02

Probabilities and Inference

Notice

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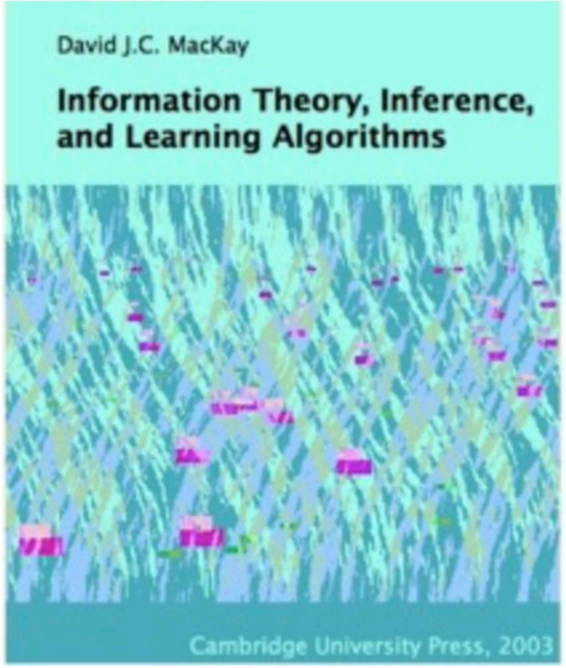
Bibliography

- **Many examples are extracted and adapted from:**



David J.C. MacKay

Information Theory, Inference, and Learning Algorithms

Cambridge University Press, 2003

Information Theory, Inference, and Learning Algorithms
David J.C. MacKay
2005, Version 7.2

- **And some slides were based on Iain Murray course**
  - http://www.inf.ed.ac.uk/teaching/courses/it/2014/

Table of Contents

Basics on Probabilities and Notation

Ensemble

- An **ensemble** X is a triple (x, A_X, P_X), where the *outcome* x is the value of a random variable, which takes on one of a set of possible values, $A_X = \{a_1, a_2, \ldots, a_i, \ldots, a_I\}$, having probabilities $P_X = \{p_1, p_2, \ldots, p_I\}$, with $P(x = a_i) = p_i$, $p_i \geq 0$ and $\sum_{a_i \in X} P(x = a_i) = 1$
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- **Probability of a subset.** If T is a subset of A_X then:

$$P(T) = P(x \in T) = \sum_{a_i \in T} P(x = a_i)$$

Ensemble: an example

- A letter that is randomly chosen from an English document.
- There are twenty-seven possible letters:
  - a–z,
  - space
  - character ‘-’.

i	a_i	p_i
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6	f	0.0173
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23	w	0.0119
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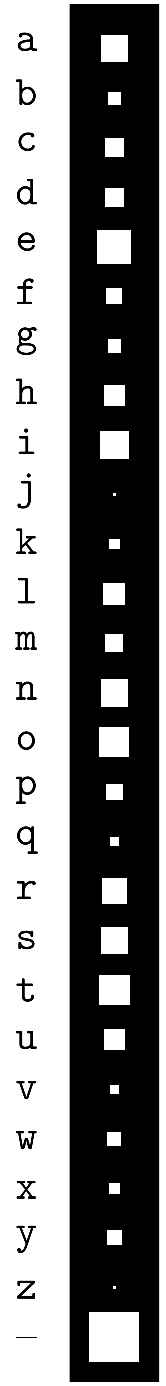
27	–	0.1928



* estimated from “The Frequently Asked Questions Manual for Linux”

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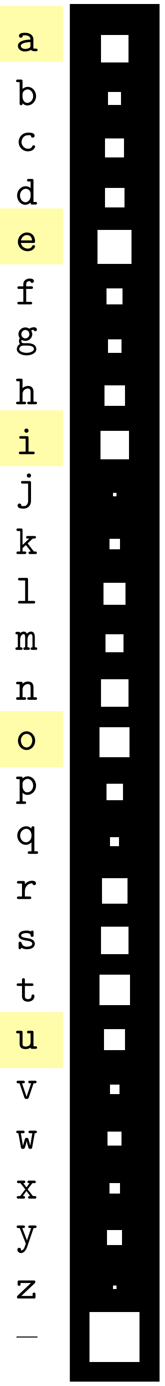

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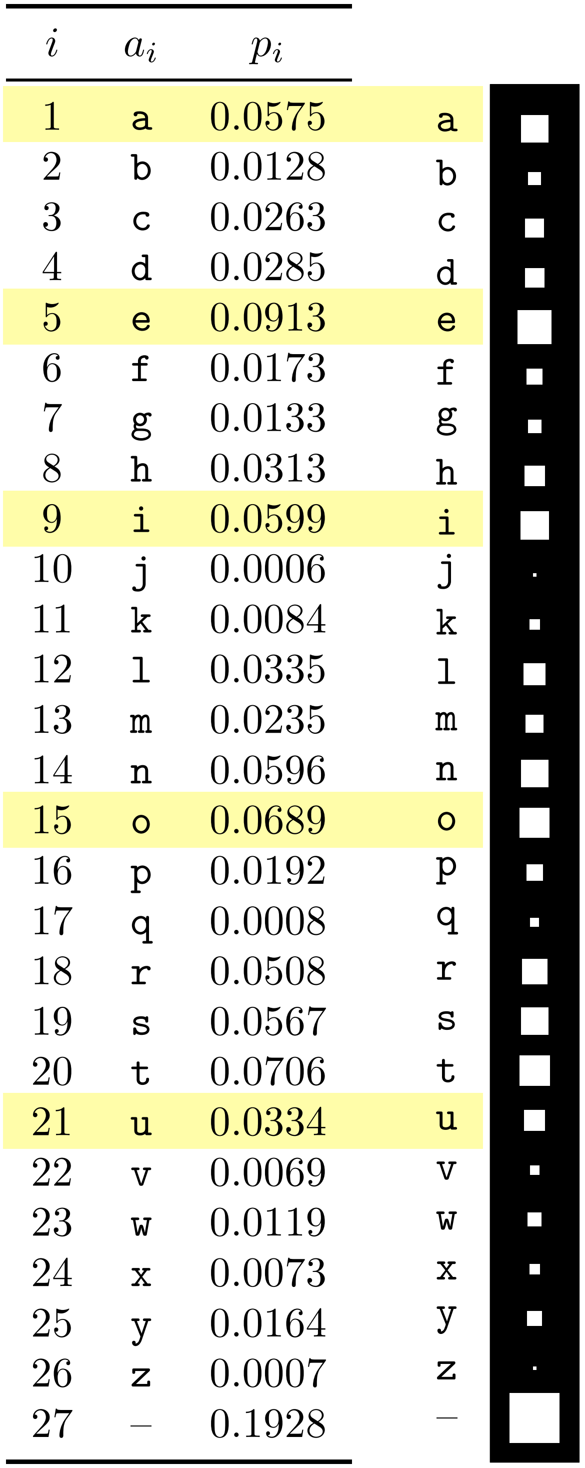
  - $P(V) = 0.06+0.09+0.06+0.07+0.03=0.31$



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Joint Ensemble

- An **joint ensemble** ***XY*** is an ensemble in which each outcome is an **ordered** pair x, y with

  $x \in A_x = \{a_1, \ldots, a_I\}$ and $y \in A_y = \{b_1, \ldots, a_J\}$.

  We call $P(x, y)$ the joint probability of x and y.

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  - xy «-» x, y
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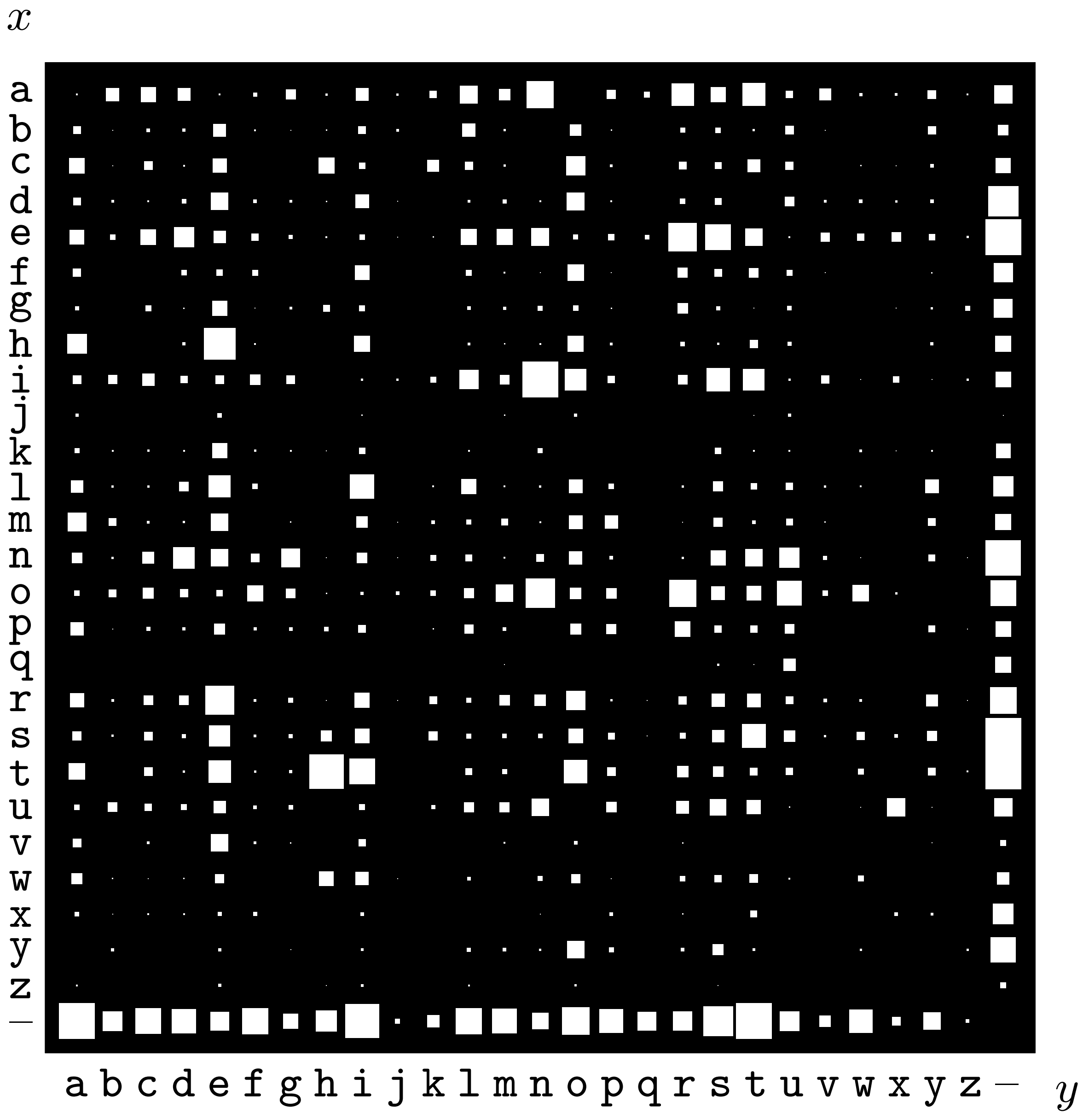

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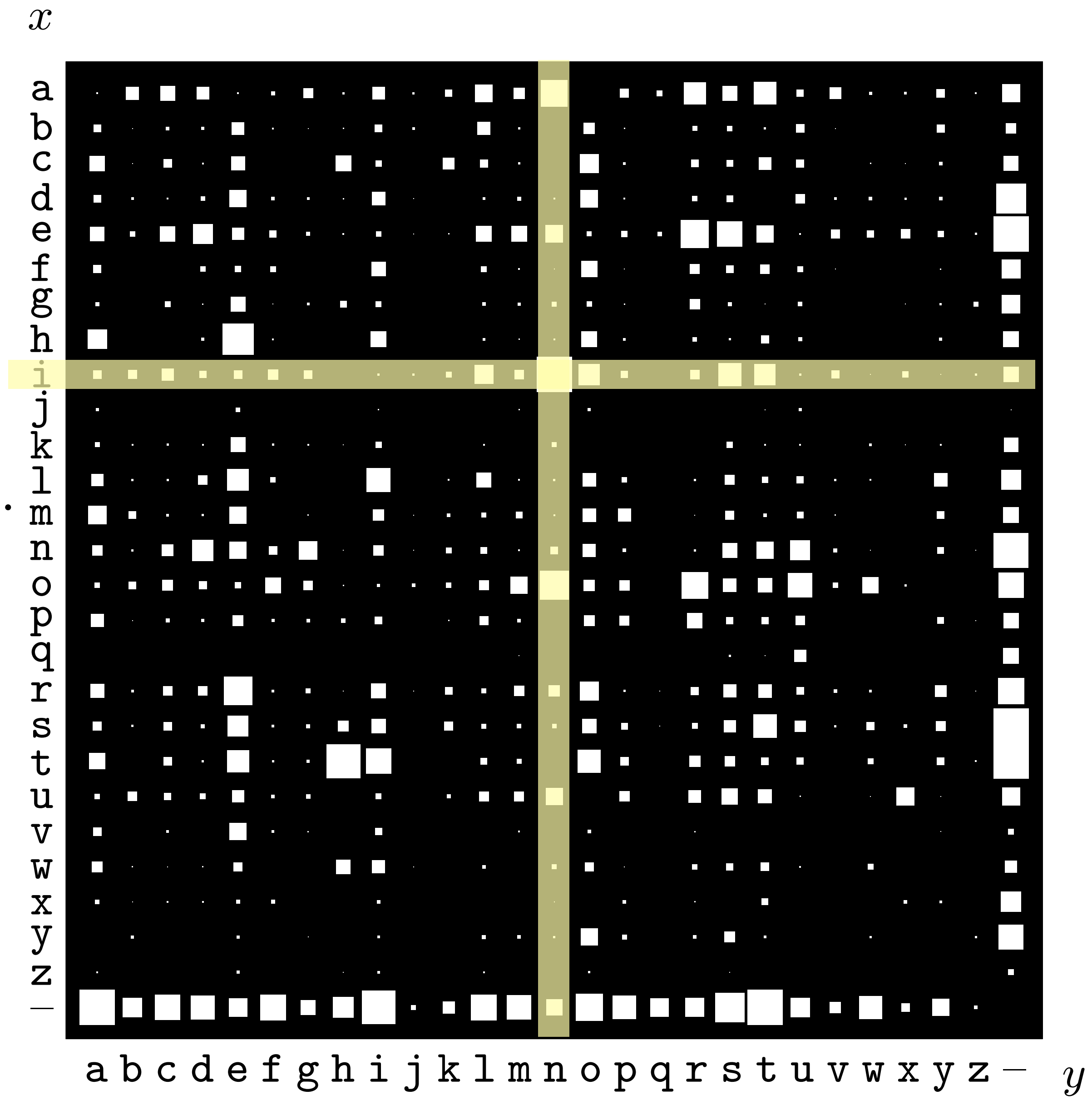

