Resolução do recurso de AM2E (24/06/16)

Título da nota 05/07/2016

1. a) $y' + \frac{y}{2} = 2 + x \Leftrightarrow y' + \frac{1}{2} y = 2 + x$.

Seja $G(x) = \int \frac{1}{2}\, dx = \frac{1}{2} x$, de que resulta o fator integrante $\mu = e^{\frac{1}{2}x}$.

Temos então

$$e^{\frac{1}{2}x} y + \frac{y}{2} e^{\frac{1}{2}x} = e^{\frac{1}{2}x}(2+x) \Leftrightarrow$$

$$\left(e^{\frac{1}{2}x} y\right)' = e^{\frac{1}{2}x}(2+x) \Leftrightarrow$$

$$e^{\frac{1}{2}x} y = \int e^{\frac{1}{2}x}(2+x)\, dx \Leftrightarrow e^{\frac{1}{2}x} y = 4 \int \frac{1}{2} e^{\frac{1}{2}x} dx + \int x e^{x} dx$$

$$e^{\frac{1}{2}x} y = 4 e^{\frac{1}{2}x} + 2 \int x \tfrac{1}{2} e^{\frac{1}{2}x} dx \Leftrightarrow$$

$$e^{\frac{1}{2}x} y = 4 e^{\frac{1}{2}x} + 2\left(x e^{\frac{1}{2}x} - \int e^{\frac{1}{2}x} dx \right) \Leftrightarrow$$

$$e^{\frac{1}{2}x} y = 4 e^{\frac{1}{2}x} + 2x e^{\frac{1}{2}x} - 4 e^{\frac{1}{2}x} + C \Leftrightarrow$$

$$y = 2x + C e^{-\frac{1}{2}x}, \quad C \in \mathbb{R}.$$

Dada a condição inicial $y(0) = 0$ vem $C = 0$. Logo a solução é $y = 2x$.

b) $2y(x+1)y' = x \Leftrightarrow 2y\dfrac{dy}{dx} = \dfrac{x}{x+1} \Leftrightarrow$

$$2y\,dy = \frac{x}{x+1}\,dx \Leftrightarrow$$

$$\int 2y\,dy = \int \frac{x}{x+1}\,dx \Leftrightarrow$$

$$2\frac{y^2}{2} = \int\left(1 - \frac{1}{x+1}\right)dx$$

$$y^2 = x - \log|x+1| + C$$

$$y = \pm\sqrt{x - \log|x+1| + C}\ ,\ C \in \mathbb{R}$$

2. $y' = 3y + \sqrt[3]{y^2}$, com a substituição $y = u^3$. Temos

$3u^2 \cdot u' = 3u^3 + u^2 \Leftrightarrow \dfrac{du}{dx} = \dfrac{3u^3 + u^2}{3u^2} \Leftrightarrow \dfrac{du}{dx} = u + \dfrac{1}{3} \Leftrightarrow$

$\dfrac{1}{u+1/3}\,du = 1\,dx \Leftrightarrow \displaystyle\int \frac{1}{u+1/3}\,du = \int 1\,dx \Leftrightarrow \log\left|u+\frac{1}{3}\right| = x + C$

$$e^{\log|x+\frac{1}{3}|} = e^{x+c} \iff \left|x+\tfrac{1}{3}\right| = e^{x}\cdot e^{c}, \ c \in \mathbb{R} \iff$$

$$\iff \left|x+\tfrac{1}{3}\right| = c e^{x}, \ c>0 \iff$$

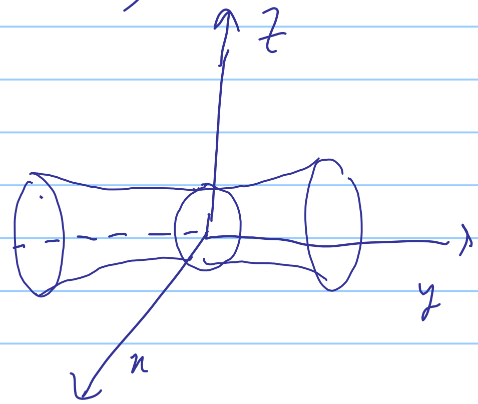
$$\iff x = c e^{x} - \tfrac{1}{3}, \ c \neq 0.$$

