Análise Matemática II E

Exame de Recurso

(28/06/2019)

Nota: Esta é apenas uma proposta de resolução de entre muitas outras possibilidades.

1

$$y^3 y' (x^2+4) = (x+1)(1+y^4)$$

$$\Leftrightarrow \quad \frac{y^3}{1+y^4}\, y' = \frac{x+1}{x^2+4}$$

Sendo a equação de variáveis separáveis temos:

$$\int \frac{y^3}{1+y^4}\, y' \, dx = \int \frac{x+1}{x^2+4}\, dx$$

$$\Leftrightarrow \int \frac{y^3}{1+y^4}\, dy = \int \frac{x}{x^2+4} + \frac{1/2}{\left(\frac{x}{2}\right)^2+1}\, \frac{1}{2}\, dx$$

$$\Leftrightarrow \frac{1}{4}\log(1+y^4) + c_1 = \frac{1}{2}\log(x^2+4) + \operatorname{arctg}(x/2)\,\frac{1}{2} + c_2,$$

$$c_1, c_2 \in \mathbb{R}$$

$$\Leftrightarrow \log(1+y^4) = 2\log(x^2+4) + 2\operatorname{arctg}(x/2) + c_3,$$

$$c_3 \in \mathbb{R}$$

$$\Leftrightarrow 1+y^4 = c_4 e^{2\log(x^2+4) + 2\operatorname{arctg}(x/2)}, \quad c_4 \in \mathbb{R}^+$$

$$\Leftrightarrow y^4 = c_4 e^{2\log(x^2+4) + 2\operatorname{arctg}(x/2)} - 1, \quad c_4 \in \mathbb{R}^+$$

2

a) $Y(x)$ - tempo, em milisegundos, que leva a executar a simulação da vida de uma estrela com x giga-anos, recorrendo ao código computacional CESAM

$$Y' = kY \Leftrightarrow Y' - kY = 0 \qquad \mu(x) = e^{\int -k\,dx} = e^{-kx}$$

$$\Leftrightarrow e^{-kx} Y' - k e^{-kx} Y = 0$$

$$\Leftrightarrow \frac{d}{dx}\left(e^{-kx} Y\right) = 0 \Leftrightarrow$$

$$\Leftrightarrow e^{-kx} Y = c, \; c \in \mathbb{R} \Leftrightarrow$$

$$\Leftrightarrow Y = c\, e^{kx}, \; c \in \mathbb{R}$$

b) $\begin{cases} Y(1) = 10 \\ Y(2) = 100 \end{cases} \Leftrightarrow \begin{cases} 10 = c\,e^{k} \\ 100 = c\,e^{2k} \end{cases} \underset{\substack{\text{se} \\ c \neq 0}}{\Leftrightarrow} \begin{cases} \frac{10}{c} = e^{k} \\ 100 = c\left(e^{k}\right)^2 \end{cases}$

$\Leftrightarrow \begin{cases} - \\ 100 = c\,\frac{100}{c^2} \end{cases} \Leftrightarrow \begin{cases} - \\ 100 = \frac{100}{c} \end{cases} \Leftrightarrow \begin{cases} - \\ 100c - 100 = 0 \end{cases}$

$\Leftrightarrow \begin{cases} e^{k} = 10 \\ c = 1 \end{cases} \Leftrightarrow \begin{cases} k = \log 10 \\ c = 1 \end{cases}$

Se $c = 0$ $\begin{cases} 10 = 0 \\ 100 = 0 \end{cases}$ impossível

③

a) $D = \{ (x,y) \in \mathbb{R}^2 : \dfrac{\log(5-x^2-y^2)}{|x-1|} \geq 0 \ \wedge$

