mathbf{v} = \mathbf{v}(t)$ can be expressed in terms of radial and transverse components as

$$\mathbf{v} = \frac{dr}{dt}\mathbf{e}_r + r\frac{d\theta}{dt}\mathbf{e}_\theta$$

(c) Show that the acceleration function $\mathbf{a} = \mathbf{a}(t)$ can be expressed in terms of radial and transverse components as

$$\mathbf{a} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\mathbf{e}_r + \left[r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right]\mathbf{e}_\theta$$

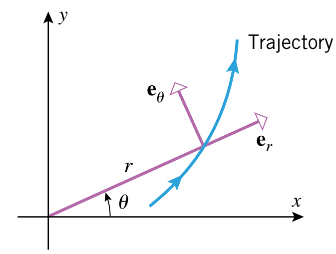


Figure Ex-13

14. As illustrated in the accompanying figure, the polar coordinates of a rocket are tracked by radar from a point that is b units from the launching pad. Show that the speed v of the rocket can be expressed in terms b, θ, and $d\theta/dt$ as

$$v = b\sec^2\theta\frac{d\theta}{dt}$$

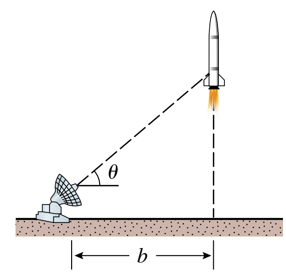


Figure Ex-14

15. Find the arc length parametrization of the line through $P(-1, 4, 3)$ and $Q(0, 2, 5)$ that has reference point P and orients the line in the direction from P to Q.

16. A player throws a ball with an initial speed of 60 ft/s at an unknown angle α with the horizontal from a point that is 4 ft above the floor of a gymnasium. Given that the ceiling of the gymnasium is 25 ft high, determine the maximum height h at which the ball can hit a wall that is 60 ft away (see the accompanying figure).

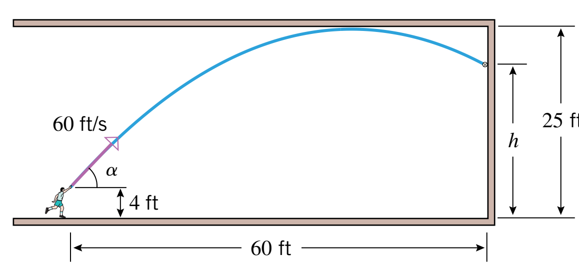


Figure Ex-16

17. Find all points on the graph of $\mathbf{r}(t) = t^3\mathbf{i} + 10t\mathbf{j} + 5t^2\mathbf{k}$ at which the tangent line is perpendicular to the tangent line at $t = 1$.

18. Solve the vector initial-value problem

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}, \quad \mathbf{r}(0) = \mathbf{r}_0$$

for the unknown vector-valued function $\mathbf{r}(t)$.

19. At time $t = 0$ a particle at the origin of an xyz-coordinate system has a velocity vector of $\mathbf{v}_0 = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$. The acceleration function of the particle is $\mathbf{a}(t) = 2t^2\mathbf{i} + \mathbf{j} + \cos 2t\mathbf{k}$.
(a) Find the position function of the particle.
(b) Find the speed of the particle at time $t = 1$.

20. Let $\mathbf{v} = \mathbf{v}(t)$ and $\mathbf{a} = \mathbf{a}(t)$ be the velocity and acceleration vectors for a particle moving in 2-space or 3-space. Show that the rate of change of its speed can be expressed as

$$\frac{d}{dt}(\|\mathbf{v}\|) = \frac{1}{\|\mathbf{v}\|}(\mathbf{v} \cdot \mathbf{a})$$