$$\iff y = \sqrt[3]{c e^{x} - \tfrac{1}{3}}, \ c \neq 0.$$

3

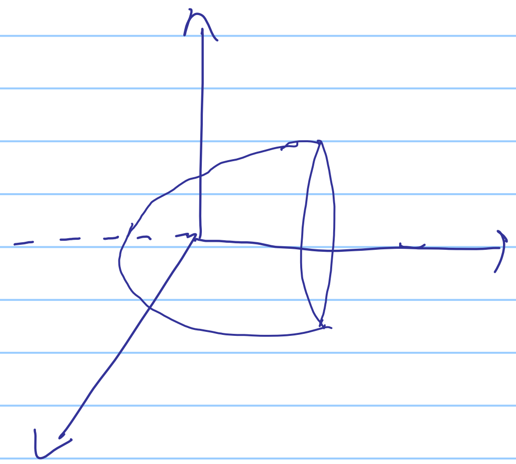
a) $9 - x^2 + \frac{y^2}{3} - z^2 = 0 \iff \frac{x^2}{9} - \frac{y^2}{(\sqrt{27})^2} + \frac{z^2}{3^2} = 1$

Hiperbolóide de 1 folha



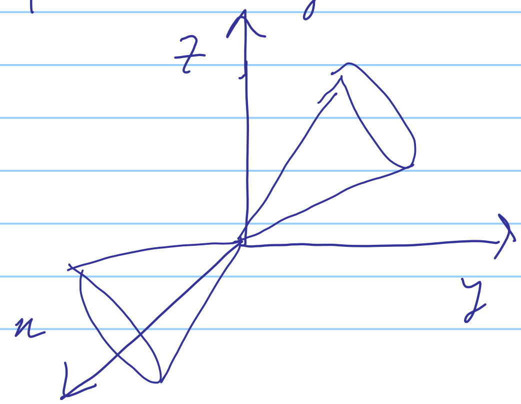
$$y - x^2 + 1 = z^2 \iff y + 1 = x^2 + z^2$$

Parabolóide circular

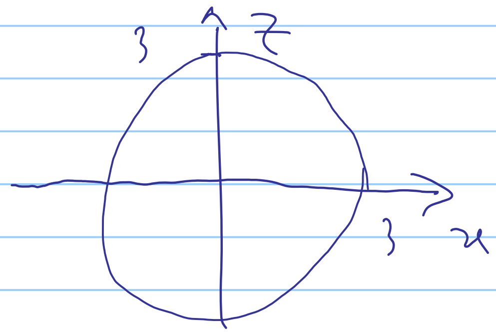


$$z^2 - \frac{x^2}{9} + \frac{y^2}{4} = 0 \iff \frac{x^2}{9} = \frac{y^2}{4} + z^2 \iff x^2 = \frac{y^2}{\left(\frac{2}{3}\right)^2} + z^2$$

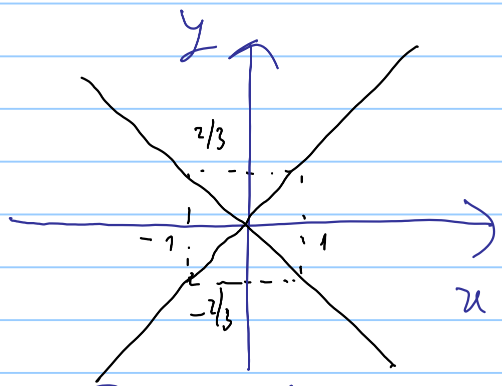
Superfície cónica dupla ao longo do eixo dos xx.



b) secção $y=0$ de S_1 $\longrightarrow$ $9 - x^2 - z^2 = 0 \Leftrightarrow x^2 + z^2 = 3^2$