$\wedge\ 5-x^2-y^2 > 0 \ \wedge\ |x-1| \neq 0 \ \wedge\ x^2+y^2 \neq 0 \}$

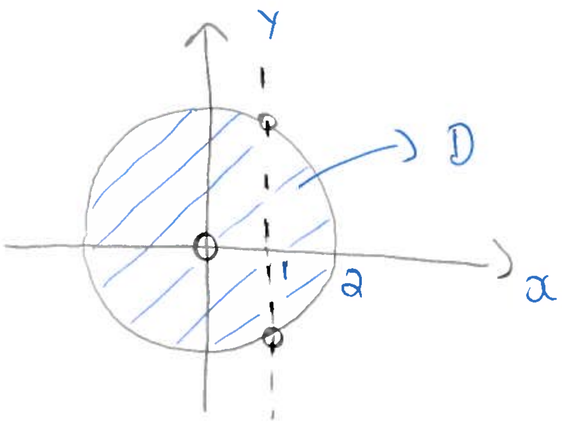
$\dfrac{\log(5-x^2-y^2)}{|x-1|} \geq 0 \iff \log(5-x^2-y^2) \geq 0 \ \wedge\ x \neq 1$

$\iff 5-x^2-y^2 \geq 1 \ \wedge\ x \neq 1 \iff x^2+y^2 \leq 4 \ \wedge\ x \neq 1$

$5-x^2-y^2 > 0 \iff x^2+y^2 < 5$

$|x-1| \neq 0 \iff x \neq 1$

$x^2+y^2 \neq 0 \iff (x,y) \neq (0,0)$



b) $\operatorname{int}(D) = \{ (x,y) \in \mathbb{R}^2 : x^2+y^2 < 4 \ \wedge\ x \neq 1 \ \wedge\ (x,y) \neq (0,0) \}$

$\operatorname{fr}(D) = \{ (x,y) \in \mathbb{R}^2 : x^2+y^2 = 4 \ \vee\ (x,y) = (0,0) \ \vee$

$\vee\ (x = 1 \ \wedge\ x^2+y^2 < 4) \}$

D não é aberto pois $D \neq \text{int}(D)$. Por exemplo, $(0,2) \in D$ mas $(0,2) \notin \text{int}(D)$

D não é fechado pois $D \neq \bar{D}$. Por exemplo, $(0,0) \in \bar{D}$ mas $(0,0) \notin D$

c)

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \sqrt{\frac{\log(5-x^2-y^2)}{|x-1|}} + \frac{x^2-y^2}{x^2+y^2}\,\text{sen}\, y$$

$$\underset{*}{=} \sqrt{\log 5} + 0 = \sqrt{\log 5}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}\,\text{sen}\, y = \lim_{\substack{\rho\to 0^+\\ \theta\in[0,2\pi[}} \frac{\rho^2\cos^2\theta - \rho^2\text{sen}^2\theta}{\rho^2}\,\text{sen}(\rho\,\text{sen}\theta)$$

$$\begin{cases} x = \rho\cos\theta & \rho > 0 \\ y = \rho\,\text{sen}\theta & \theta\in[0,2\pi[\end{cases}$$

$$= \lim_{\substack{\rho\to 0^+\\ \theta\in[0,2\pi[}} (\cos^2\theta - \text{sen}^2\theta)\,\text{sen}(\rho\,\text{sen}\theta)$$

$= 0$ pois $\text{sen}\theta$ é limitada

e $\lim_{\rho\to 0^+} \rho = 0$ logo $\lim_{\rho\to 0^+} \rho\,\text{sen}\theta = 0$. Assim

$\lim_{\rho\to 0^+} \text{sen}(\rho\,\text{sen}\theta) = 0$. Como $\cos^2\theta - \text{sen}^2\theta$ é limitada

temos que $\lim_{\rho\to 0^+} (\cos^2\theta - \text{sen}^2\theta)\,\text{sen}(\rho\,\text{sen}\theta) = 0$

Como o limite existe, é possível prolongar f por continuidade a $(0,0)$, sendo a função prolongamento definida por ⑤

$$\bar{f}(x,y) = \begin{cases} \sqrt{\dfrac{\log(5-x^2-y^2)}{|x-1|}} + \dfrac{x^2-y^2}{x^2+y^2}\operatorname{sen} y, & \text{se } (x,y)\in D \\ \sqrt{\log 5}, & \text{se } (x,y)=(0,0) \end{cases}$$