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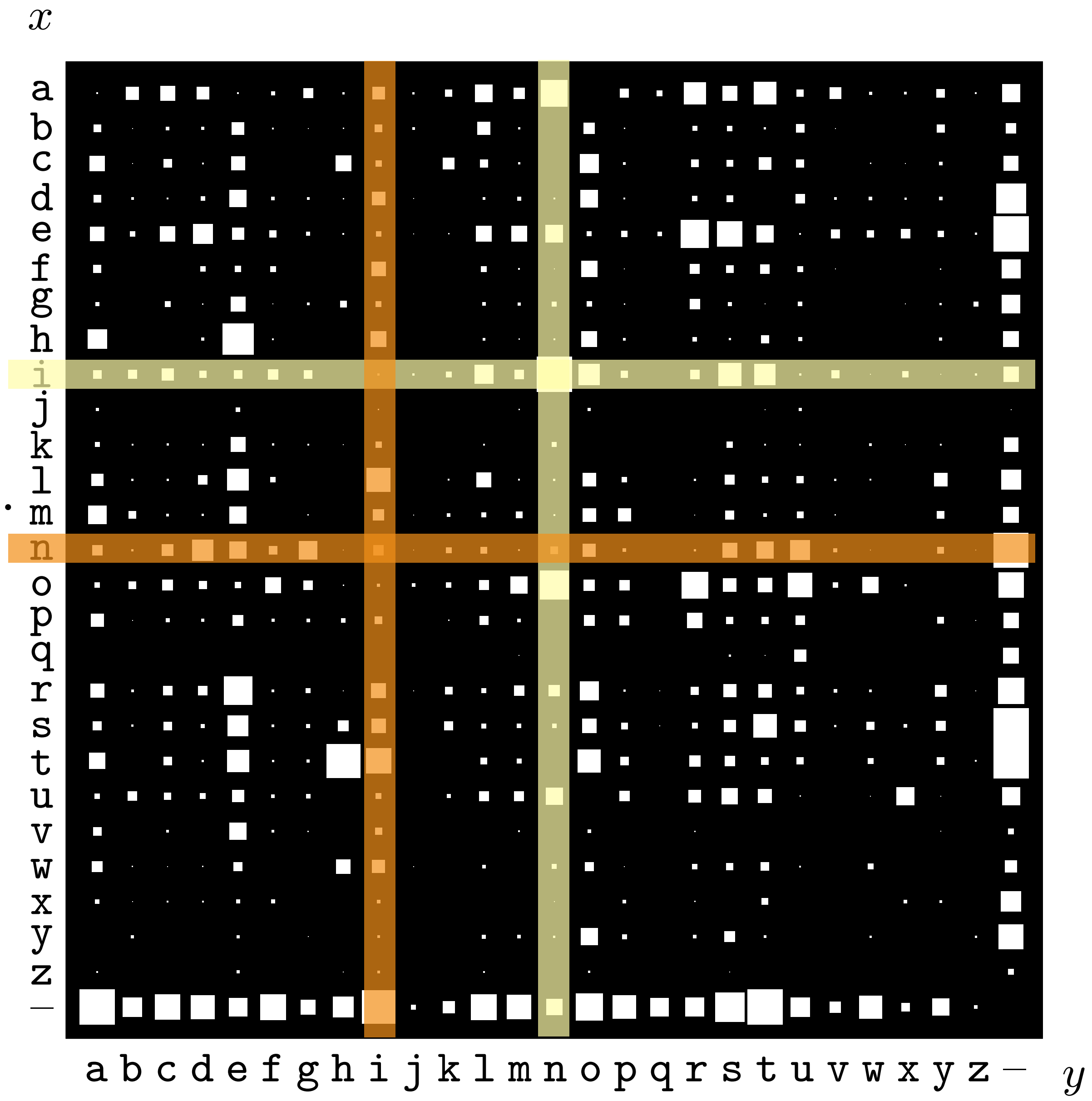

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Marginal Probability

- Marginal probability $P(x)$ from the joint probability $P(x, y)$

$$P(x=a_i)=\sum_{y\in A_Y} P(x=a_i,y) \qquad P(x)=\sum_{y\in A_Y} P(x,y)$$

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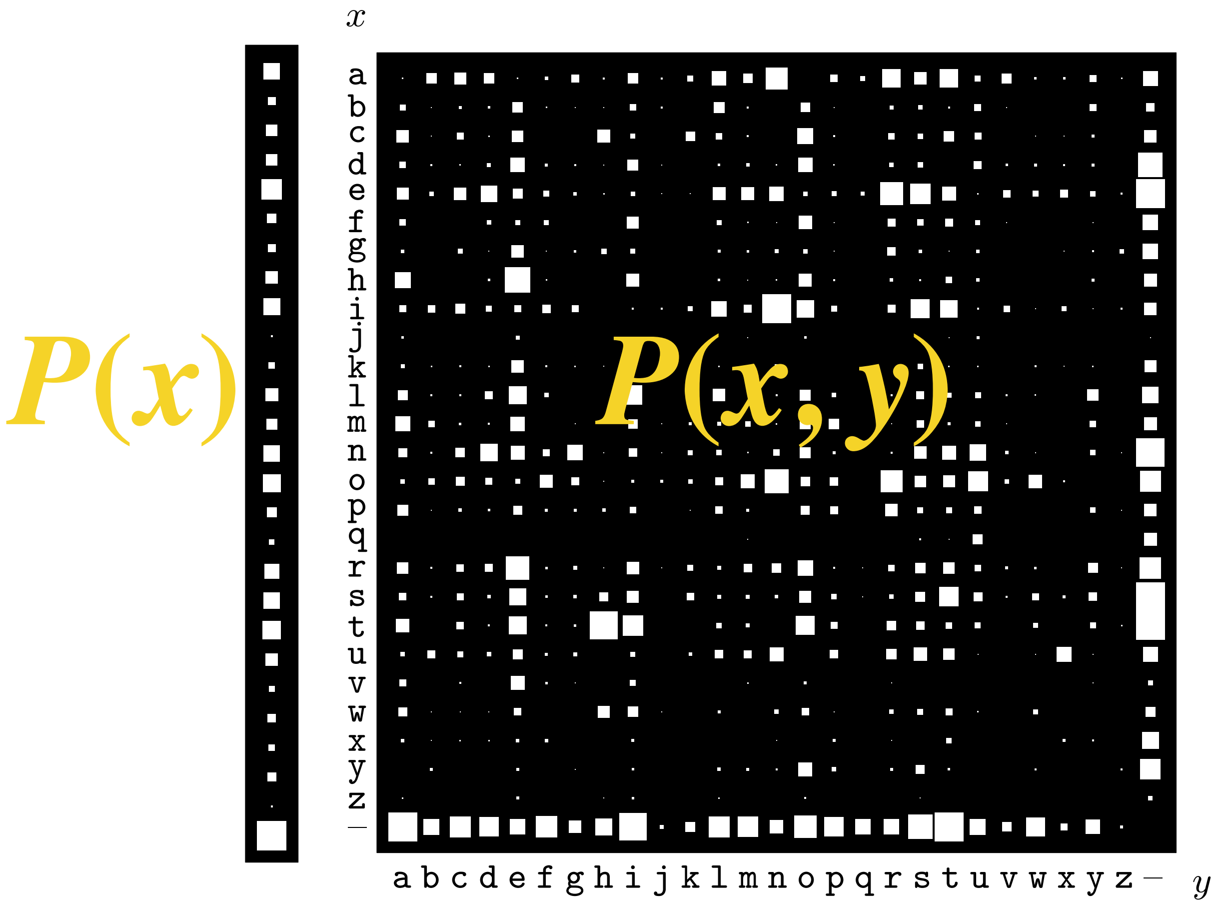

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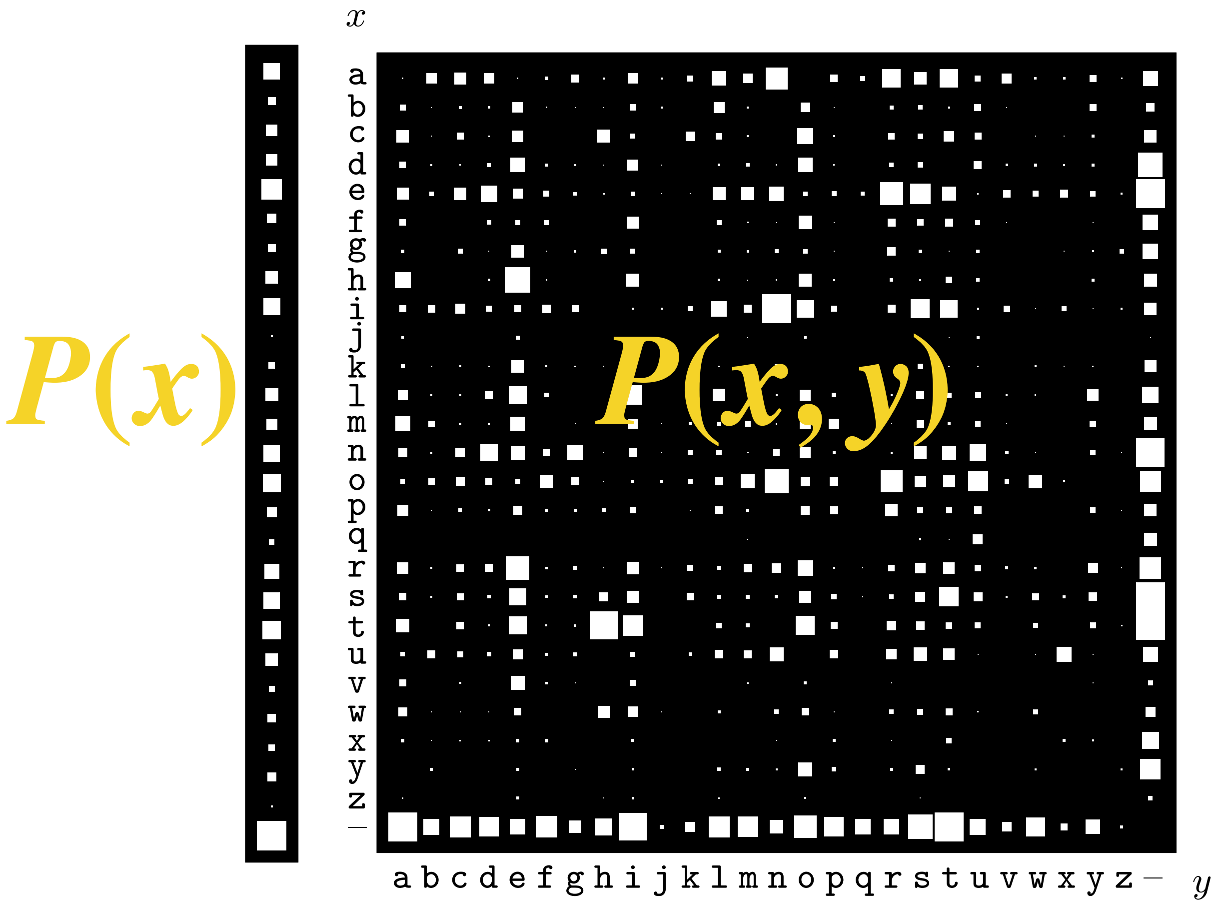
- This joint ensemble has the special property that its two marginal distributions, $P(x)$ and P(y), are identical.



Condicional Probability

- $P(x = a_i \mid y = b_j)$ is the probability that x equals a_i, given y equals b_j:

$$P(x=a_i \mid y=b_j) = \frac{P(x=a_i, y=b_j)}{P(y=b_j)} \quad if \quad P(y=b_j) \neq 0$$

  - **If $P(y = b_j) = 0$ then $P(x = a_i \mid y = b_j)$ is undefined**

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  - To obtain $P(x \mid y)$ we normalize the columns by dividing each $P(x, y)$ in a column by $P(y)$.

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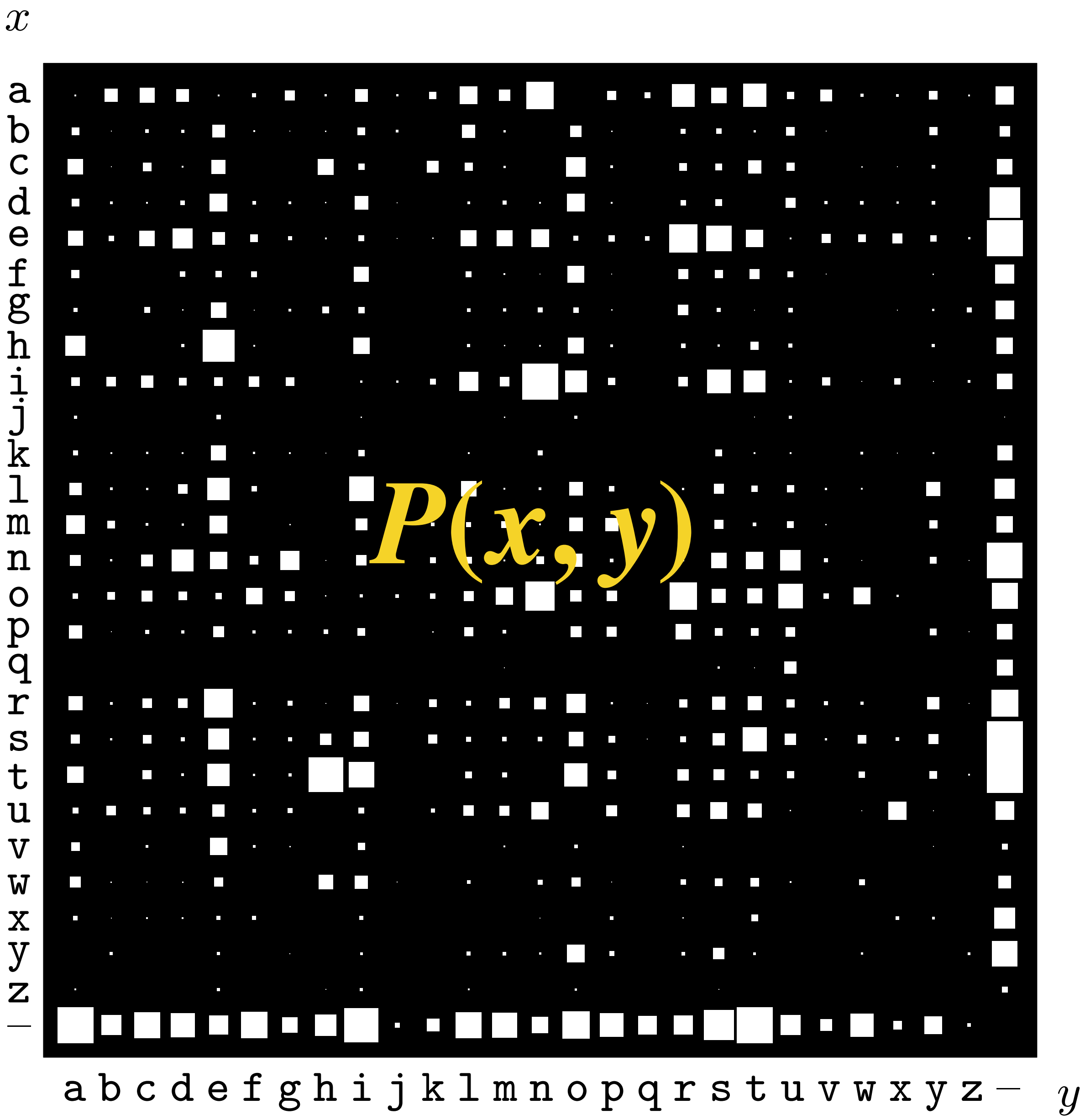
Condicional Probability: Example

- To obtain $P(y \mid x)$ we normalize the rows by **dividing each $P(x, y)$ in a row by $P(x)$**.