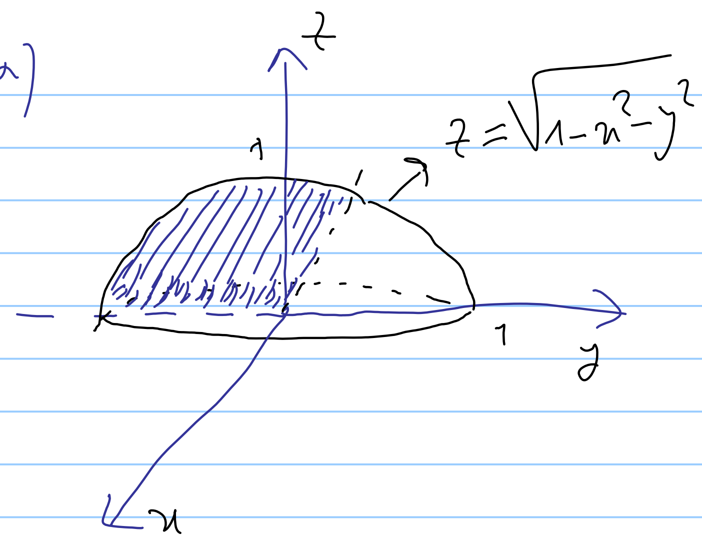


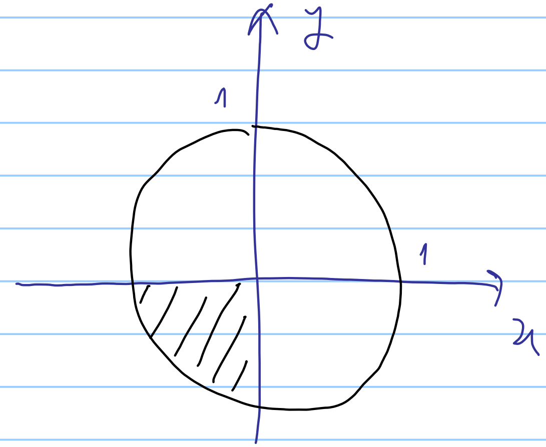
secção $z=0$ de $_3$ $\longrightarrow$ $\frac{y^2}{4} - \frac{x^2}{9} = 0 \Leftrightarrow y^2 = \frac{4}{9}x^2 \Leftrightarrow y = \pm \frac{2}{3}x$



c) $z^2 = r(\operatorname{sen}\theta - r\cos\theta) + 1$.

4. a)





"sombra" do sólido no plano XOY ($z=0$).

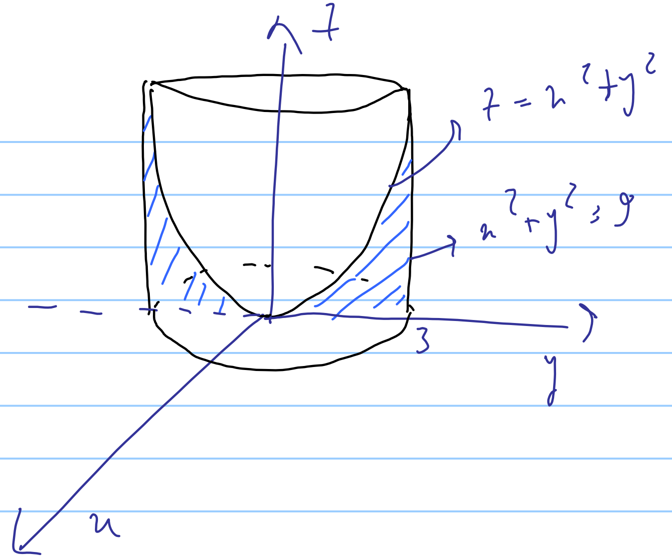
$z \geq 0 \Rightarrow \phi \in [0, \pi/2]$

$z \leq \sqrt{1-x^2-y^2} \Rightarrow \rho \in [0,1]$

$x, y \leq 0 \Rightarrow \theta \in [\pi, 3\pi/2]$.