④

a) Seja $u_1 \neq 0$ e $u_2 \neq 0$

$$g'_{(u_1,u_2)}(0,0) = \lim_{t\to 0} \frac{g(tu_1,tu_2)-g(0,0)}{t}$$

$$= \lim_{t\to 0} \frac{\cancel{t}\, u_1 \dfrac{e^{t^2u_1u_2}-1}{t^2(u_1^2+u_2^2)}}{\cancel{t}} = \lim_{t\to 0} u_1 \frac{e^{t^2u_1u_2}-1}{t^2u_1u_2}\,\frac{u_1u_2}{u_1^2+u_2^2}$$

$$= \frac{u_1^2u_2}{u_1^2+u_2^2} \quad \text{pois } \lim_{a\to 0}\frac{e^a-1}{a}=1$$

Seja $u_1 \neq 0$

$$g'_{(u_1,0)}(0,0) = \lim_{t\to 0}\frac{g(tu_1,0)-g(0,0)}{t} =$$

$$= \lim_{t\to 0}\frac{tu_1\dfrac{1-1}{t^2u_1^2}}{t} = \lim_{t\to 0} 0 = 0$$

Seja $u_2 \neq 0$

$$g'_{(0,u_2)}(0,0) = \lim_{t \to 0} \frac{g(0,tu_2) - g(0,0)}{t} =$$

$$= \lim_{t \to 0} \frac{0 \, \frac{1-1}{t^2 u_2^2}}{t} = \lim_{t \to 0} 0 = 0$$

b)

$$\nabla g(0,0) = \begin{bmatrix} g'_{(1,0)}(0,0) \\ g'_{(0,1)}(0,0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Se g fosse diferenciável em (0,0) então

$$g'_{(u_1,u_2)}(0,0) = \nabla g(0,0)^T \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} =$$

$$= [0 \;\; 0] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0 \, , \; \forall (u_1,u_2) \in \mathbb{R}^2$$

Pela alínea anterior sabemos que, por exemplo,

$$g'_{(1,1)}(0,0) = \frac{1}{2} \neq 0$$

Logo g não é diferenciável em (0,0).

5 $\qquad\qquad$ 7

$$\mathbb{R}^2 \to \mathbb{R} \to \mathbb{R}$$
$$(x,y) \to r \to z$$

$$\frac{dz}{dx} = \frac{dz}{dr}\frac{dr}{dx} = \frac{dz}{dr}\frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{dz}{dy} = \frac{dz}{dr}\frac{\partial r}{\partial y} = \frac{dz}{dr}\frac{y}{\sqrt{x^2+y^2}}$$

$$\frac{d^2z}{dx^2} = \frac{d^2z}{dr^2}\frac{dr}{dx}\frac{x}{\sqrt{x^2+y^2}} + \frac{dz}{dr}\frac{\sqrt{x^2+y^2} - x\frac{x}{\sqrt{x^2+y^2}}}{x^2+y^2}$$

$$= \frac{d^2z}{dr^2}\frac{x^2}{x^2+y^2} + \frac{dz}{dr}\frac{\cancel{x^2}+y^2-\cancel{x^2}}{(x^2+y^2)\sqrt{x^2+y^2}} =$$

$$= \frac{d^2z}{dr^2}\frac{x^2}{x^2+y^2} + \frac{dz}{dr}\frac{y^2}{(x^2+y^2)\sqrt{x^2+y^2}}$$

$$\frac{d^2z}{dy^2} = \frac{d^2z}{dr^2}\frac{dr}{dy}\frac{y}{\sqrt{x^2+y^2}} + \frac{dz}{dr}\frac{\sqrt{x^2+y^2} - y\frac{y}{\sqrt{x^2+y^2}}}{x^2+y^2} =$$

$$= \frac{d^2z}{dr^2}\frac{y^2}{x^2+y^2} + \frac{dz}{dr}\frac{x^2+\cancel{y^2}-\cancel{y^2}}{(x^2+y^2)\sqrt{x^2+y^2}}$$

$$= \frac{d^2z}{dr^2}\frac{y^2}{x^2+y^2} + \frac{dz}{dr}\frac{x^2}{(x^2+y^2)\sqrt{x^2+y^2}}$$

Se $$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0 \Leftrightarrow$$

$$\Leftrightarrow \frac{d^2 z}{d x^2} + \frac{dz}{dx}\frac{1}{\sqrt{x^2+y^2}} = 0 \Leftrightarrow$$

$$\Leftrightarrow x\frac{d^2 z}{d x^2} + \frac{dz}{dx} = 0$$

6) Seja $e_i = (0,\dots,0,1,0,\dots,0)$, com $i \in \{1,\dots,m\}$.
$\hookrightarrow$ posição i