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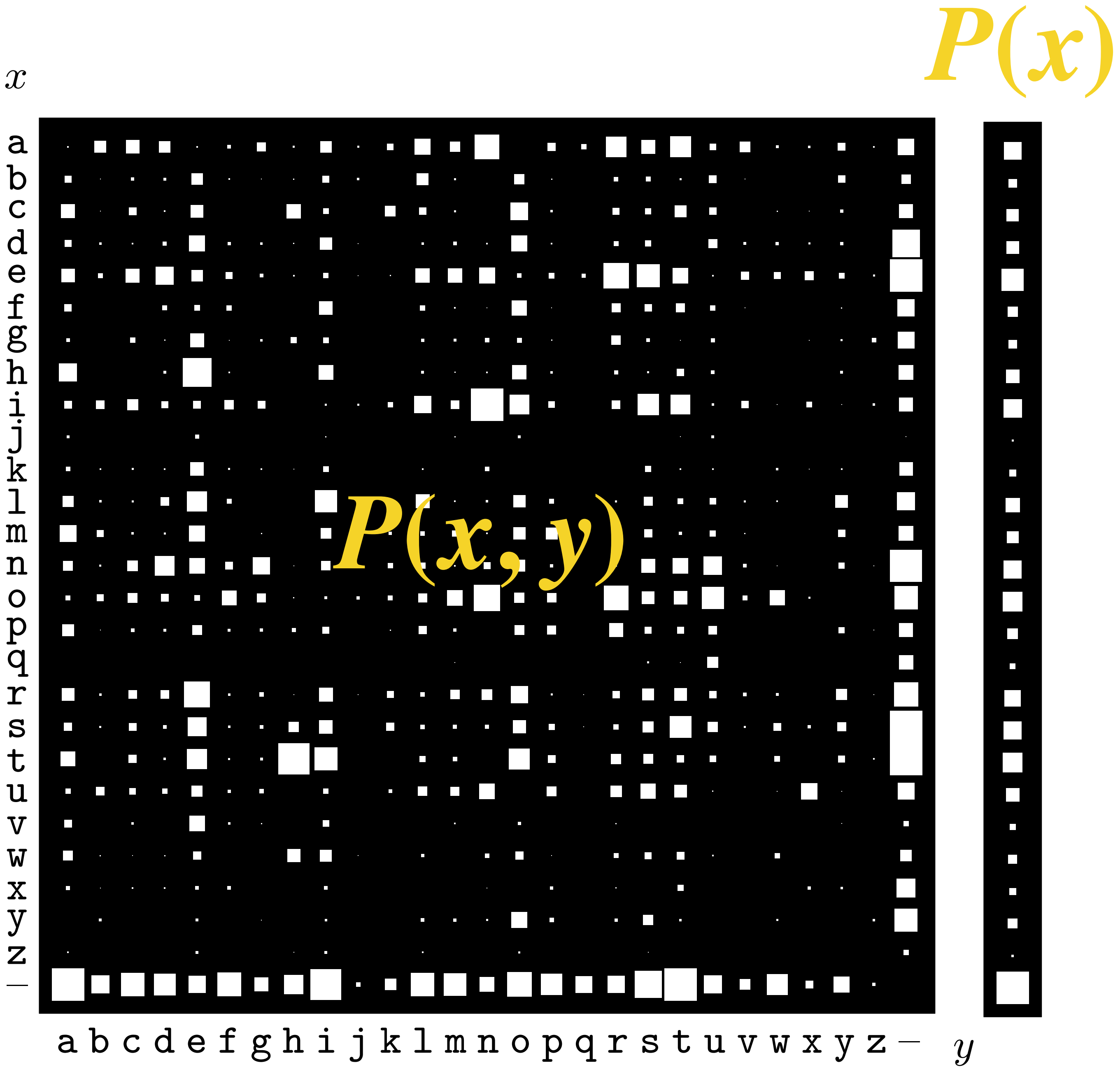

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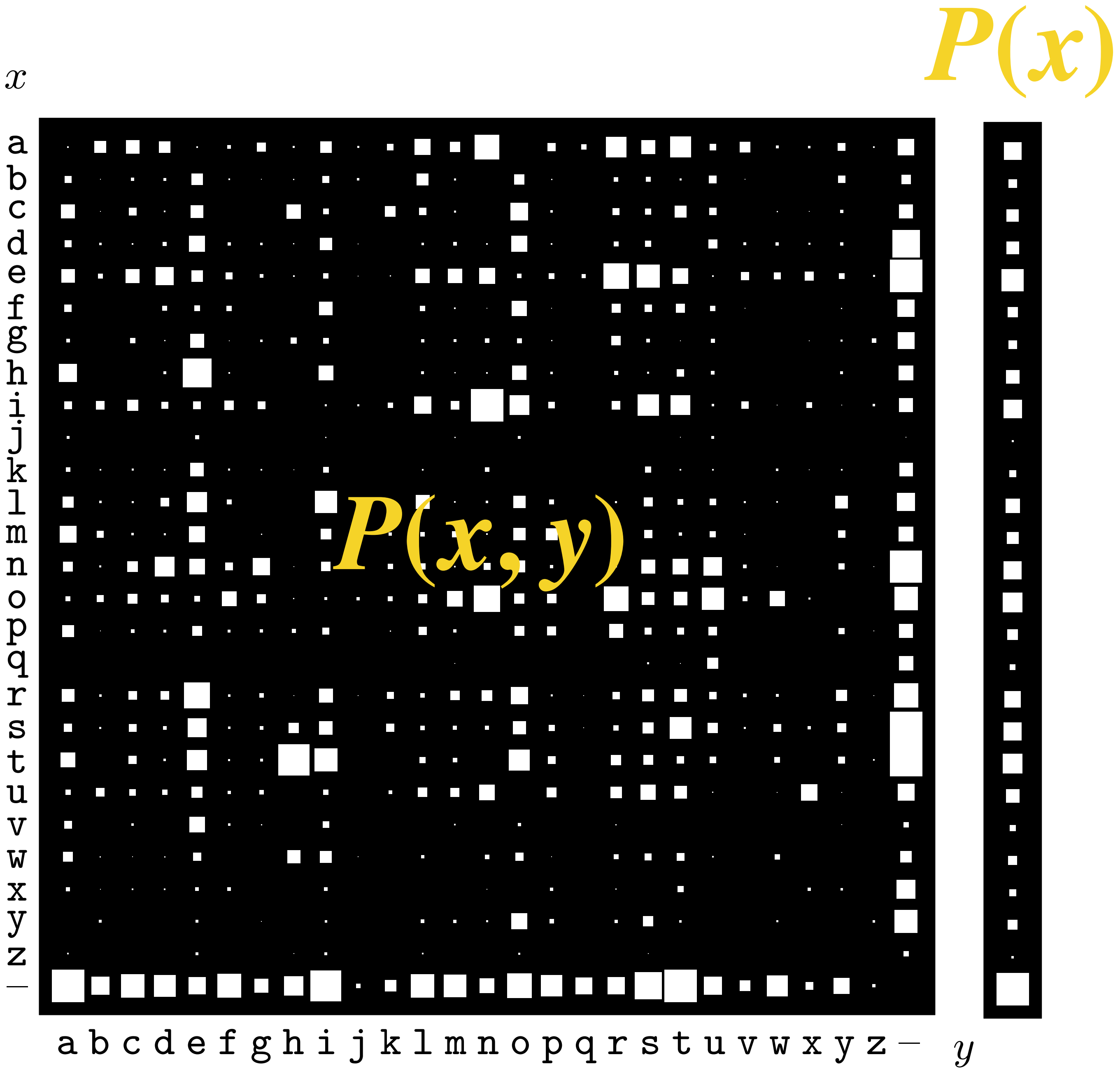

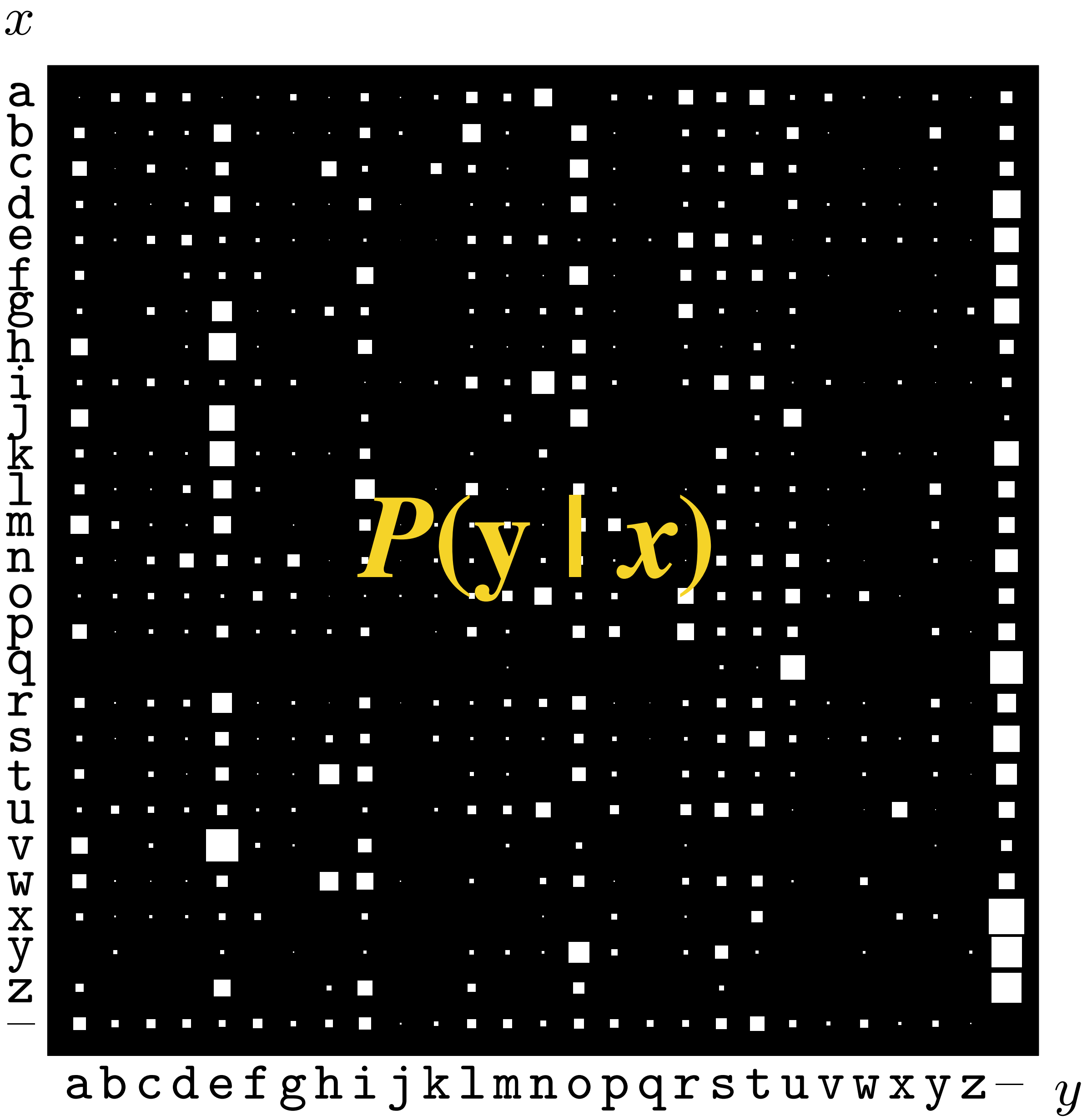

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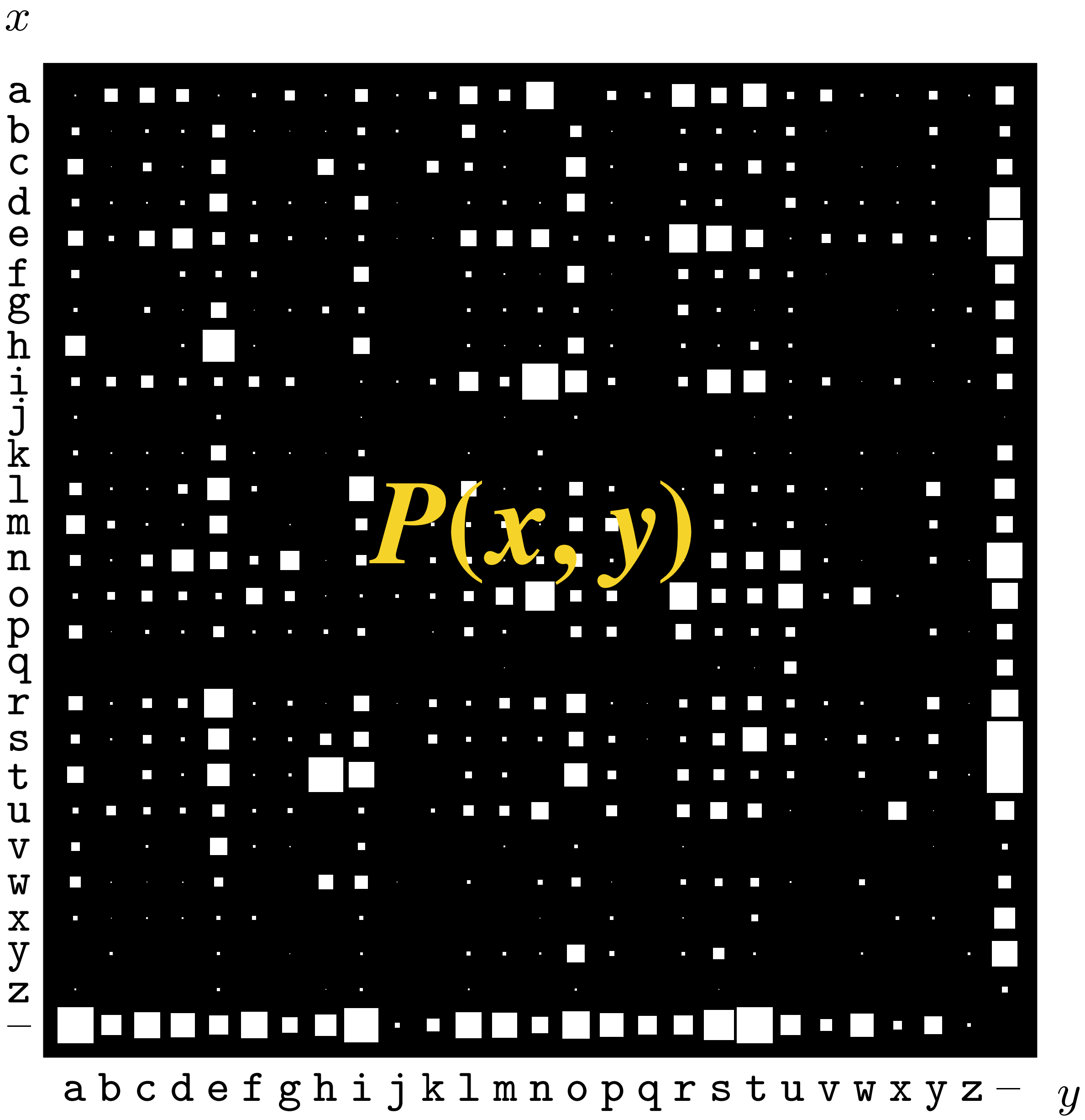

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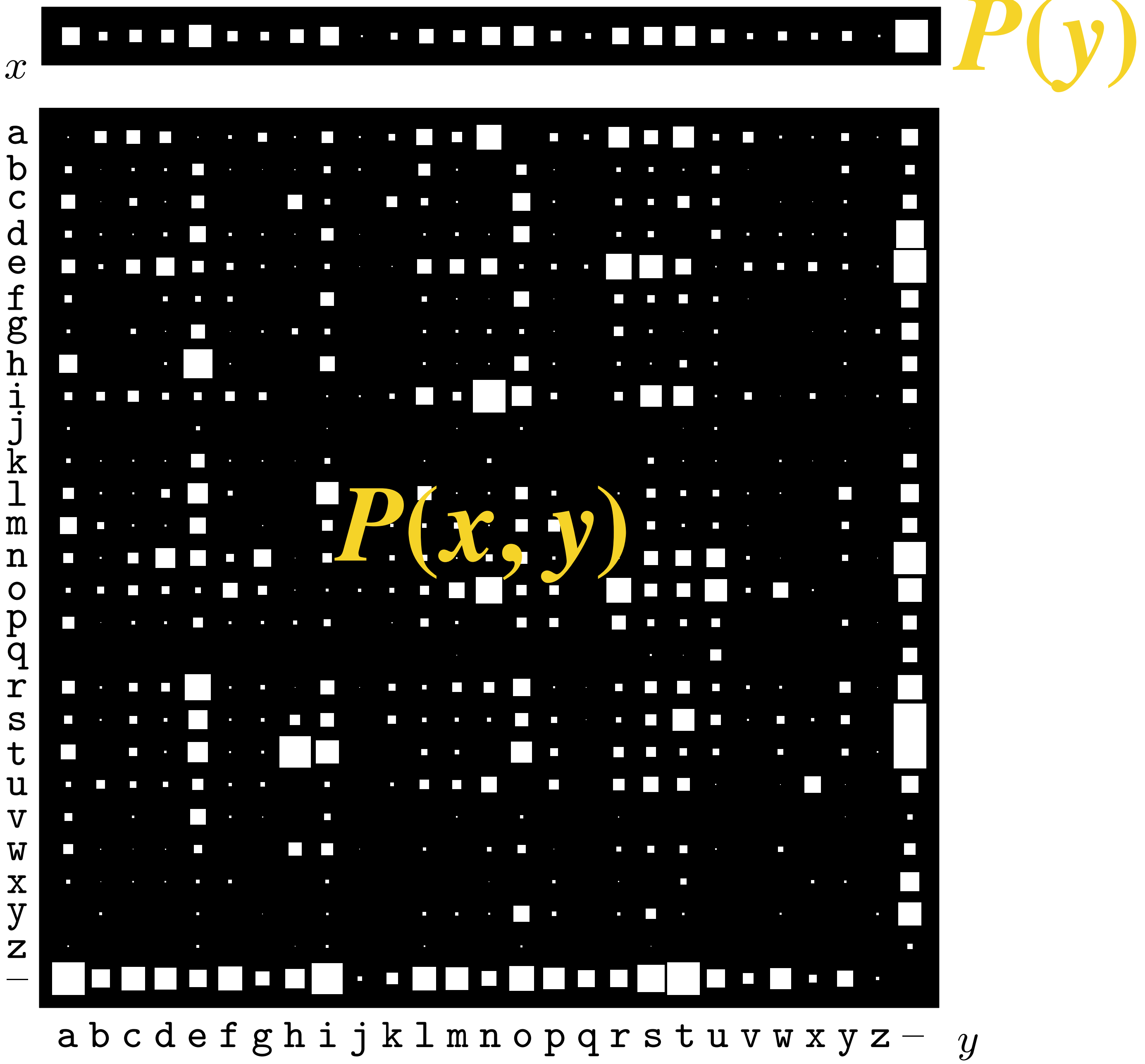