Temos então $C = \{(\rho, \theta, \phi): 0 \leq \rho \leq 1, \pi \leq \theta \leq 3\pi/2, 0 \leq \phi \leq \pi/2\}$

b)



$\{(r,\theta,z): 0 \le r \le 3,\ 0 \le \theta \le 2\pi,\ 0 \le z \le r^2\}$

5. Temos $\frac{\partial f}{\partial x}(x,y) = e^x \cos(x+3y) - e^x \sin(x+3y) + \frac{y^2}{1+xy^2}$

e $\frac{\partial f}{\partial x}(0,0) = 1$. Por outro lado temos

$\frac{\partial f}{\partial y}(x,y) = -3e^x \sin(x+3y) + \frac{2xy}{1+xy^2}$ e $\frac{\partial f}{\partial y}(0,0) = 0$.

Temos ainda $f(0,0) = 1$.

Então a aproximação linear em torno do ponto $(0,0)$ é dada por

$$L(x,y) = f(0,0) + \frac{\partial f}{\partial x}(0,0)(x-0) + \frac{\partial f}{\partial y}(0,0)(y-0)$$

$$= 1 + x \text{. Pronto}$$

$$L(-0.02, -0.03) = 1 - 0.02 = 0.98.$$

$(e^5, \cos(\pi e^5), \operatorname{sen} \pi e^5)$

6. a) Temos

- $r(5) - r(0) = (e^5 - 1, \cos(\pi e^5) + 1, \operatorname{sen}(\pi e^5))$, vector deslocamento em $t=5$;
- $r'(t) = (e^t, -\pi e^t \operatorname{sen}(\pi e^t), \pi e^t (\cos(\pi e^t)))$, vector velocidade no instante t;
- $r'(5) = (e^5, -\pi e^5 \operatorname{sen}(\pi e^5), \pi e^5 \cos(\pi e^5))$, vector velocidade no instante $t=5$;
- $\|r'(t)\| = \sqrt{e^{2t} + \pi^2 e^{2t}}$, velocidade no instante t;
- $\|r'(5)\| = \sqrt{e^{10} + \pi^2 e^{10}}$, velocidade no instante $t=5$.

b) $r(t) = (1,-1,0) \Leftrightarrow t=0$

Uma equação da reta tangente à curva no ponto $(1,-1,0)$ é dada por

$$(x,y,z) = (1,-1,0) + \lambda r'(0) \quad , \lambda \in \mathbb{R},$$

ou seja $(x,y,z) = (1,-1,0) + \lambda(1,0,-\pi) \quad , \lambda \in \mathbb{R}.$

Ora $(0,-1,\pi) = (1,-1,0) + (-1)(1,0,-\pi)$, donde o ponto $(0,-1,\pi)$ pertence à referida reta tangente.

e) $$[0,40] \longrightarrow [0,10]$$
$$s \longmapsto t \quad , \text{ onde } t = s/4 = \varphi(s)$$

Logo $\tilde{r}(s) = r(\varphi(s)) = \left(e^{s/4}, \cos(\pi e^{s/4}), \operatorname{sen}(\pi e^{s/4})\right), \ s \in [0,40],$

7. a) Temos $\lim_{\substack{(x,y)\to(0,0)\\(x,y)\neq(0,0)}} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{\operatorname{sen}(x^2y)}{x^2+y^2} = \frac{0}{0}$.

Calculemos os limites direcionais $y = mx$.

Ora $\lim_{\substack{(x,y)\to(0,0)\\ y=mx}} \frac{\operatorname{sen}(x^2y)}{x^2+y^2} = \lim_{x\to 0} \frac{\operatorname{sen}(x^2 m x)}{x^2+m^2x^2} = \lim_{x\to 0} \frac{\operatorname{sen}(x^3m)}{x^2(1+m^2)}$

$= \lim_{x\to 0} \frac{xm}{1+m^2} \underbrace{\frac{\operatorname{sen}(x^3m)}{x^3m}}_{\to 1} = \lim_{x\to 0} \frac{xm}{1+m^2} = 0.$

Confirmemos que o limite de $f(x,y)$ no ponto $(0,0)$ é mesmo zero.