Como $f'_{\vec{u}}(x_*) \geq 0$, $\forall \vec{u} \in \mathbb{R}^m$ temos que

$$f'_{e_i}(x_*) \geq 0 \text{ e } f'_{-e_i}(x_*) \geq 0, \ \forall i \in \{1,\dots,m\}$$

Seja $i \in \{1,\dots,m\}$.

$$f'_{-e_i}(x_*) = \lim_{t\to 0} \frac{f(x_* + t(-e_i)) - f(x_*)}{t} =$$

$$= \lim_{t\to 0} \frac{f(x_* - te_i) - f(x_*)}{t} =$$

$$= -\lim_{-t\to 0} \frac{f(x_* - te_i) - f(x_*)}{-t} = -f'_{e_i}(x^*)$$

Logo

$$f'_{e_i}(x_*) \geq 0 \text{ e } -f'_{e_i}(x_*) \geq 0, \ \forall i \in \{1,\dots,m\}$$

Assim

$$\frac{\partial f}{\partial x_i}(x_*) = f'_{e_i}(x_*) = 0, \ \forall i \in \{1,\dots,m\}$$

Donde

$$\nabla f(x_*) = 0$$

f) $f \in C^2(D)$, com $D = \{(x,y) \in \mathbb{R}^2 : y \neq 0\}$

$$\nabla f(x,y) = \begin{bmatrix} \frac{1}{y} - y \\ -\frac{x}{y^2} - x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \frac{1}{y} - y = 0 \\ -\frac{x}{y^2} - x = 0 \end{cases} \Leftrightarrow \begin{cases} \frac{1-y^2}{y} = 0 \\ - \end{cases} \Leftrightarrow \begin{cases} y = \pm 1 \\ - \end{cases}$$

Se $y = 1$ ou $y = -1$

$$-x - x = 0 \Leftrightarrow -2x = 0 \Leftrightarrow x = 0$$

Pontos estacionários

$$(0,1) \text{ e } (0,-1)$$

$$\text{Hess} f(x,y) = \begin{bmatrix} 0 & -\frac{1}{y^2} - 1 \\ -\frac{1}{y^2} - 1 & \frac{2x}{y^3} \end{bmatrix}$$

$$|\text{Hess} f(0,\pm 1)| = \begin{vmatrix} 0 & -2 \\ -2 & 0 \end{vmatrix} = -4 < 0 \text{ logo}$$

$(0,1)$ e $(0,-1)$ são pontos de sela (não são extremos de f).

8

a) Seja
$$f(x,y,z) = \arcsin(3xy) + \cos(zy^2)$$
Começamos por verificar que f satisfaz as condições do teorema da função implícita numa vizinhança de $(0,1,\pi/2)$.

1. $f(0,1,\pi/2) = \arcsin 0 + \cos \pi/2 = 0$

2.
$$\left.\begin{aligned} \frac{\partial f}{\partial x}(x,y,z) &= \frac{3y}{\sqrt{1-9x^2y^2}} \\ \frac{\partial f}{\partial y}(x,y,z) &= \frac{3x}{\sqrt{1-9x^2y^2}} - \sin(zy^2)\, 2zy \\ \frac{\partial f}{\partial z}(x,y,z) &= -\sin(zy^2)\, y^2 \end{aligned}\right\} \text{funções contínuas em}$$

$$D = \{(x,y,z) \in \mathbb{R}^3 : -1 \leq 3xy \leq 1\}.$$

Como $3 \times 0 \times 1 = 0$ temos que $f \in C^1$ numa vizinhança de $(0,1,\pi/2)$.