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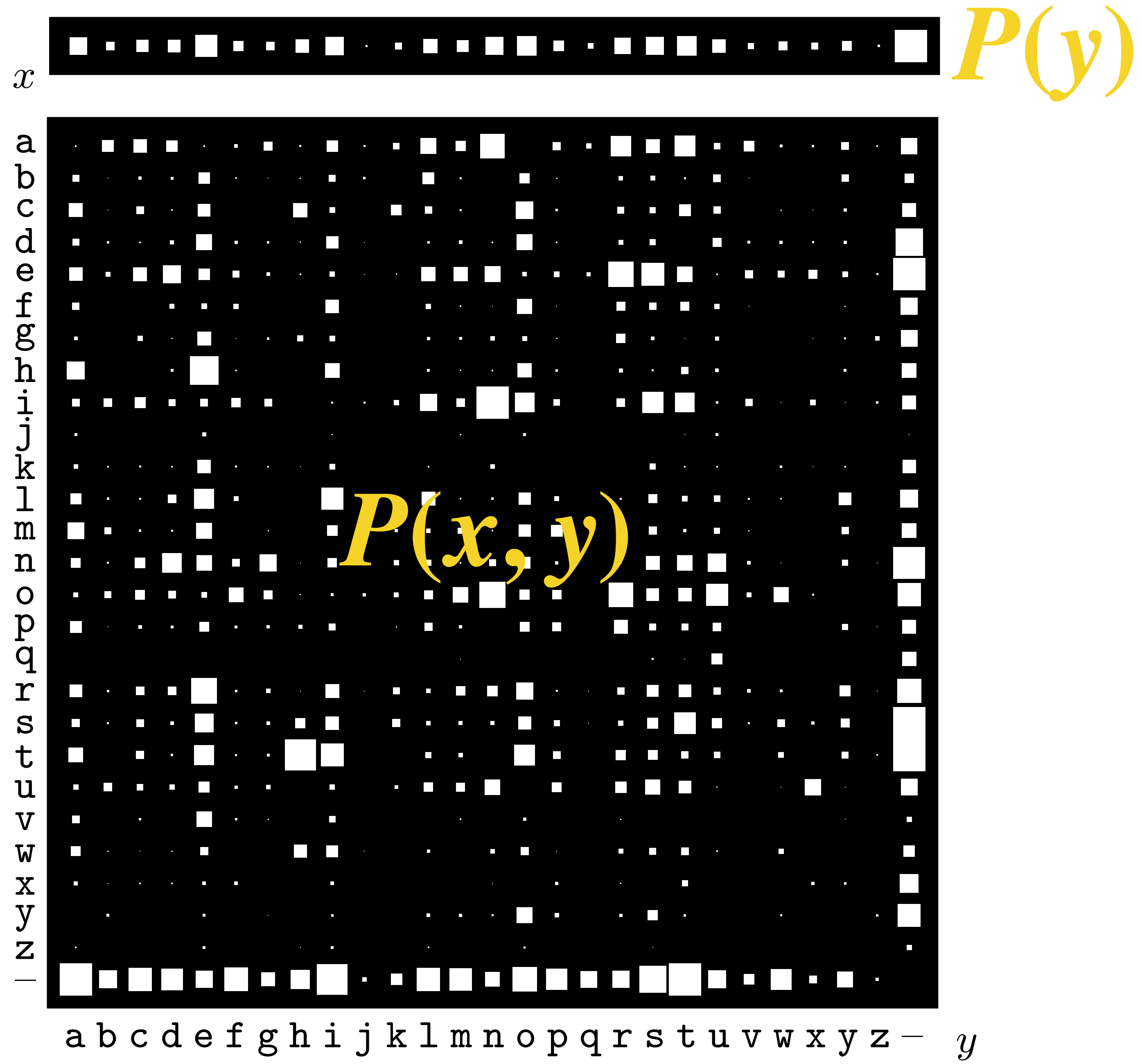
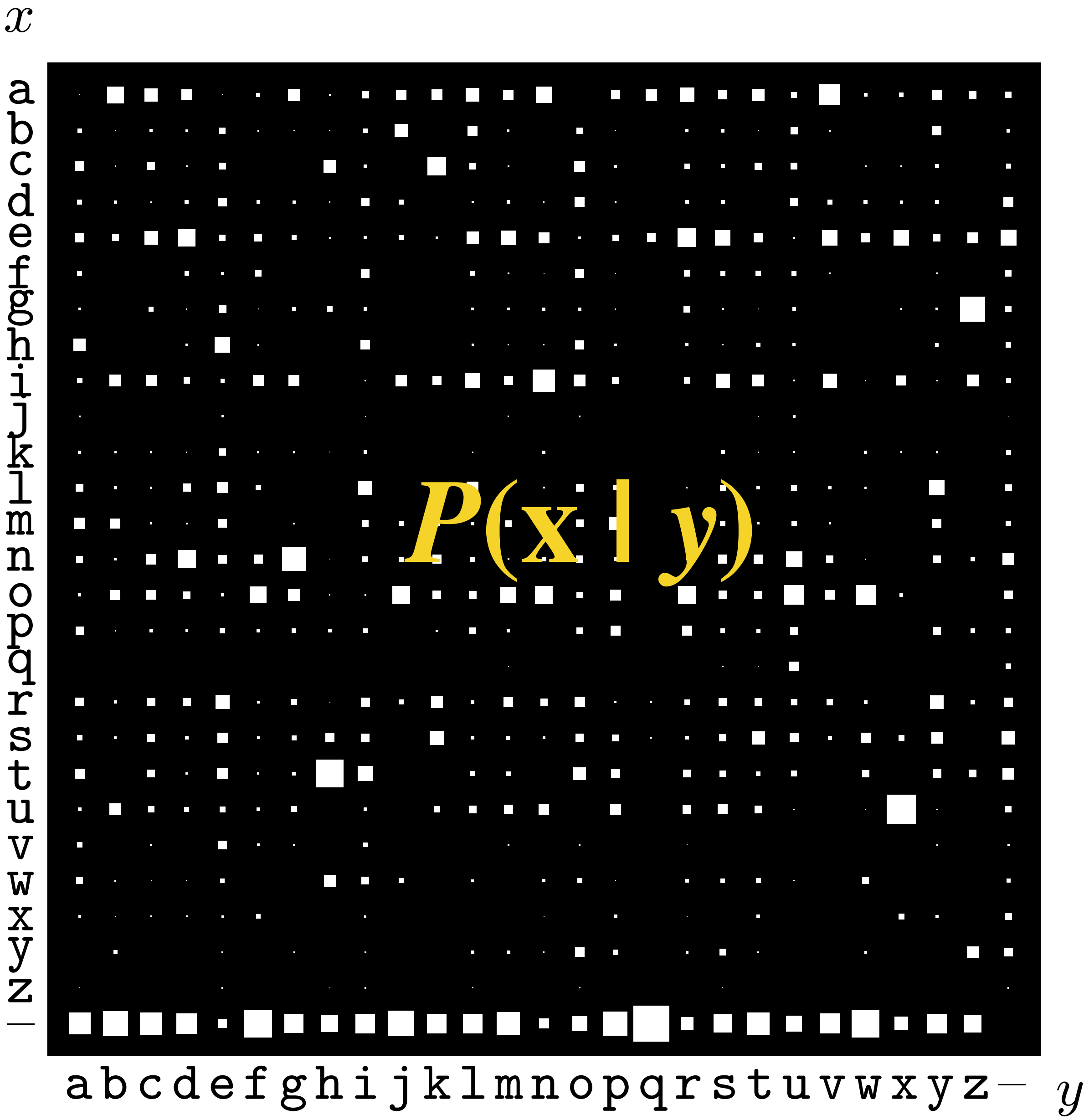
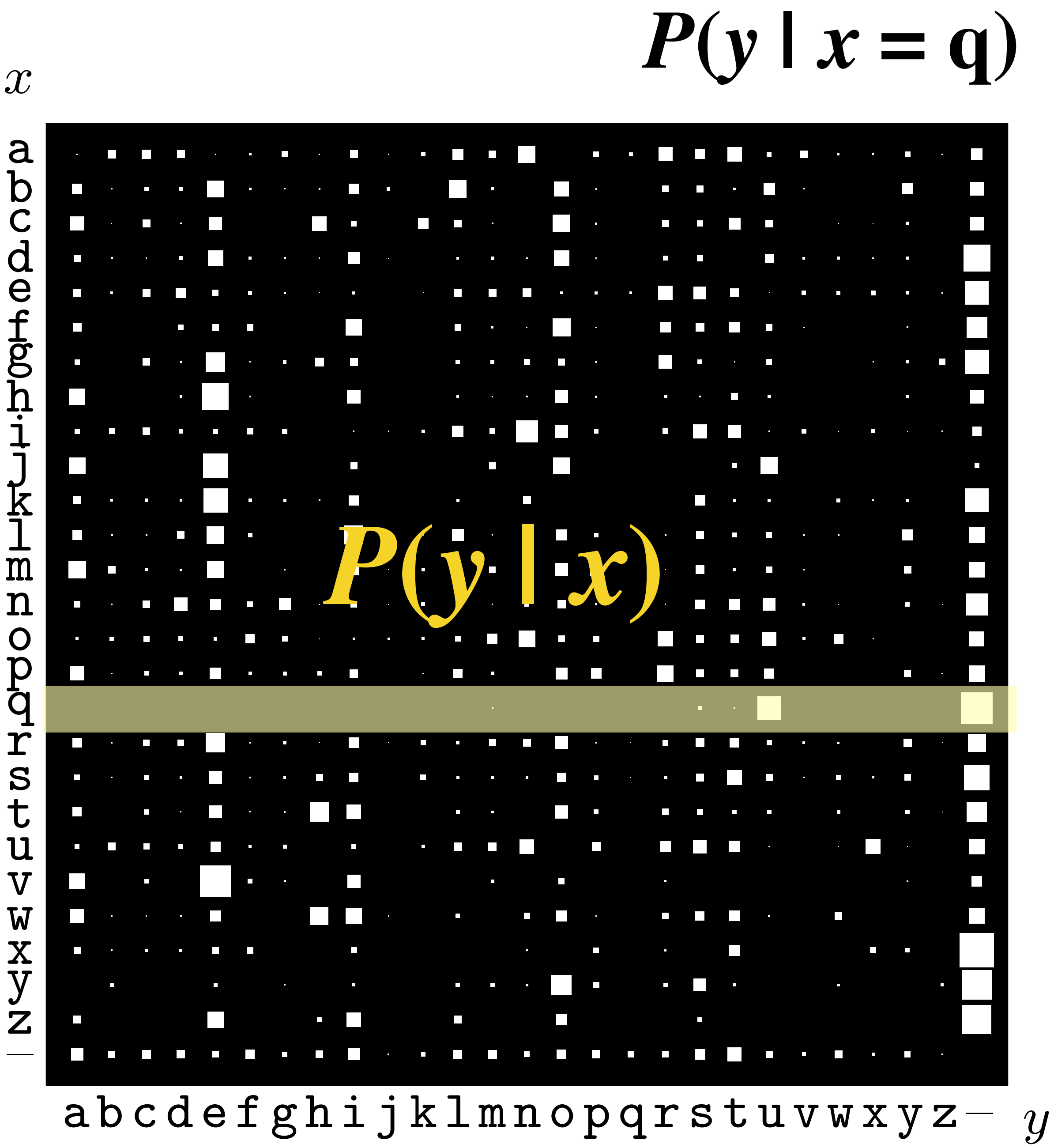




Condicional Probability: Example

- For each x, $P(y \mid x)$ is a probability distribution.



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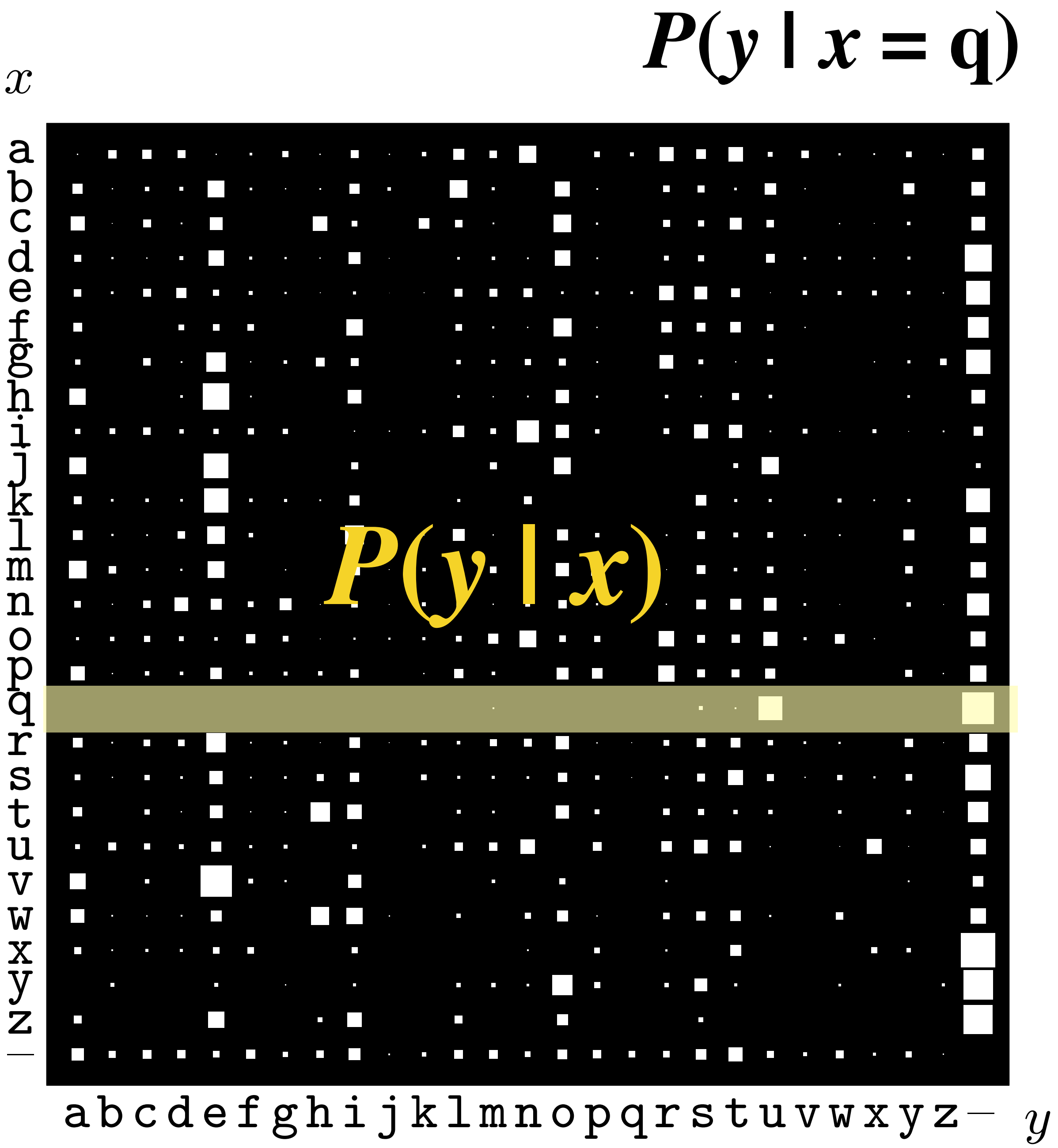

- The two most probable values for the second letter *y* given that the first letter *x* is **q** are **u** and **-**.