Temos $0 \leq \left| \frac{\operatorname{sen}(x^2y)}{x^2+y^2} - 0 \right| \leq \frac{|x^2y|}{x^2+y^2} \leq \frac{(x^2+y^2)\sqrt{x^2+y^2}}{x^2+y^2} = \sqrt{x^2+y^2} \xrightarrow[(x,y)\to(0,0)]{} 0$

Logo $\lim_{\substack{(x,y)\to(0,0)\\(x,y)\neq(0,0)}} f(x,y) = 0 = f(0,0)$, e assim f é contínua em $(0,0)$.

b) Temos $\frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h \to 0} \frac{f(h,0)}{h} =$

$= \lim_{h \to 0} \frac{\operatorname{sen}(h^2 \cdot 0)}{h^2} = \lim_{h \to 0} \frac{0}{h^2} = 0.$

Por outro lado $\frac{\partial f}{\partial y}(0,0) = \lim_{k \to 0} \frac{f(0,k)}{k} = \lim_{k \to 0} \frac{0}{k^2} = 0.$

Além disso vem

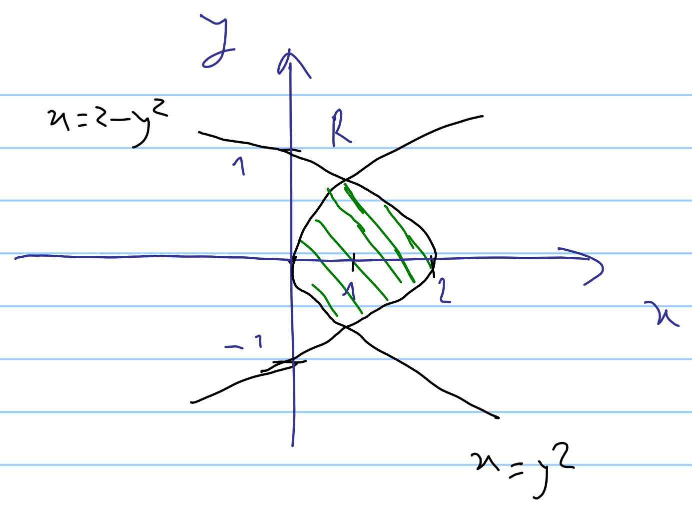
$D_{(1,1)} f(0,0) = \lim_{t \to 0} \frac{f((0,0)+t(1,1)) - f(0,0)}{t} =$

$= \lim_{t \to 0} \frac{f(t,t)}{t} = \lim_{t \to 0} \frac{\frac{\operatorname{sen}(t^3)}{2t^2}}{t} = \lim_{t \to 0} \frac{\operatorname{sen}(t^3)}{2t^3} = \frac{1}{2}.$

c) Temos grad $f(0,0) \cdot (1,1) = (0,0) \cdot (1,1) = 0.$

Como a derivada de f em $(0,0)$ segundo o vector $(1,1)$ é $\frac{1}{2} \neq 0$ concluímos que f não é diferenciável no ponto $(0,0)$.

8.

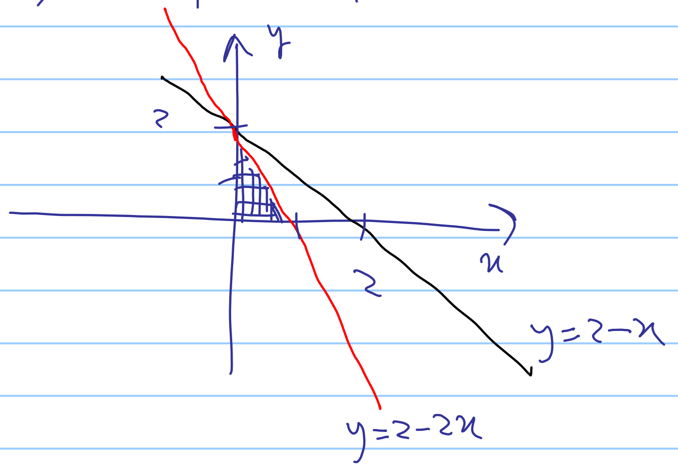


$$\begin{cases} x = y^2 \\ x = 2 - y^2 \end{cases} \iff \begin{cases} x = 1 = y \\ \text{ou} \\ x = -1 = y \end{cases}$$