3. $$\frac{\partial f}{\partial y}(0,1,\pi/2) = -\sin(\pi/2)\,\pi = -\pi \neq 0$$

Pelo Teorema da função implícita concluímos que existe U vizinhança de $(0, \pi/2)$ e V vizinhança de 1 e $\phi : U \to V$ tal que

$$\forall (x,z) \in U, \forall y \in V, \; f(x,y,z) = 0 \Leftrightarrow y = \phi(x,z)$$

Mais $\phi \in C^1(U)$ e

$$\frac{\partial \phi}{\partial x}(x,z) = - \frac{\frac{\partial f}{\partial x}(x,y,z)}{\frac{\partial f}{\partial y}(x,y,z)} \quad \text{logo} \quad \frac{\partial \phi}{\partial x}(0, \pi/2) = - \frac{\frac{3}{\sqrt{1-0}}}{-\pi} = \frac{3}{\pi}$$

$$\frac{\partial \phi}{\partial z}(x,z) = - \frac{\frac{\partial f}{\partial z}(x,y,z)}{\frac{\partial f}{\partial y}(x,y,z)} \quad \text{logo} \quad \frac{\partial \phi}{\partial z}(0, \pi/2) = \frac{+1}{-\pi} = -\frac{1}{\pi}$$

b)

Como $\phi \in C^1(U)$ sabemos que ϕ é diferenciável em $(0, \pi/2)$. Assim

$$y'_{\vec{u}}(0, \pi/2) = \nabla \phi (0, \pi/2)^T \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{\pi} & -\frac{1}{\pi} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} =$$

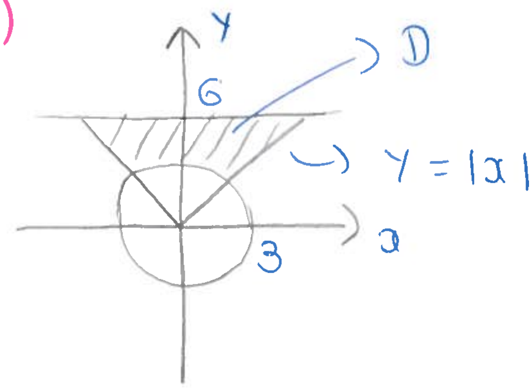
$$= \frac{3}{\pi} u_1 - \frac{1}{\pi} u_2 = 0 \Leftrightarrow u_2 = 3u_1$$

A família de vectores pretendida será então

$$(u_1, \; 3u_1), \text{ com } u_1 \in \mathbb{R}$$

9

a)



$$\text{Área} = \iint_D 1 \, dx\, dy =$$

$$= \int_{\pi/4}^{3\pi/4} \int_{3}^{\frac{6}{\operatorname{sen}\theta}} \rho \, d\rho \, d\theta =$$

$$|Jac| = \rho$$

$$\begin{cases} x = \rho\cos\theta \\ y = \rho \operatorname{sen}\theta \end{cases} \qquad \frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}, \quad 3 \leq \rho \leq \frac{6}{\operatorname{sen}\theta}$$

$$y = x \Leftrightarrow \rho \operatorname{sen}\theta = \rho\cos\theta \Leftrightarrow \underset{\rho \neq 0}{\Leftrightarrow} \operatorname{tg}\theta = 1 \Rightarrow \theta = \pi/4$$

$$y = -x \Leftrightarrow \rho \operatorname{sen}\theta = -\rho\cos\theta \Leftrightarrow \underset{\rho \neq 0}{\Leftrightarrow} \operatorname{tg}\theta = -1 \Rightarrow \theta = 3\pi/4$$

$$y = 6 \Leftrightarrow \rho \operatorname{sen}\theta = 6 \Leftrightarrow \rho = \frac{6}{\operatorname{sen}\theta}$$

b)

$$= \int_{\pi/4}^{3\pi/4} \left[\frac{\rho^2}{2}\right]_{3}^{\frac{6}{\operatorname{sen}\theta}} d\theta =$$

$$= \int_{\pi/4}^{3\pi/4} \frac{18}{\operatorname{sen}^2\theta} - \frac{9}{2} \, d\theta =$$