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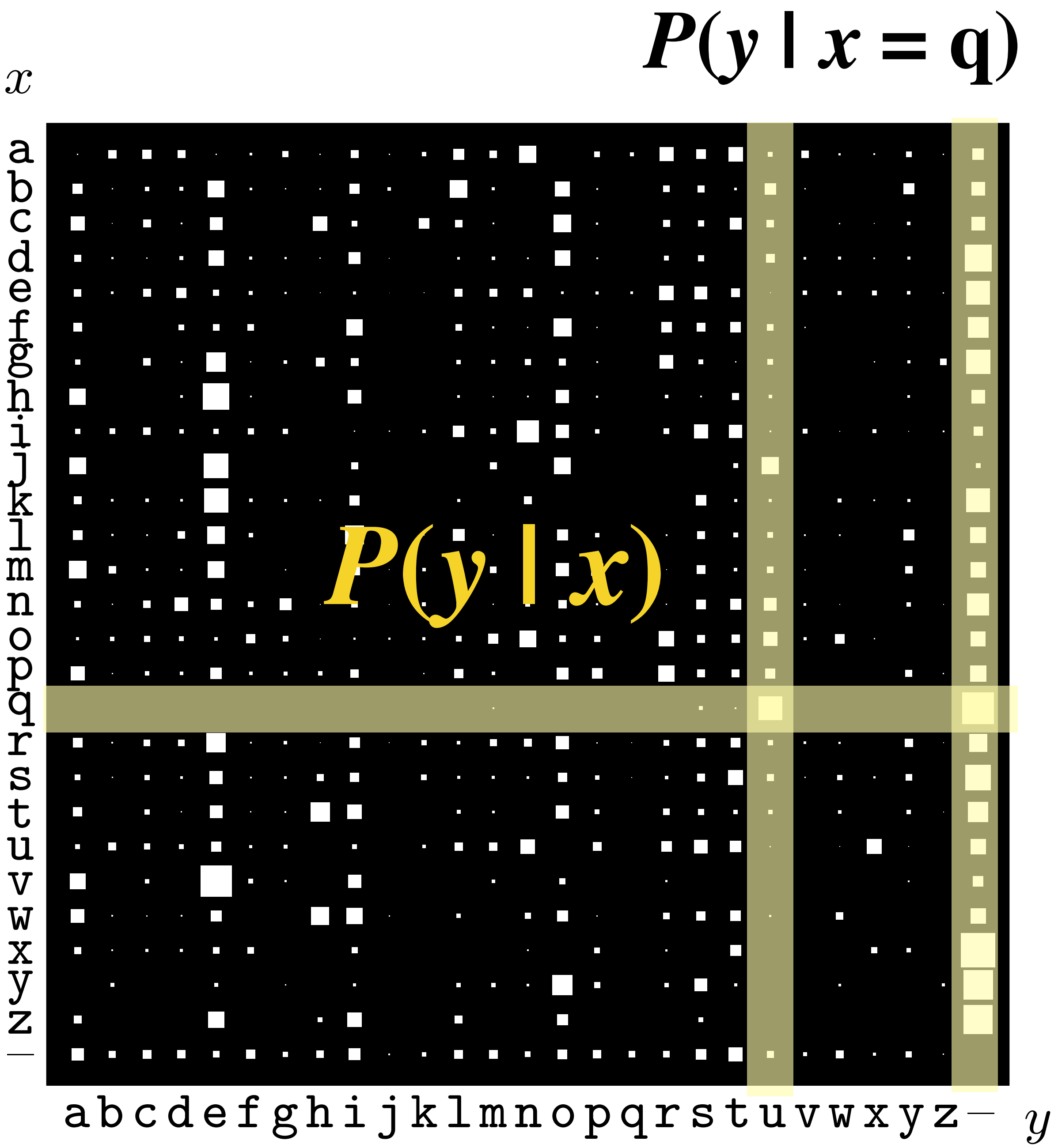

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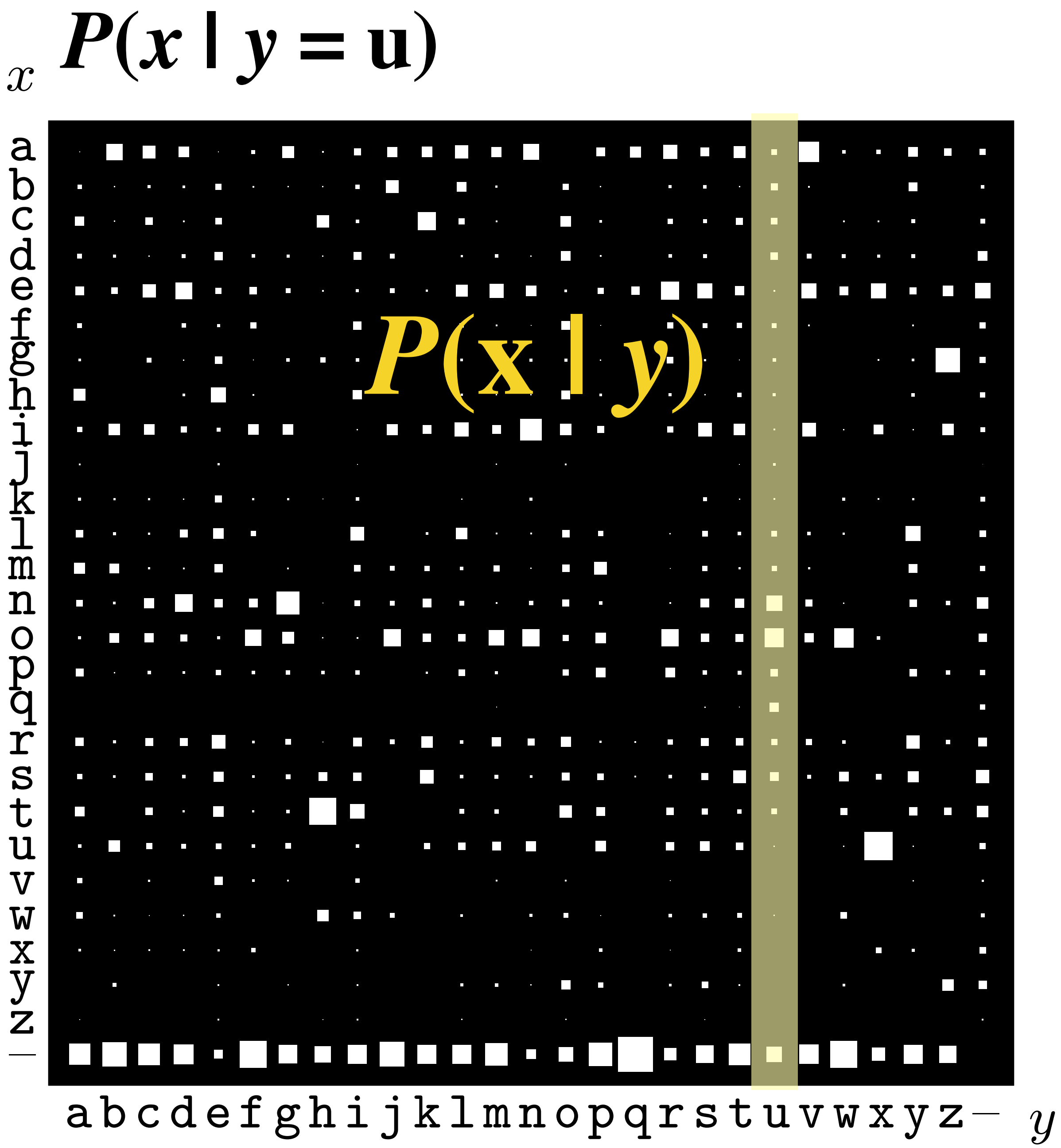

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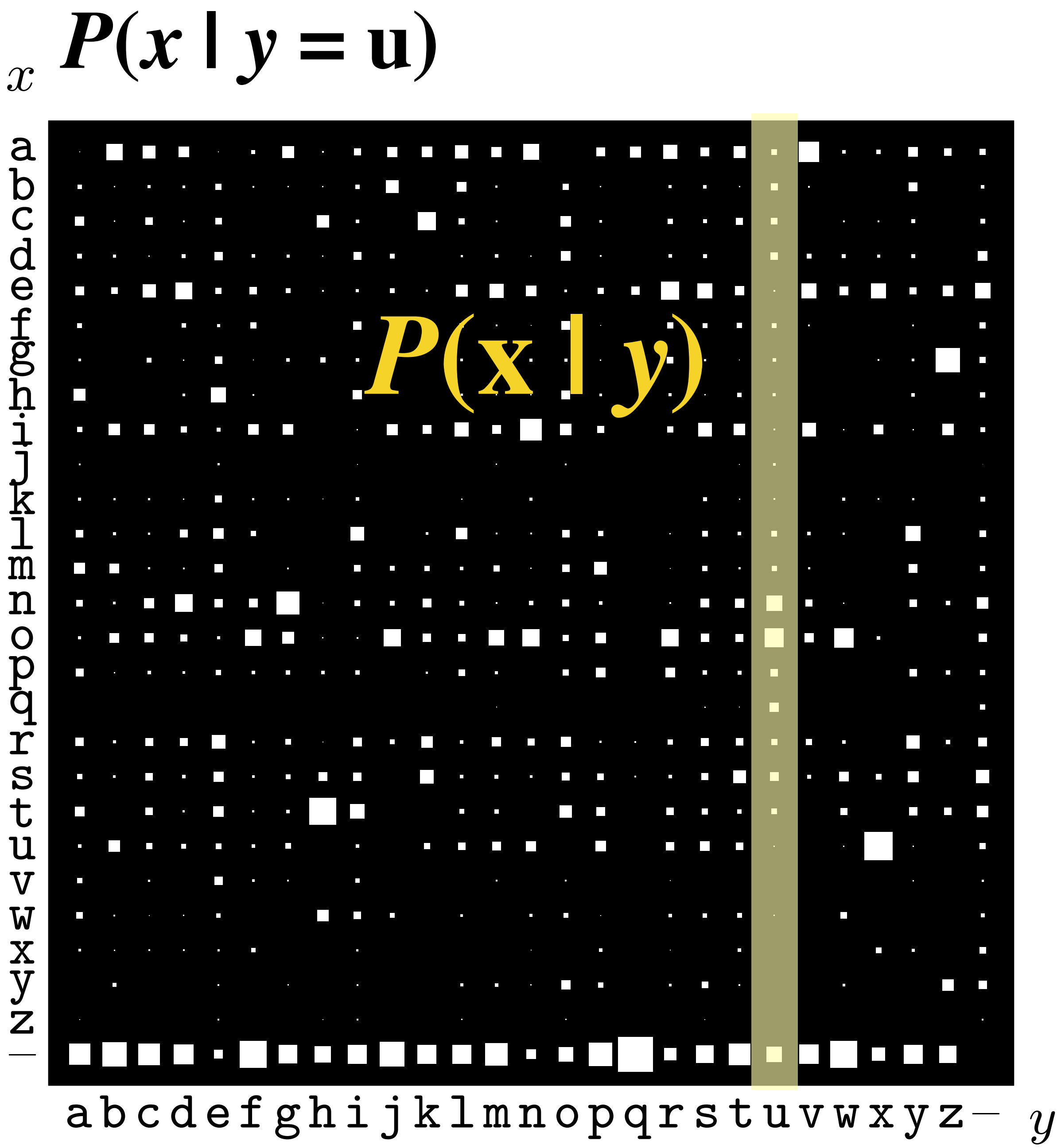

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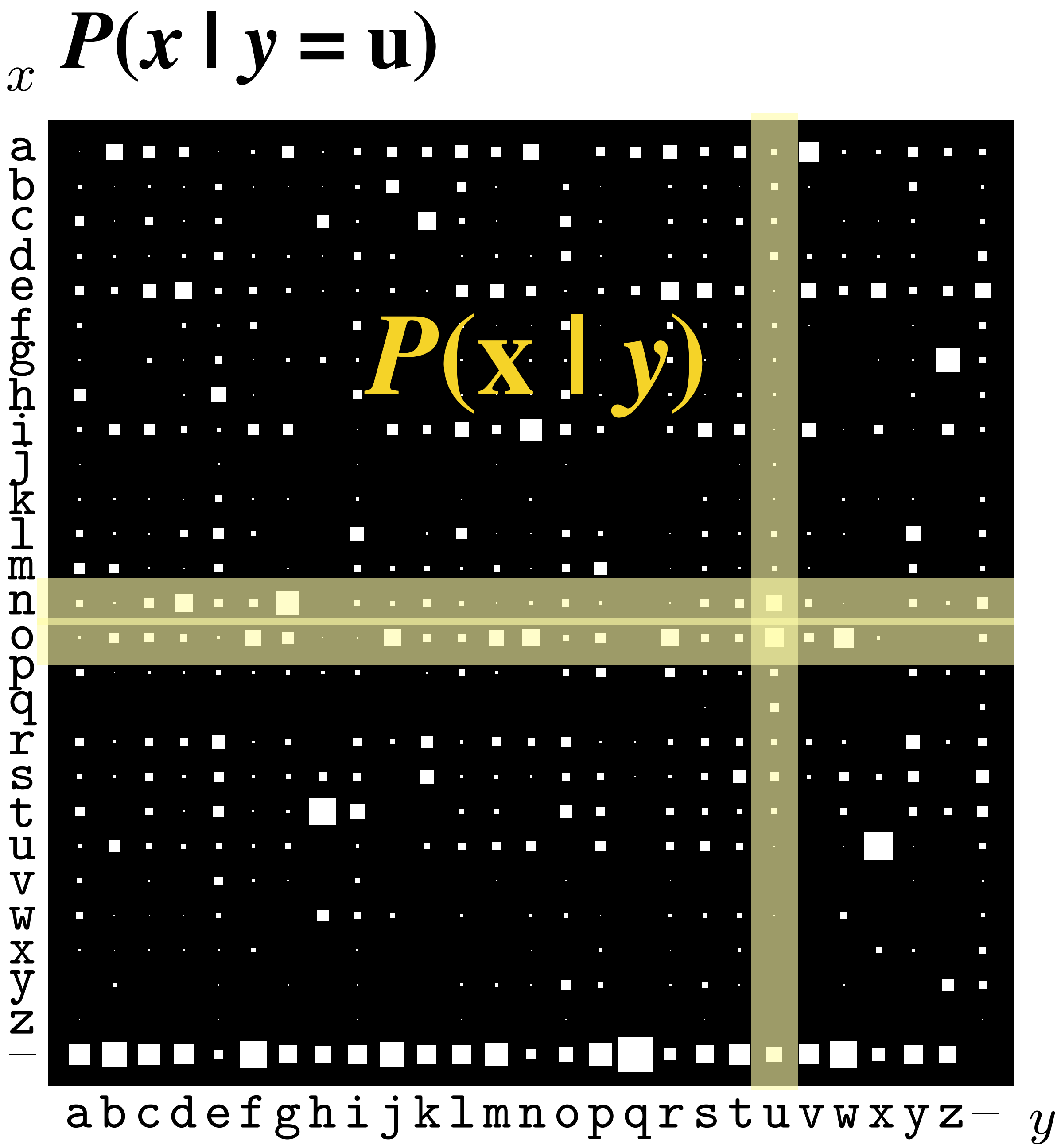

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We can define an ensemble in terms of a **collection of conditional probabilities**

Bayes' Theorem

- Essential to integrate new pieces of evidence.

$$P(y \mid x, \boldsymbol{H}) = \frac{P(x \mid y, \boldsymbol{H}) P(y \mid \boldsymbol{H})}{P(x \mid \boldsymbol{H})}$$

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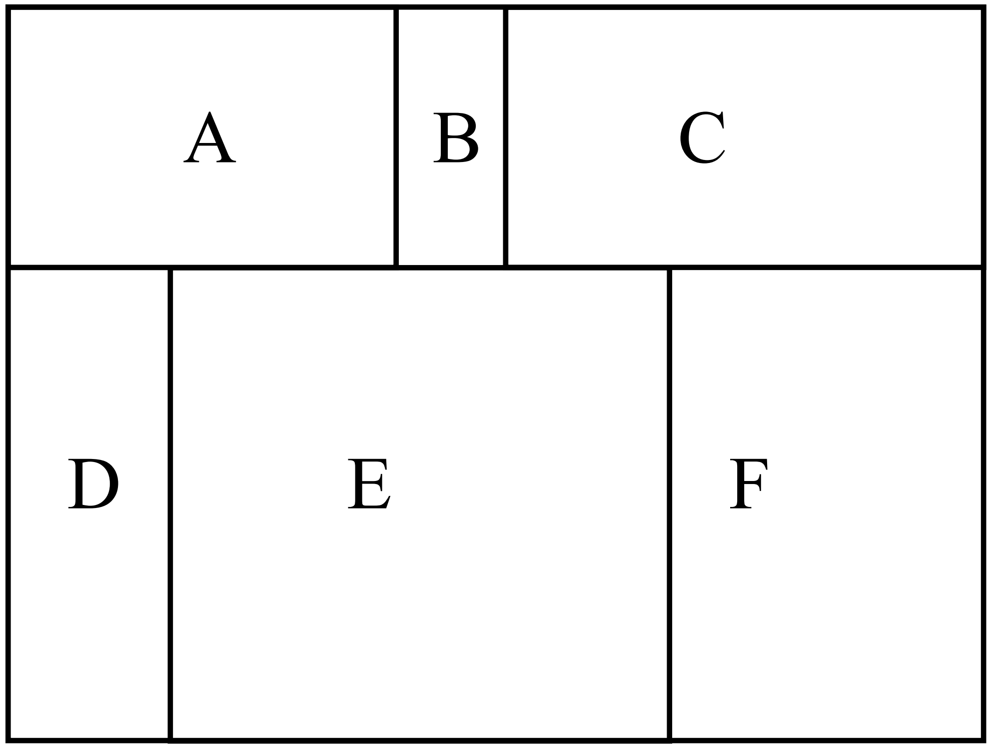
Bayes' Theorem: a geometric interpretation

