Uma resolução do 2º teste - 1º semestre 2016.

1 a) Temos $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2(x+y)}{x^2+y^2} = \frac{0}{0}$.

Estudemos os limites direcionais da forma $y=mx$, $x\in\mathbb{R}$.

Ora $\lim_{\substack{(x,y)\to(0,0)\\ y=mx}} \frac{x^2(x+y)}{x^2+y^2} = \lim_{x\to 0} \frac{x^2(x+mx)}{x^2+m^2x^2} = \lim_{x\to 0}\frac{x+mx}{1+m^2} = \frac{0}{1+m^2} = 0.$

Provemos que o limite é de facto zero.

Tem-se

$$0 \le \left|\frac{x^2(x+y)}{x^2+y^2}\right| = \frac{x^2|x+y|}{x^2+y^2} \le \frac{(x^2+y^2)|x+y|}{x^2+y^2} = |x+y| \underset{(x,y)\to(0,0)}{\longrightarrow} 0.$$

Portanto $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

Vejamos agora nos pontos (a,b) tais que $ab\neq 0$. Temos

$\lim_{(x,y)\to(a,b)} \frac{x^2(x+y)}{x^2+y^2} = \frac{a^2(a+b)}{a^2+b^2} \in \mathbb{R}$ pois dado que

$a\neq 0$ ou $b\neq 0$ vem $a^2+b^2\neq 0$.

b) Temos $\lim_{\substack{(x,y)\to(1,1)\\ x\neq y}} g(x,y) = \lim_{(x,y)\to(1,1)} x+y = 2$ e

$$\lim_{\substack{(x,y)\to(1,1)\\ x=y}} g(x,y) = \lim_{(x,y)\to(1,1)} x+1 = 2$$

Como $\{(x,y)\in\mathbb{R}^2 : x\neq y\} \cup \{(x,y)\in\mathbb{R}^2 : x=y\} = \mathbb{R}^2$

e os limites são iguais concluímos que $\lim_{(x,y)\to(1,1)} f(x,y) = 2 = f(1,1)$

e portanto f é contínua em $(1,1)$.

Quanto à diferenciabilidade temos

$$\frac{\partial g}{\partial x}(1,1) = \lim_{h\to 0} \frac{g(1+h,1) - g(1,1)}{h} = \lim_{h\to 0} \frac{1+h+1-2}{h} =$$

$$= \lim_{h\to 0} \frac{h}{h} = 1.$$

Por outro lado temos

$$\frac{\partial g}{\partial y}(1,1) = \lim_{k\to 0} \frac{g(1,1+k) - g(1,1)}{k} = \lim_{k\to 0} \frac{1+1+k-2}{k} = 1.$$

Ora temos

$$\lim_{(h,k)\to(0,0)} \frac{g(1+h,1+k) - g(1,1) - h\frac{\partial g}{\partial x}(1,1) - k\frac{\partial g}{\partial y}(1,1)}{\sqrt{h^2+k^2}} =$$

$$\lim_{(h,k)\to(0,0)} \frac{g(1+h,1+k) - 2 - h - k}{\sqrt{h^2+k^2}}.$$

Considerando limites relativos nem

$$\lim_{\substack{(h,k)\to(0,0)\\ h\neq k}} \frac{g(1+h,1+k) - 2 - h - k}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{1+h+1+k-2-h-k}{\sqrt{h^2+k^2}} =$$

$$= \lim_{(h,k)\to(0,0)} \frac{0}{\sqrt{h^2+k^2}} = 0 \quad e$$

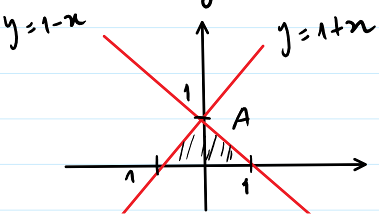
$$\lim_{\substack{(h,k)\to(0,0)\\ h= k}} \frac{g(1+h,1+k) - g(1,1) - h - k}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{1+h+1-1-h-k}{\sqrt{h^2+k^2}} =$$

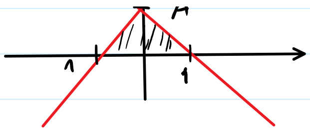
$$= \lim_{(h,k)\to(0,0)} \frac{1-k}{\sqrt{h^2+k^2}} = \frac{1}{0^+} = +\infty.$$

Logo g não é diferenciável no ponto $(1,1)$.

2.

a) Representemos geometricamente o conjunto A.