$$A = \iint_R 1 \, dA = \int_{-1}^{1} \int_{y^2}^{2-y^2} 1 \, dx \, dy = \int_{-1}^{1} (2 - y^2 - y^2) \, dy = \int_{-1}^{1} (2 - 2y^2) \, dy$$

$$= 2 \cdot 2 - 2 \left[\frac{y^3}{3} \right]_{-1}^{1} = 4 - 2 \left(\frac{1}{3} + \frac{1}{3} \right) = \frac{8}{3}$$

9. O sólido A é limitado superiormente pelo plano $z = 2 - x - y$ e inferiormente pelo plano $z = x$. A "sombra" (projecção) do sólido no plano $z = 0$ tem a seguinte representação geométrica:



$z = 0 \wedge z = 2 - x - y \Rightarrow y = 2 - x$

$z = 2 - x - y \wedge z = x \Rightarrow x = \dfrac{2-y}{2}$

Donde $V(A) = \iiint_A 1 \, dV = \iint_R \int_x^{2-x-y} 1 \, dz \, dA = \int_0^1 \int_0^{2-2x} \int_x^{2-x-y} 1 \, dz$

$= \int_0^1 \int_0^{2-2x} (2 - x - y - x) \, dy \, dx = \int_0^1 \int_0^{2-2x} (2 - 2x - y) \, dy \, dx =$

$$= \int_0^1 2(2-2x) - 2x(2-2x) - \left[\frac{y^2}{2}\right]_0^{2-2x} dx$$

$$= \int_0^1 \left(4 - 4x - 4x + 4x^2 - \frac{1}{2}(2-2x)^2\right) dx =$$

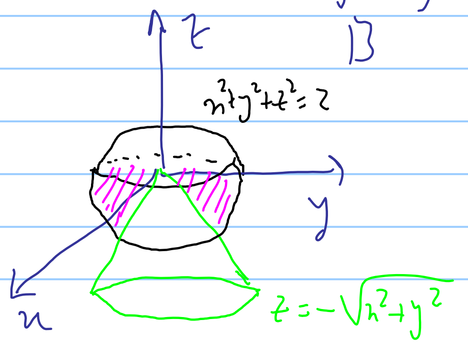
$$= \int_0^1 4 - 8x + 4x^2 - \frac{1}{2}(4 - 8x + 4x^2)\, dx =$$

$$= \int_0^1 (2 - 4x + 2x^2)\, dx = 2 - 4\left[\frac{x^2}{2}\right]_0^1 + 2\left[\frac{x^3}{3}\right]_0^1 = \frac{2}{3}$$

10.

$$\iiint_B z\, dV = \int_0^{\sqrt{2}}\int_0^{2\pi}\int_{\pi/2}^{3\pi/4} \rho \cos\phi\, \rho^2 \operatorname{sen}\phi\, d\phi\, d\theta\, d\rho = \int_0^{\sqrt{2}} \rho^3 \int_0^{2\pi}\int_{\pi/2}^{3\pi/4} \cos\phi \operatorname{sen}\phi\, d\phi\, d\theta\, d\rho$$

$$= \int_0^{\sqrt{2}} \rho^3 \int_0^{2\pi} \left[\frac{\operatorname{sen}^2\phi}{2}\right]_{\pi/2}^{3\pi/4} d\theta\, d\rho = \int_0^{\sqrt{2}} \rho^3 \pi\left(\frac{1}{2} - 1\right) d\rho = -\frac{\pi}{2}\left[\frac{\rho^4}{4}\right]_0^{\sqrt{2}} = -\frac{\pi}{2}.$$