- Consider this modified version of a Venn Diagram where the area of each event A, …, F is proportional to its probability



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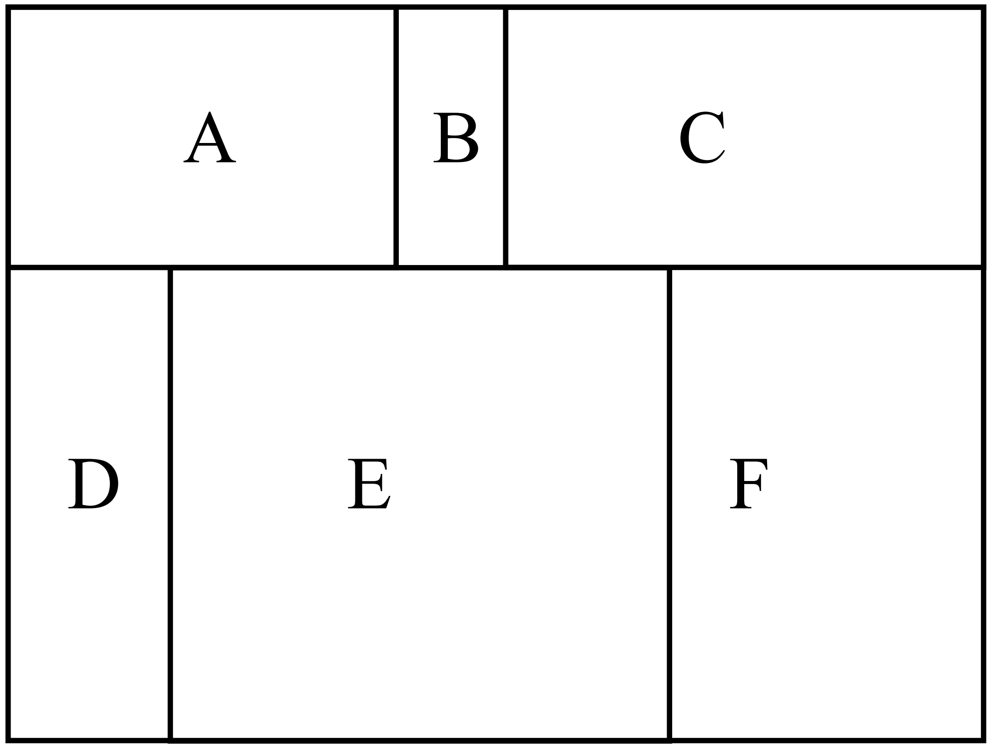

- $P(\text{A}) + \ldots . P(\text{F}) = 1$
- P(E) is the most probable event
- P(B) is the less probable event

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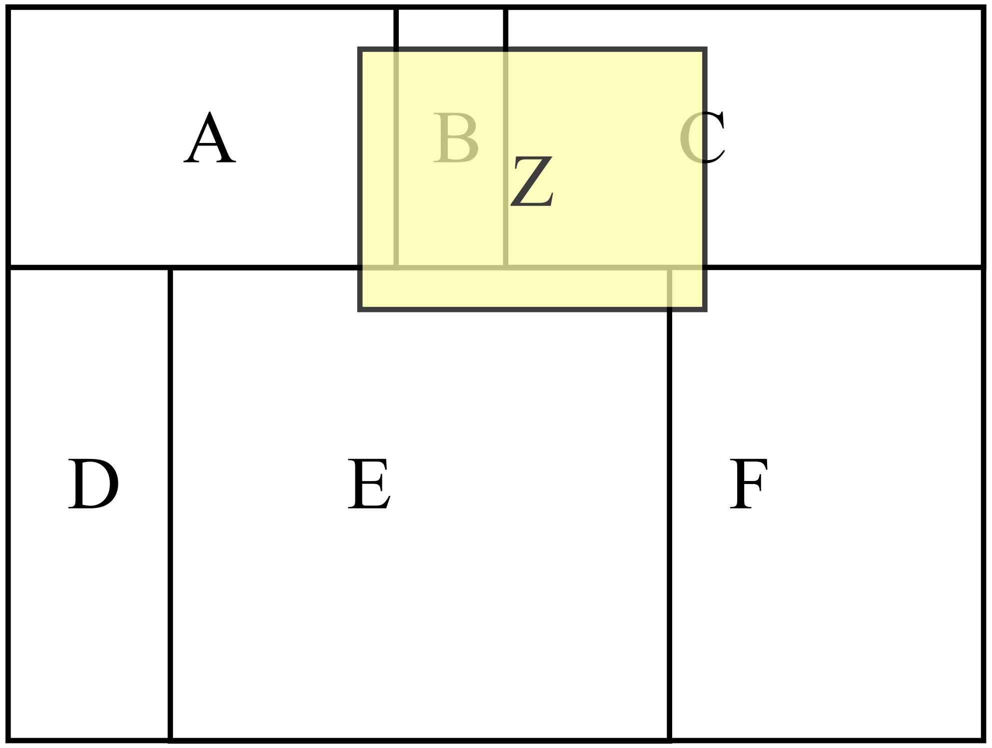

- P(A) + …. P(F) = 1
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- P(B) is the less probable event
- Now lets get a new evidence.
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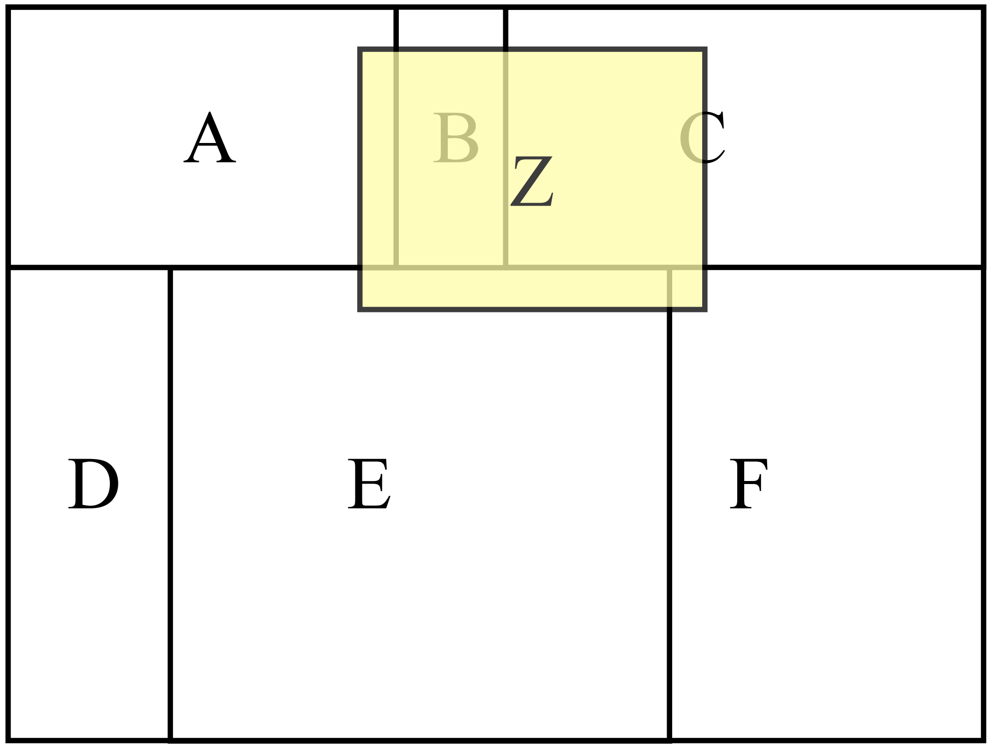

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- Now lets get a new evidence.
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- What are now the new probabilities?

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- If we now that Z happen, then $P(Z) = 1$
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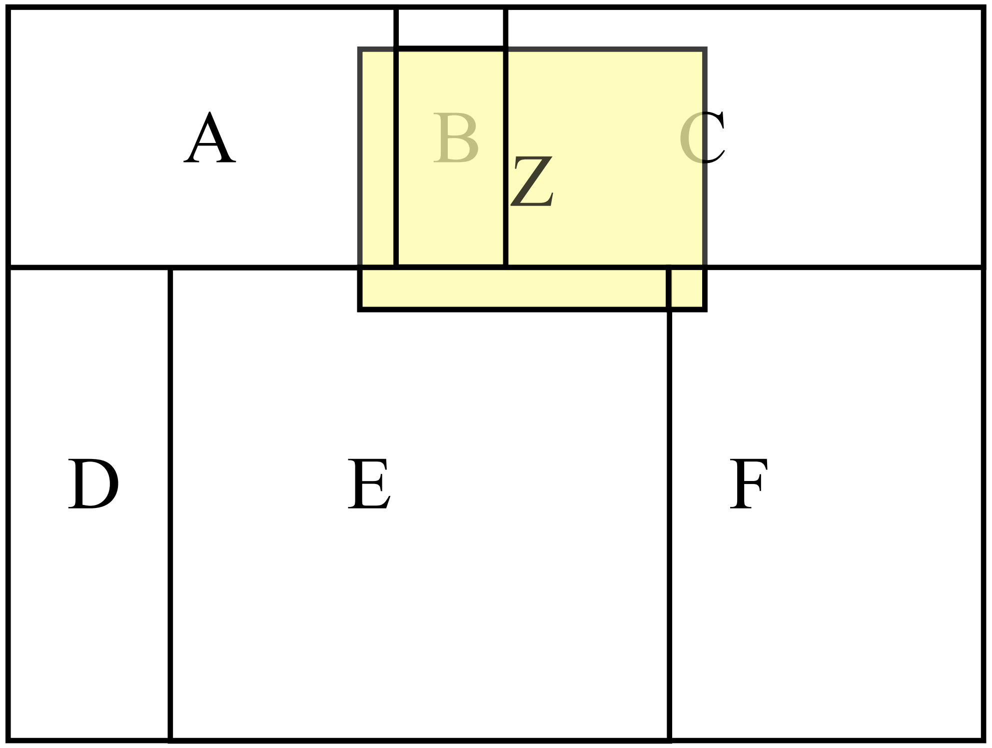

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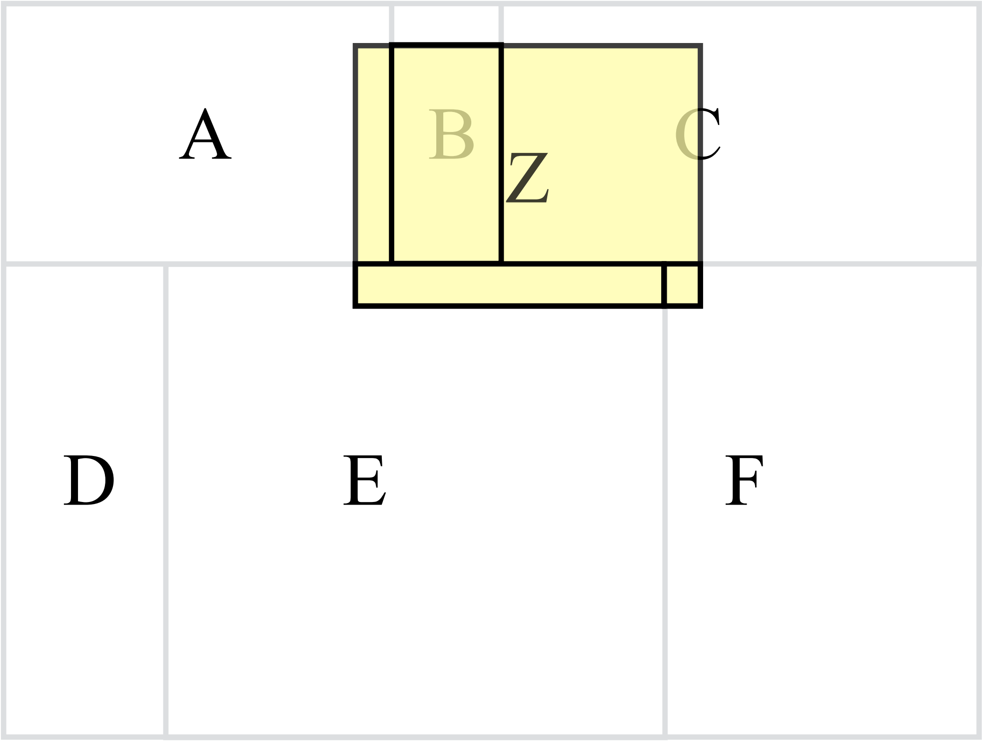

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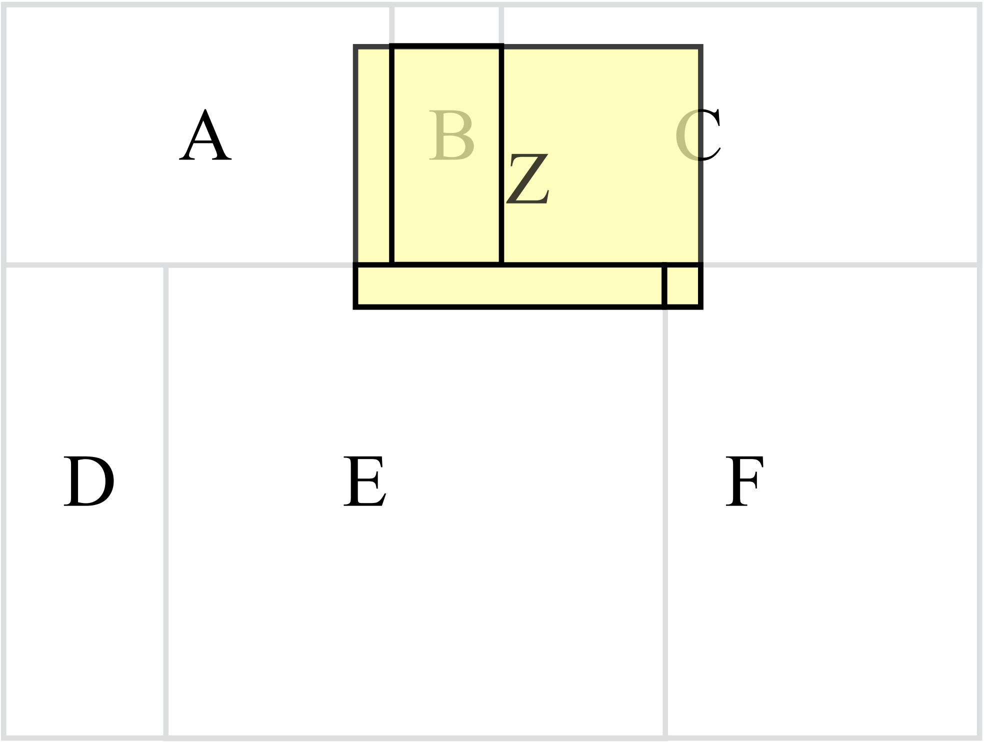