Então temos

$$\text{Área } A = \iint_A 1 \, dA = \int_0^1 \int_{y-1}^{1-y} 1 \, dx \, dy =$$

$$= \int_0^1 (2-2y) \, dy = 2 - 2\left[\frac{y^2}{2}\right]_0^1 = 1.$$

b) Temos

$$\iint_A 3e^x \, dA = \int_0^1 \int_{y-1}^{1-y} 3e^x \, dx \, dy = 3\int_0^1 \left[e^x\right]_{y-1}^{1-y} dy$$

$$= 3\int_0^1 \left(e^{1-y} - e^{y-1}\right) dy = 3\left(-\left[e^{1-y}\right]_0^1 - \left[e^{y-1}\right]_0^1\right)$$

$$= 3\left(-(1-e) - (e - e^{-1})\right) = -3 + \frac{3}{e} + 2e$$

3.

a) A equação do plano tangente ao gráfico de g no ponto referido é

$$z = g(1,1) + \frac{\partial g}{\partial x}(1,1)(x-1) + \frac{\partial g}{\partial y}(1,1)(y-1)$$

Ora $\frac{\partial g}{\partial x}(x,y) = \dfrac{1}{\frac{xy}{x+y}} = \dfrac{y(x+y) - 1(xy)}{(x+y)^2}$

$$= \frac{y^2}{(xy)(x+y)} \quad \text{donde } \frac{\partial g}{\partial x}(1,1) = \frac{1}{2} \text{ e analogamente}$$

$$\frac{\partial g}{\partial y}(x,y) = \frac{x^2}{xy(x+y)} \quad \text{donde } \frac{\partial g}{\partial y}(1,1) = \frac{1}{2}.$$

Logo a equação do plano tangente é

$$z = \log\left(\frac{1}{2}\right) + \frac{1}{2}(x-1) + \frac{1}{2}(y-1).$$

b) Temos

$$g(0.9, 1.1) \simeq \log\left(\frac{1}{2}\right) + (0.9-1)\frac{1}{2} + (1.1-1)\frac{1}{2} =$$

$$= \log\left(\frac{1}{2}\right).$$

4.

a) Seja $h = g\left(\frac{x}{y}, \frac{z}{x}\right)$

Esquematicamente temos

$$(x,y,z) \longrightarrow \left(\underbrace{\frac{x}{y}}_{u}, \underbrace{\frac{z}{x}}_{v}\right) \longrightarrow g(u,v)$$

Temos então

$$\frac{\partial h}{\partial x}(x,y,z) = \frac{\partial g}{\partial u}(u,v) \cdot \frac{\partial}{\partial x}\left(\frac{x}{y}\right) + \frac{\partial g}{\partial v}(u,v) \cdot \frac{\partial}{\partial x}\left(\frac{z}{x}\right)$$

$$= \frac{\partial g}{\partial u}(u,v) \cdot \frac{1}{y} + \frac{\partial g}{\partial v}(u,v)\left(-\frac{z}{x^2}\right),$$

$$\frac{\partial g}{\partial y}(x,y,z) = \frac{\partial g}{\partial u}(u,v) \cdot \frac{\partial}{\partial y}\left(\frac{x}{y}\right) + \frac{\partial g}{\partial v}(u,v) \cdot \frac{\partial}{\partial y}\left(\frac{z}{x}\right)$$

$$= \frac{\partial g}{\partial u}(u,v) \cdot \left(-\frac{x}{y^2}\right) \quad \text{e}$$

$$\frac{\partial g}{\partial z}(x,y,z) = \frac{\partial g}{\partial u}(u,v) \cdot \frac{\partial}{\partial z}\left(\frac{x}{y}\right) + \frac{\partial g}{\partial v}(u,v) \cdot \frac{\partial}{\partial z}\left(\frac{z}{x}\right)$$

$$= \frac{\partial g}{\partial v}(u,v) \cdot \frac{1}{x}.$$

Assim o gradiente de g no ponto $(1,1,0)$ é dado por

$$\left(\frac{\partial g}{\partial u}(1,0) \cdot 1 + \frac{\partial g}{\partial v}(1,0)(-1), \frac{\partial g}{\partial v}(1,0)(-1), \frac{\partial g}{\partial v}(1,0) \cdot 1\right)$$

$$= (2-1, -2, 2) = (1,-1,2),$$

dado que $\operatorname{grad} g(1,0) = \left(\frac{\partial g}{\partial u}(1,0), \frac{\partial g}{\partial v}(1,0)\right) = (2,1)$ e que quando $(x,y,z) = (1,1,0)$ se tem $(u,v) = (1,0)$.

b) Temos

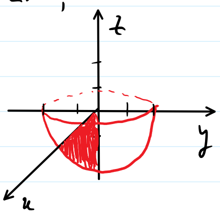
$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=x\left(2x\,g\left(\frac{x}{y},\frac{z}{x}\right)+x^2\left(\frac{\partial g}{\partial u}\,\frac{1}{y}+\frac{\partial g}{\partial v}\left(-\frac{z}{x^2}\right)\right)\right)$$

$$+\,yx^2\left(\frac{\partial g}{\partial u}\cdot\left(-\frac{x}{y^2}\right)\right)+zx^2\left(\frac{\partial g}{\partial v}\cdot\frac{1}{x}\right)=$$