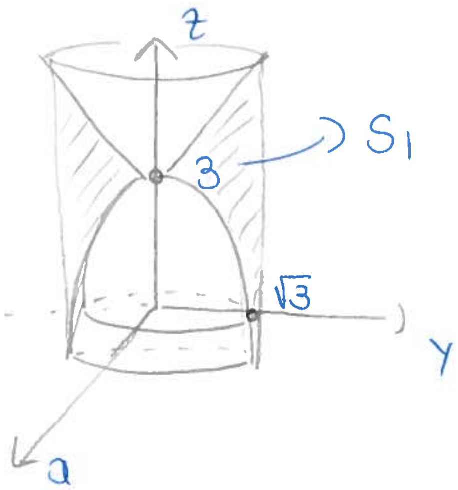
$$= \left[-18 \operatorname{cotg}\theta - \frac{9}{2}\theta\right]_{\pi/4}^{3\pi/4} =$$

$$= 18 - \frac{27}{8}\pi + 18 + \frac{9\pi}{8} = 36 - \frac{18}{8}\pi =$$

$$= 36 - \frac{9}{4}\pi$$

(10)

a)



$$\begin{cases} x = r\cos\theta & 0 \le \theta \le 2\pi \\ y = r\sin\theta & 0 \le r \le 2 \\ z = z & 3 - r^2 \le z \le 3 + r \end{cases}$$

$$|J| = r$$

$$\text{volume} = \iiint_{S_1} 1 \, dx\,dy\,dz =$$

$$= \int_0^{2\pi} \int_0^{2} \int_{3-r^2}^{3+r} r \, dz\, dr\, d\theta$$

b)

$$= 2\pi \int_0^2 \cancel{3r} + r^2 - \cancel{3r} + r^3 \, dr =$$

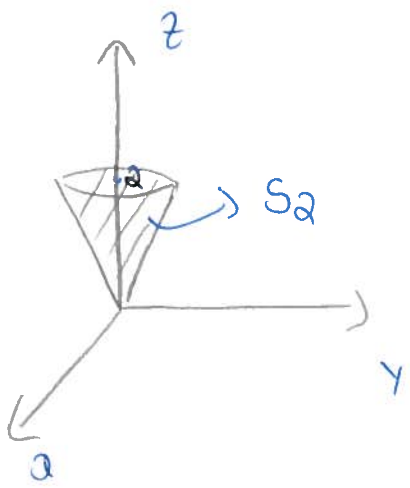
$$= 2\pi \int_0^2 r^3 + r^2 \, dr =$$

$$= 2\pi \left[\frac{r^4}{4} + \frac{r^3}{3} \right]_0^2 =$$

$$= 2\pi \left(4 + \frac{8}{3} \right) = 2\pi \frac{20}{3} = \frac{40}{3}\pi$$

(11)

a)



$$\begin{cases} x = \rho \cos\theta \, \operatorname{sen}\varphi & 0 \le \theta \le 2\pi \\ y = \rho \operatorname{sen}\theta \, \operatorname{sen}\varphi & 0 \le \varphi \le \pi/3 \\ z = \rho \cos\varphi & 0 \le \rho \le \frac{2}{\cos\varphi} \end{cases}$$

$$|J_{ac}| = \rho^2 \operatorname{sen}\varphi$$

$$z = \sqrt{\frac{x^2+y^2}{3}} \Leftrightarrow \rho \cos\varphi = \sqrt{\frac{\rho^2 \operatorname{sen}^2\varphi}{3}} \Leftrightarrow$$

$$\underset{\varphi\in[0,\pi]}{\Leftrightarrow} \rho\cos\varphi = \frac{\rho \operatorname{sen}\varphi}{\sqrt{3}} \underset{\rho\neq 0}{\Leftrightarrow} \operatorname{tg}\varphi = \sqrt{3} \Rightarrow \varphi = \pi/3$$

$$z = 2 \Leftrightarrow \rho\cos\varphi = 2 \Leftrightarrow \rho = \frac{2}{\cos\varphi}$$

Como S_2 é homogéneo temos que

$$d(x,y,z) = k > 0 \quad , \forall (x,y,z) \in S_2$$

$$\text{Massa} = \iiint_{S_2} k \, dx\,dy\,dz = \int_0^{2\pi}\int_0^{\pi/3}\int_0^{\frac{2}{\cos\varphi}} k \, \rho^2 \operatorname{sen}\varphi \, d\rho\, d\varphi\, d\theta$$

b)