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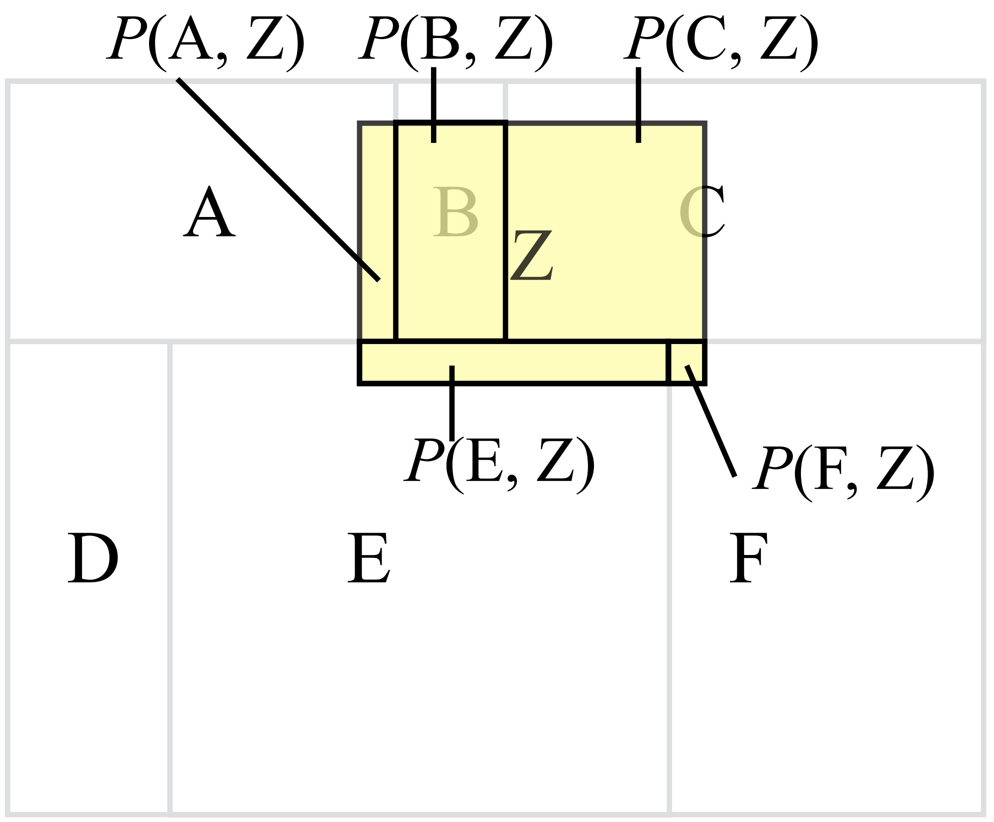

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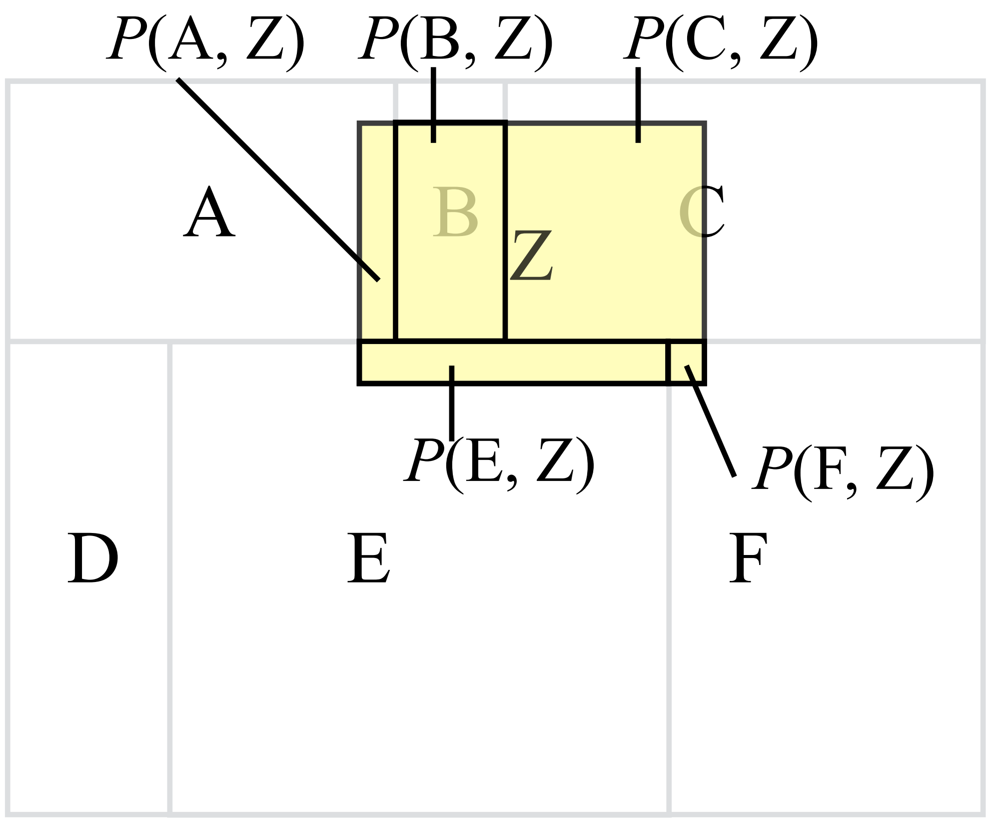

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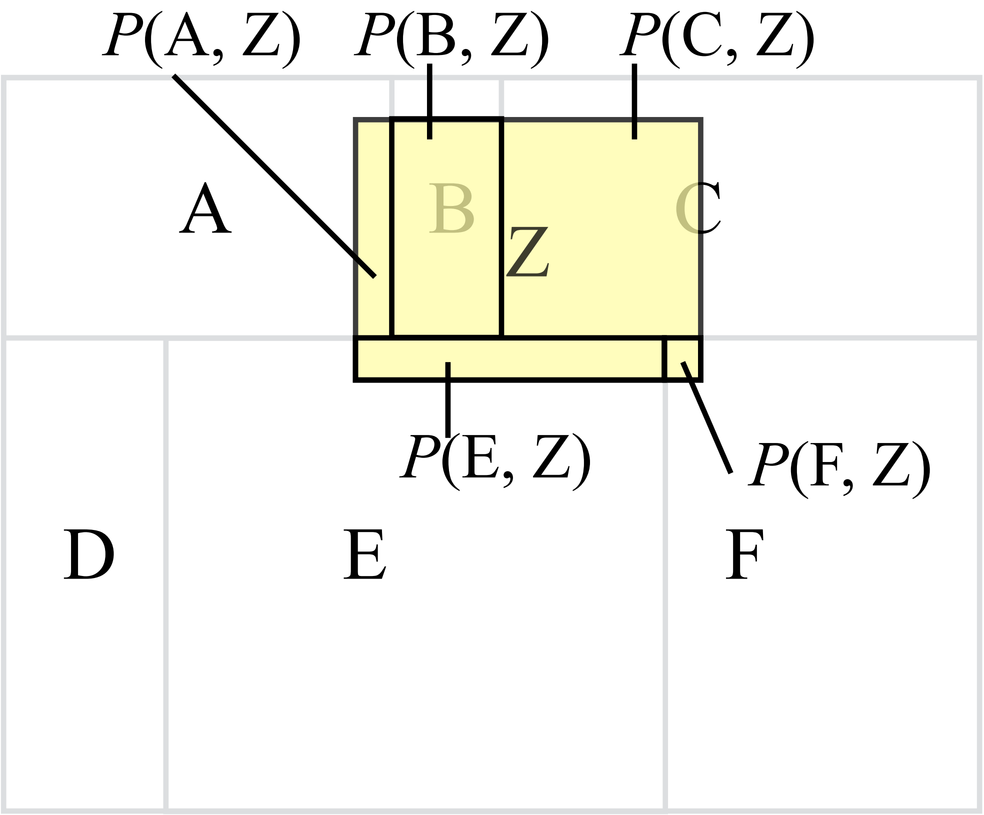

- What are the probabilities A, …., F, that are inside Z, before Z happen?
- Notice that ∑P(_, Z) = P(Z)
- Now, We just have to scale those probabilities such that P(Z) = 1, i.e, such that P(A, Z) + P(B, Z) + P(C, Z) + P(E, F) + P(F, Z) = 1

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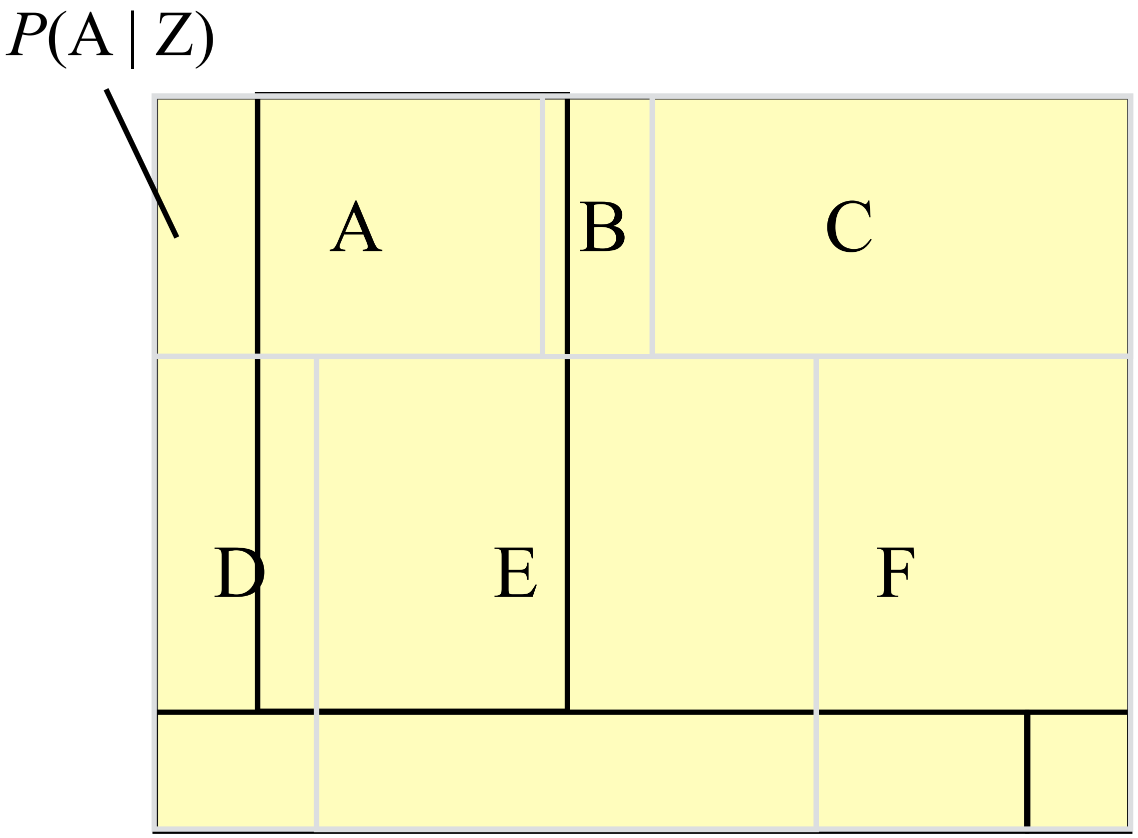

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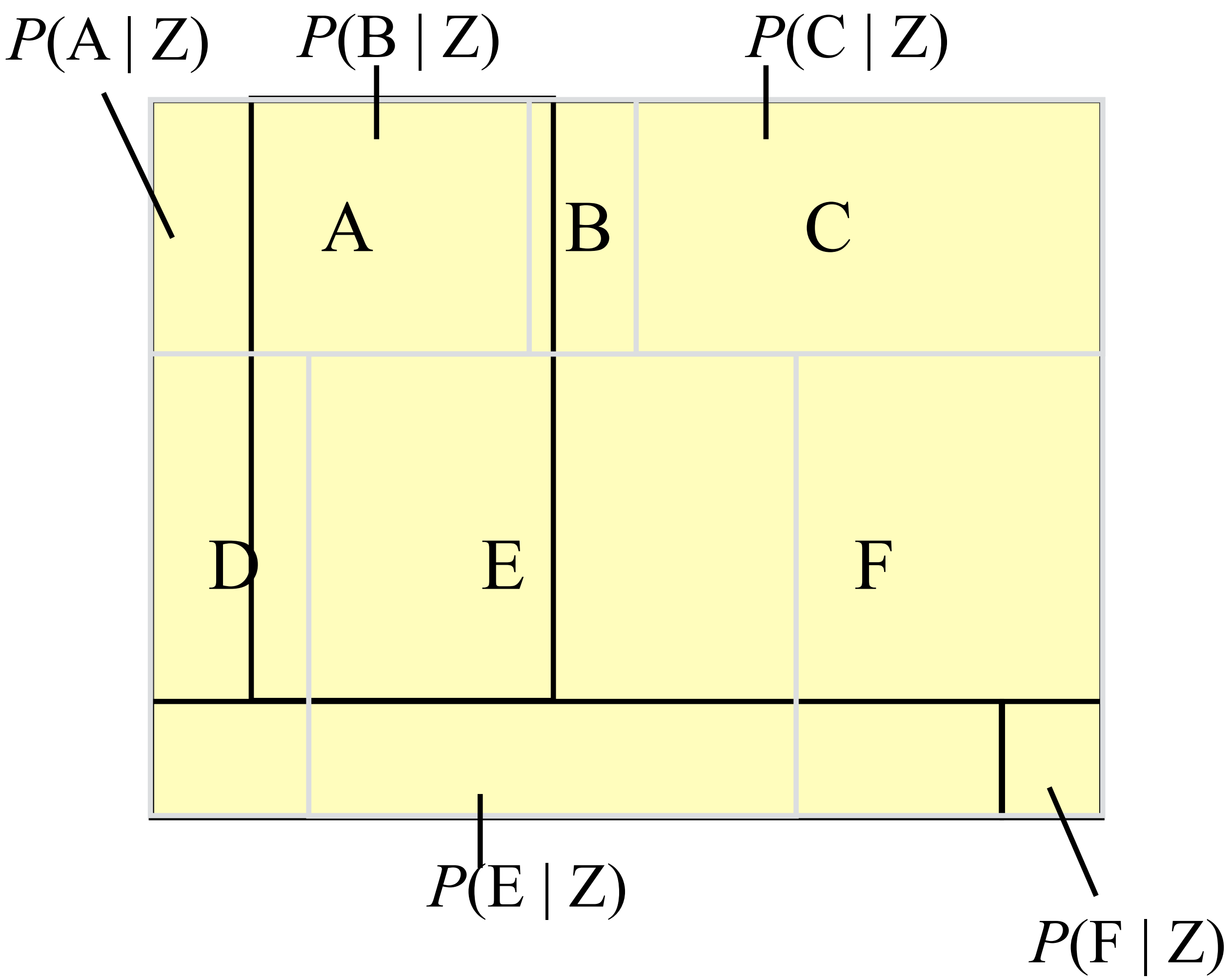

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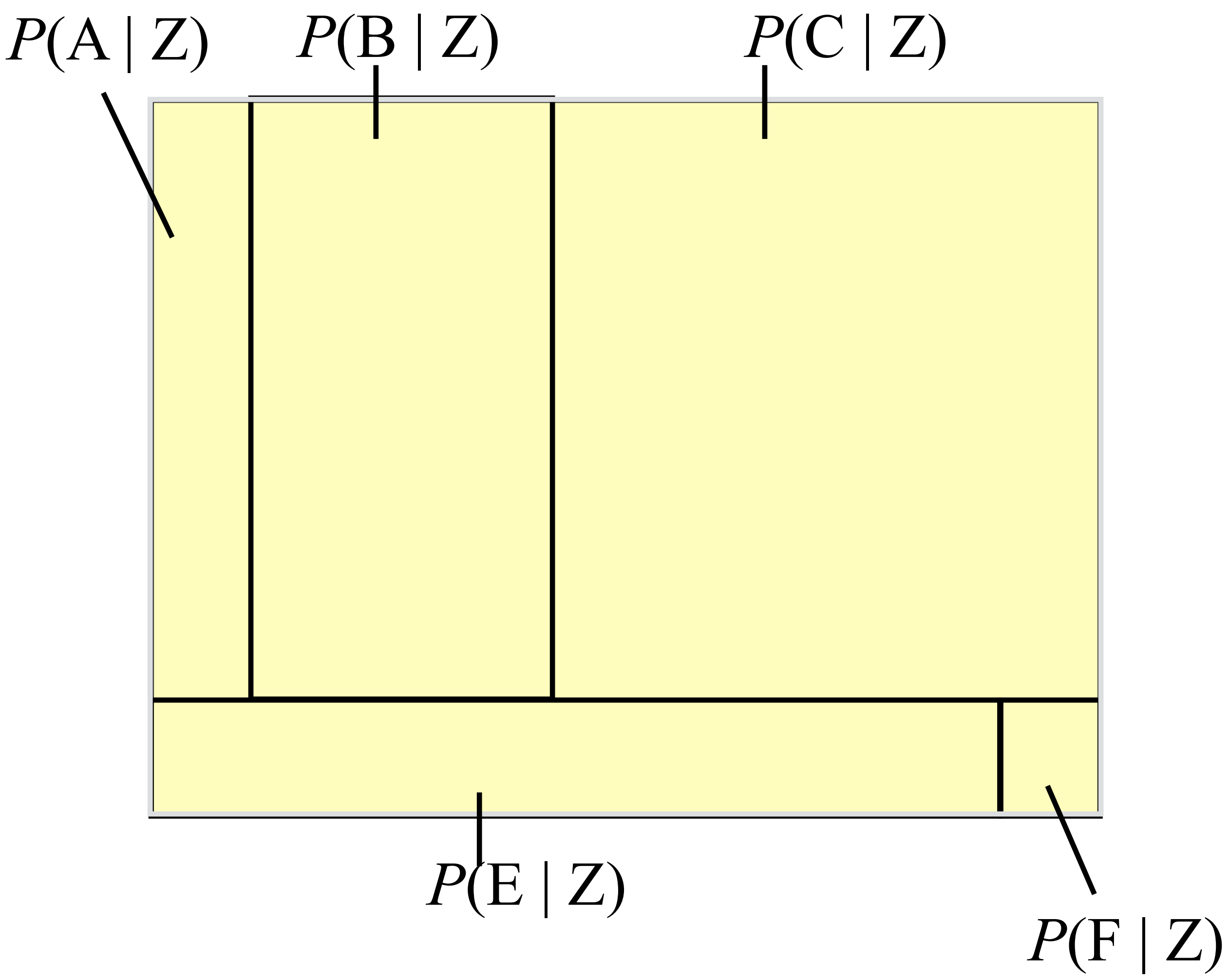