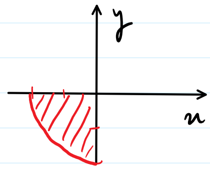
$$=2x^2g\left(\frac{x}{y},\frac{z}{x}\right)+\cancel{\frac{x^3}{y}\frac{\partial g}{\partial u}}-\cancel{zx\frac{\partial g}{\partial v}}-\cancel{\frac{x^3}{y}\frac{\partial g}{\partial u}}+\cancel{zx\frac{\partial g}{\partial v}}$$

$$=2f(x,y,z).$$

5 a) Esboço do sólido M



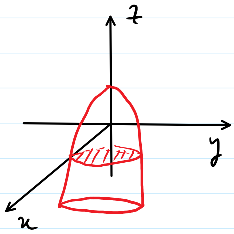
Esboço da "sombra" do sólido M no plano OXY.

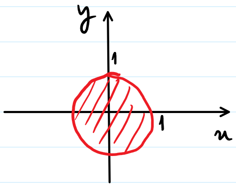


Temos portanto

$$V(M) = \iiint_M 1 \, dV = \int_0^2 \int_{\pi}^{3\pi/2} \int_{\pi/2}^{\pi} \rho^2 \operatorname{sen}\phi \, d\phi \, d\theta \, d\rho = \int_0^2 \rho^2 \int_{\pi}^{3\pi/2} \left[-\cos\phi\right]_{\pi/2}^{\pi} d\theta \, d\rho = \frac{\pi}{2}\left[\frac{\rho^3}{3}\right]_0^2 = \frac{4}{3}\pi .$$

b)





$$\begin{cases} z=-1 \\ z = 1-2x^2-2y^2 \end{cases} \Leftrightarrow \begin{cases} z=-1 \\ x^2+y^2=1 \end{cases}$$

Temos

$$\iiint_N \sqrt{x^2+y^2} \, dV = \int_0^{2\pi}\int_0^1\int_{-1}^{1-2r^2} r \cdot r \, dz \, d\theta \, dr$$

$$= \int_0^{2\pi}\int_0^1 r^2\left(1-2r^2+1\right) d\theta \, dr = \int_0^{2\pi}\int_0^1 2r^2-2r^4 \, dr \, d\theta$$

$$= 4\pi\left(\left[\frac{r^3}{3}\right]_0^1 - \left[\frac{r^5}{5}\right]_0^1\right) = 4\pi\left(\frac{1}{3}-\frac{1}{5}\right) = \frac{8}{15}\pi .$$

6. Dado que f é uma função contínua em M que é um conjunto limitado e fechado concluímos que f tem um máximo e mínimo absoluto em M.

Usamos o método dos multiplicadores de Lagrange para determinar os "candidatos" a máx. e mín. absolutos em M.

Seja $h(x,y,z,\lambda) = f(x,y,z) - \lambda\left((x-3)^2+y^2-2\right)$

$= x^2+2xy+y^2 - \lambda\left((x-3)^2+y^2-2\right)$

Temos

$$\begin{cases} \frac{\partial h}{\partial x} = 0 \\ \frac{\partial h}{\partial y} = 0 \\ (x-3)^2+y^2=2 \end{cases} \Leftrightarrow \begin{cases} 2x+2y-\lambda . 2(x-3) = 0 \\ 2y+2x-\lambda . 2y = 0 \\ (x-3)^2+y^2=2 \end{cases} \Leftrightarrow$$

$$\begin{cases} 2\lambda(x-3) = 2\lambda y \\ \text{——} \\ \text{——} \end{cases} \underset{\lambda \neq 0}{\Leftrightarrow} \begin{cases} x-3 = y \\ \text{——} \\ \text{——} \end{cases} \Leftrightarrow \begin{cases} \text{——} \\ \text{——} \\ y^2+y^2=2 \end{cases} \Leftrightarrow \begin{cases} \text{——} \\ \text{——} \\ y^2=1 \end{cases}$$

$\Leftrightarrow \begin{cases} \_\_\_\_ \\ \_\_\_\_ \\ y = \pm 1 \end{cases}$

Se $y=1$ vem $x=4$ e obtemos o ponto $(4,1)$.

Se $y=-1$ vem $x=2$ e obtemos o ponto $(2,-1)$.

Assim os pontos onde podemos obter o máximo e o mínimo de f em M são $(4,1)$ e $(2,-1)$. Comparemos as suas imagens. Temos

$f(4,1) = (4+1)^2 = 25$ e

$f(2,-1) = (2-1)^2 = 1^2 = 1$.

Assim o máximo é 25 e o mínimo é 1 que são obtidos respetivamente nos pontos $(4,1)$ e $(2,-1)$.