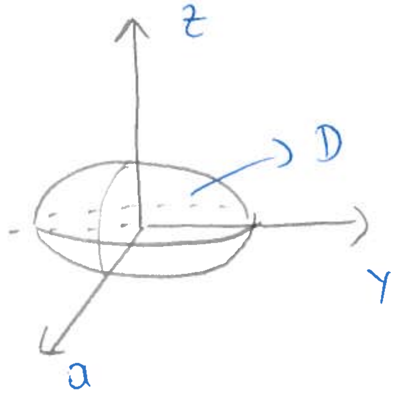
$$= 2\pi k \int_0^{\pi/3} \left[\frac{\rho^3}{3} \operatorname{sen}\varphi\right]_0^{\frac{2}{\cos\varphi}} d\varphi =$$

$$= \frac{2\pi}{3} k \int_0^{\pi/3} \frac{8 \operatorname{sen}\varphi}{\cos^3\varphi} \, d\varphi = \frac{8\pi k}{3} \left[\cos^{-2}\varphi\right]_0^{\pi/3} =$$

$$= \frac{8\pi}{3} k \, (4-1) = 8\pi k$$

(12)

$$\text{volume} = \iiint_D 1\, dx\, dy\, dz$$



$$T: \mathbb{R}^3 \to \mathbb{R}^3$$

$$(\rho, \theta, \varphi) \to (x, y, z) = (a\rho \cos\theta \operatorname{sen}\varphi,\ B\rho \operatorname{sen}\theta \operatorname{sen}\varphi,\ C\rho\cos\varphi)$$

$$\text{Jac}\,T(\rho,\theta,\varphi) = \begin{array}{ccc} \rho & \theta & \varphi \\ \end{array}$$

$$\text{Jac}\,T(\rho,\theta,\varphi) = \begin{bmatrix} a\cos\theta \operatorname{sen}\varphi & -a\rho \operatorname{sen}\theta \operatorname{sen}\varphi & a\rho\cos\theta\cos\varphi \\ B\operatorname{sen}\theta\operatorname{sen}\varphi & B\rho\cos\theta\operatorname{sen}\varphi & B\rho\operatorname{sen}\theta\cos\varphi \\ C\cos\varphi & 0 & -C\rho\operatorname{sen}\varphi \end{bmatrix}$$

$$\text{det}(\text{Jac}\,T(\rho,\theta,\varphi)) = C\cos\varphi \begin{vmatrix} -a\rho\operatorname{sen}\theta\operatorname{sen}\varphi & a\rho\cos\theta\cos\varphi \\ B\rho\cos\theta\operatorname{sen}\varphi & B\rho\operatorname{sen}\theta\cos\varphi \end{vmatrix}$$

$$-C\rho\operatorname{sen}\varphi \begin{vmatrix} a\cos\theta\operatorname{sen}\varphi & -a\rho\operatorname{sen}\theta\operatorname{sen}\varphi \\ B\operatorname{sen}\theta\operatorname{sen}\varphi & B\rho\cos\theta\operatorname{sen}\varphi \end{vmatrix}$$

$$= C\cos\varphi\,(-aB\rho^2\operatorname{sen}\varphi\cos\varphi) - C\rho\operatorname{sen}\varphi\,(aB\rho\operatorname{sen}^2\varphi)$$

$$= -aBC\rho^2\operatorname{sen}\varphi$$

$$|\det(\text{Jac}\,T(\rho,\theta,\varphi))| = aBC\rho^2\operatorname{sen}\varphi$$

$$\begin{cases} a = a\rho\cos\theta\sin\varphi & 0 \le \rho \le 1 \\ y = B\rho\sin\theta\sin\varphi & 0 \le \theta \le 2\pi \\ z = C\rho\cos\varphi & 0 \le \varphi \le \pi \end{cases}$$

$$\text{Volume} = \int_0^{2\pi}\int_0^{\pi}\int_0^{1} aBC\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta =$$

$$= 2\pi aBC \int_0^{\pi}\left[\frac{\rho^3}{3}\sin\varphi\right]_0^1 d\varphi =$$

$$= \frac{2\pi}{3} aBC \int_0^{\pi}\sin\varphi\,d\varphi =$$

$$= \frac{2\pi}{3} aBC \left[-\cos\varphi\right]_0^{\pi} =$$

$$= \frac{2\pi}{3} aBC\,(1+1) = \frac{4\pi}{3} aBC$$