11. Tome a composição

$$(x,y,z) \longrightarrow (\underbrace{f(x,y,z)}_{u}, \underbrace{x+y+z}_{v}, \underbrace{xyz}_{w}) \longrightarrow f(u,v,w)$$

Temos

$$\frac{\partial G}{\partial x}(x,y,z) = \frac{\partial f}{\partial u}(u,v,w)\frac{\partial u}{\partial x}(x,y,z) + \frac{\partial f}{\partial v}(u,v,w)\overbrace{\frac{\partial v}{\partial x}(x,v,w)}^{1} + \frac{\partial f}{\partial w}(u,v,w)\overbrace{\frac{\partial w}{\partial x}(x,y,z)}^{yz}.$$

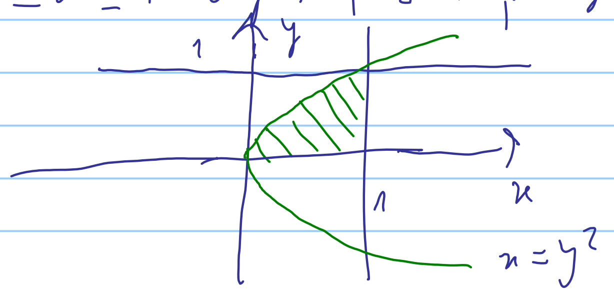
Donde

$$\frac{\partial G}{\partial x}(0,0,0) = \frac{\partial f}{\partial u}(0,0,0)\frac{\partial f}{\partial x}(0,0,0) + \frac{\partial f}{\partial v}(0,0,0).1 + \frac{\partial f}{\partial w}(0,0,0).0$$

$$= \frac{\partial f}{\partial x}(0,0,0)\frac{\partial f}{\partial x}(0,0,0) + \frac{\partial f}{\partial y}(0,0,0).1$$

$$= 2.2 + 3.1 = 7.$$

12. Tome $0 \le y \le 1$ e $y^2 \le x \le 1$ cuja representação geométrica é



$x = y^2 \Leftrightarrow y = \pm\sqrt{x}$

Então $\int_0^1 \int_{y^2}^1 \sqrt{x}\, e^{x^2}\, dx\, dy = \iint_R \sqrt{x}\, e^{x^2}\, dA =$

$$= \int_0^1 \int_0^{\sqrt{x}} \sqrt{x}\, e^{x^2}\, dy\, dx = \int_0^1 x e^{x^2}\, dx = \frac{1}{2}\left[e^{x^2}\right]_0^1 = \frac{1}{2}(e-1).$$

13. O conjunto D é um conjunto limitado, e fechado logo pelo teorema de Weierstrass f tem máximo e mínimo em D.

Por outro lado D é um conjunto definido pela igualdade $x^2+y^2+2z^2=8$

Determinemos os "candidatos" a extremo através do método dos multiplicadores de Lagrange.

Considere-se $L(x,y,z,\lambda) = x+y-\lambda(x^2+y^2+2z^2-8)$.

Temos

$$\begin{cases} \dfrac{\partial L}{\partial x} = 0 \\ \dfrac{\partial L}{\partial y} = 0 \\ \dfrac{\partial L}{\partial z} = 0 \\ x^2+y^2+2z^2=8 \end{cases} \Leftrightarrow \begin{cases} 1-2\lambda x = 0 \\ 1-2\lambda y = 0 \\ -4\lambda z = 0 \\ x^2+y^2+2z^2=8 \end{cases} \Leftrightarrow \begin{cases} x=y \\ z=0 \\ \text{---} \\ \text{---} \end{cases} \Leftrightarrow \begin{cases} (x,y,z)=(2,2,0) \\ \text{ou} \\ (x,y,z)=(-2,-2,0). \end{cases}$$

Como $f(2,2,0)=4$ e $f(-2,-2,0)=-4$.

Assim o máximo é 4 e o mínimo é -4.