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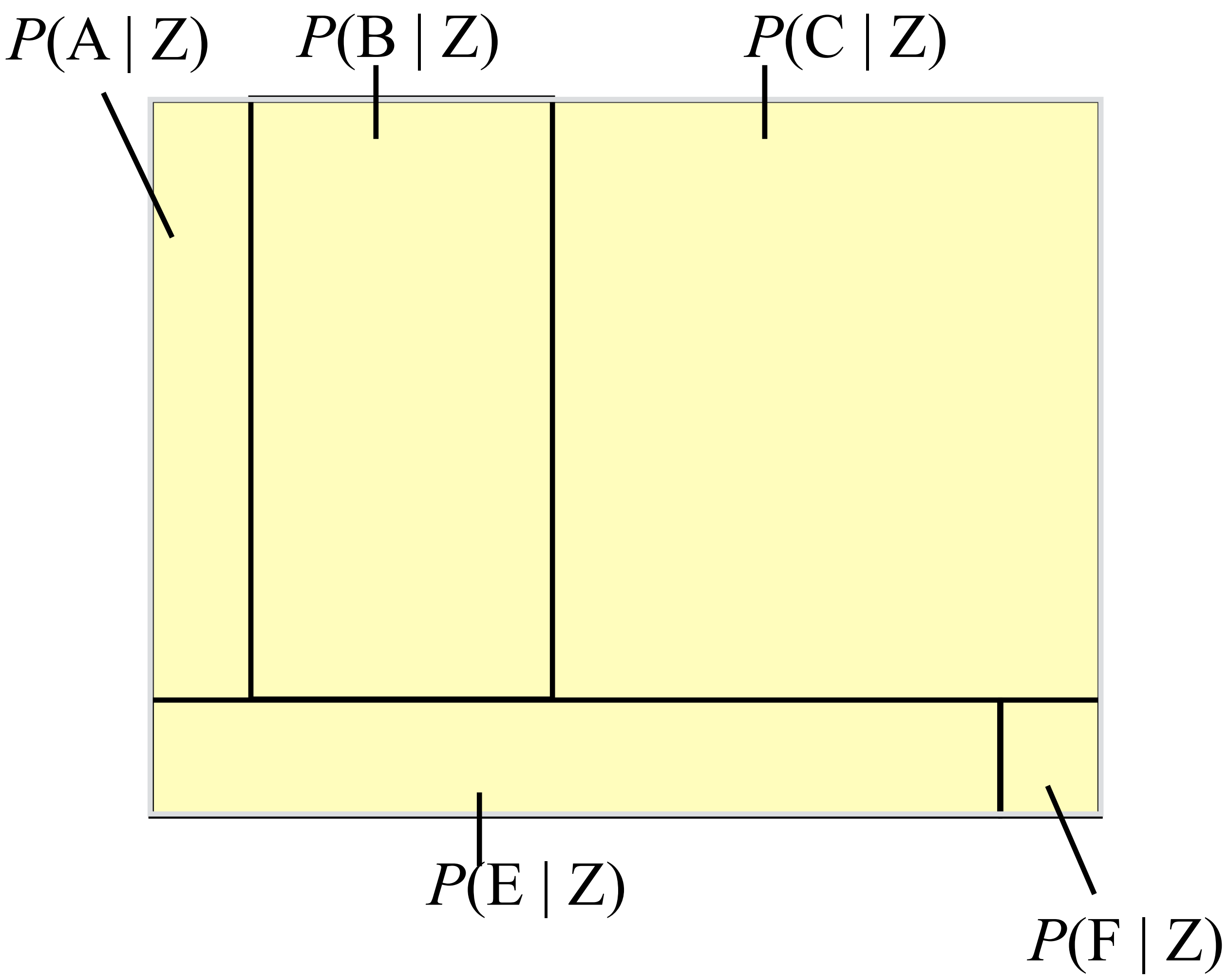

- P(A, Z) -> $P(A \mid Z) = P(A, Z) / P(Z)$
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- The disease prevalence tells us about the marginal probability of a:
  - $P(a = 1) = 0.01$
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$$P(b) = \sum_a P(a,b)$$

$P(b = 1)$	**0.059**
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An example: compute the joint probabilities … etc.

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$P(a \mid b)$	$a = 1$	$a = 0$
$b = 1$	**0.161**	**0.839**
$b = 0$	**0.001**	**0.999**

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- **Jo has the test, and the result is positive**. **What is the probability that Jo has the disease?**
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$$P(a=1) = 0.01$$

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- Two random variables *X* and *Y* are independent if and only if

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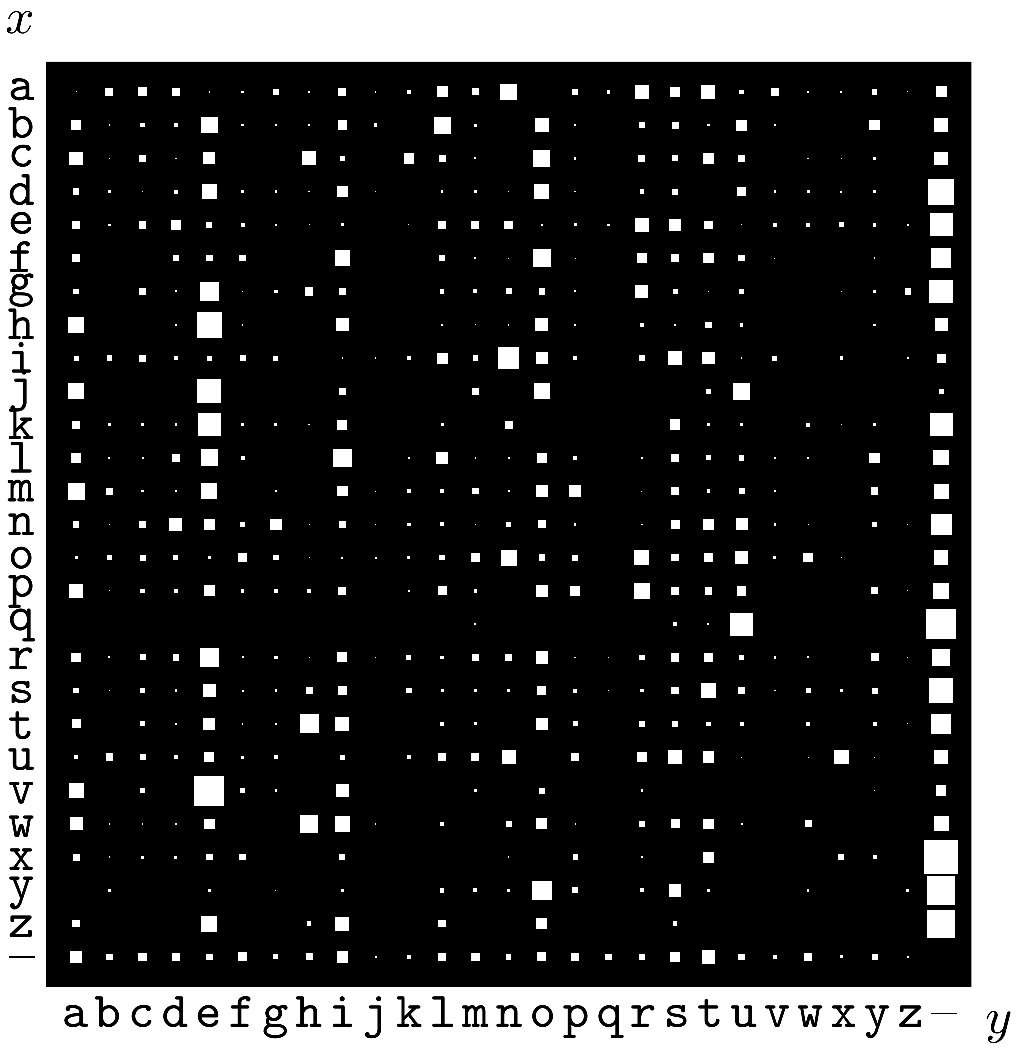
Are they independent?

$$P(\mathbf{y} \mid x)$$



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- **Inverse probability problems** involve a generative model of a process, but instead of computing the probability distribution of some quantity produced by the process, we **compute the conditional probability of one or more of the unobserved variables** in the process, given the observed variables. **This invariably requires the use of Bayes' theorem**

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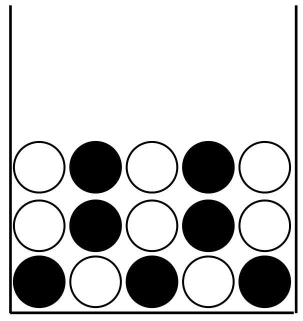

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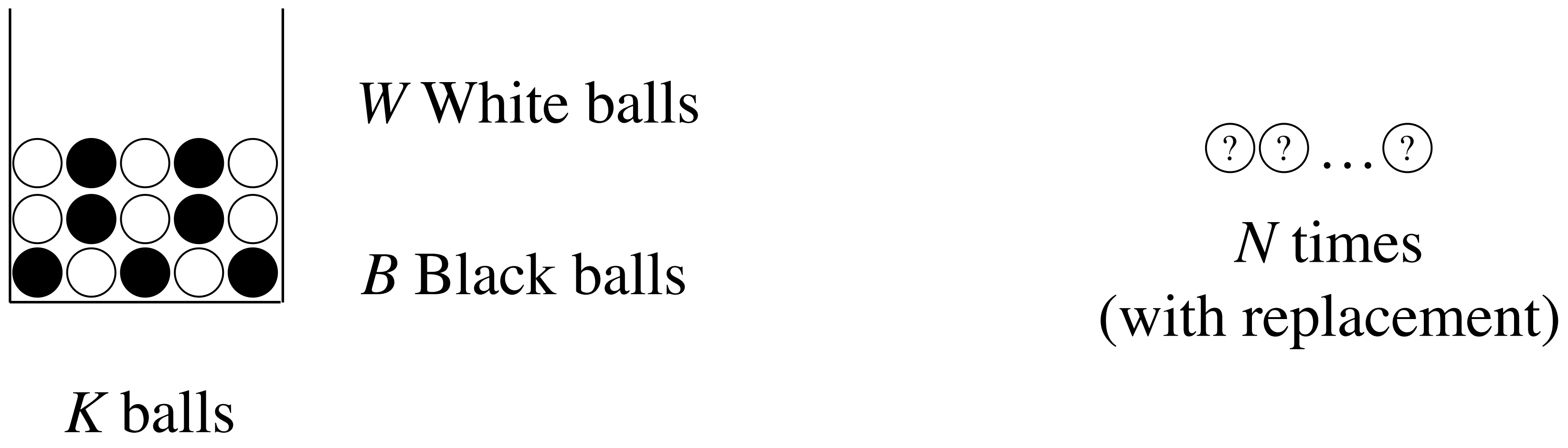

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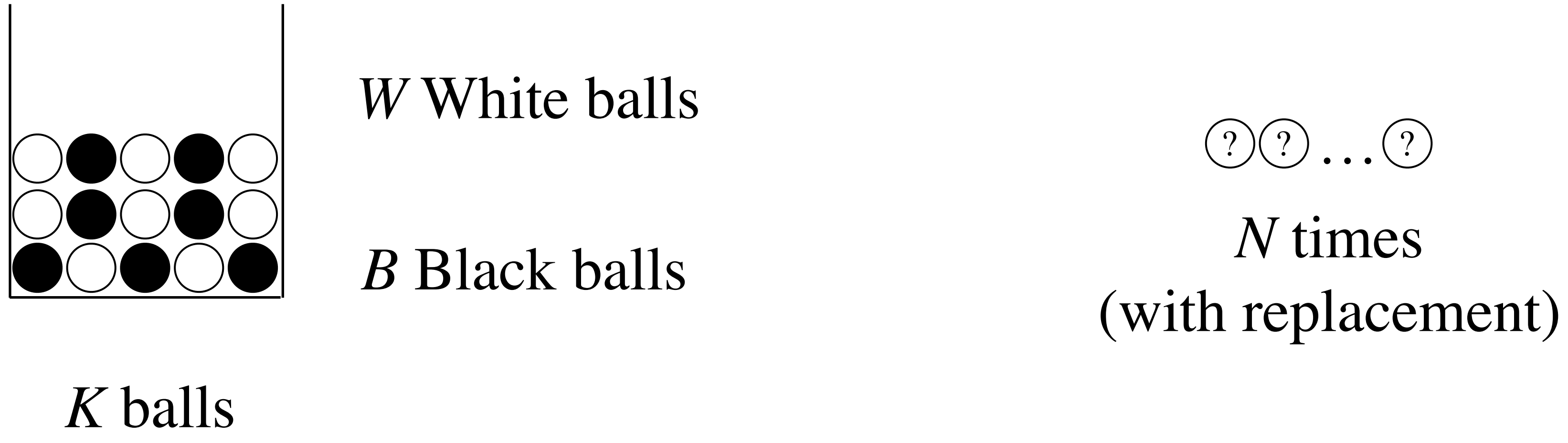

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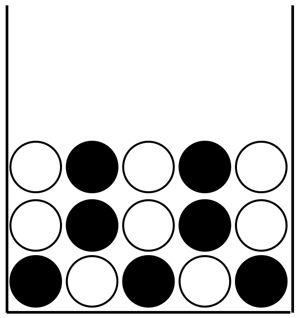

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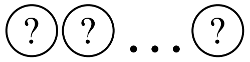

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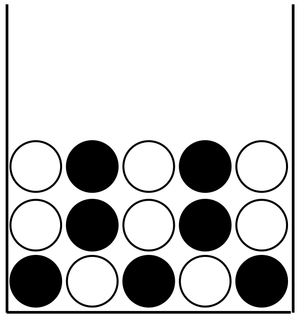

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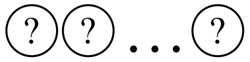

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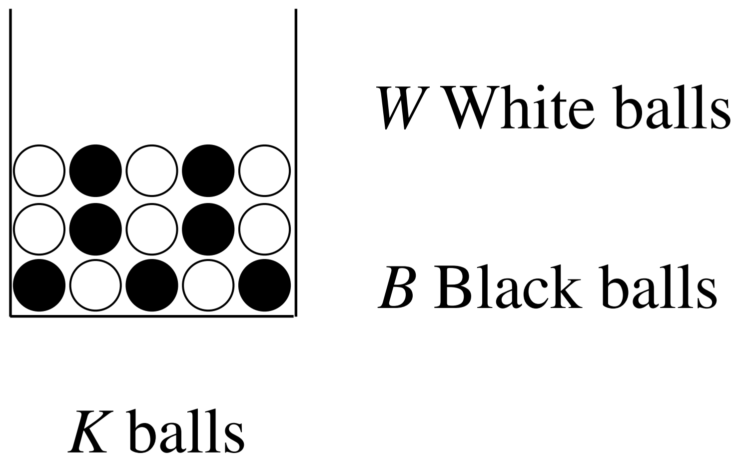

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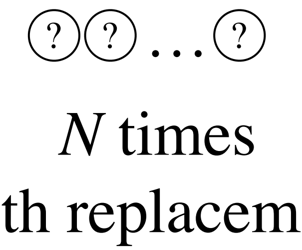

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Binomial Distribution

- Lets *f* be the probability of one outcome of a random experiment. Let *r* be a random variable that represents the number of times the outcome occurs in *N* independent experiments.

$$P(r \mid f, N) = \binom{N}{r} f^r (1-f)^{N-r}$$

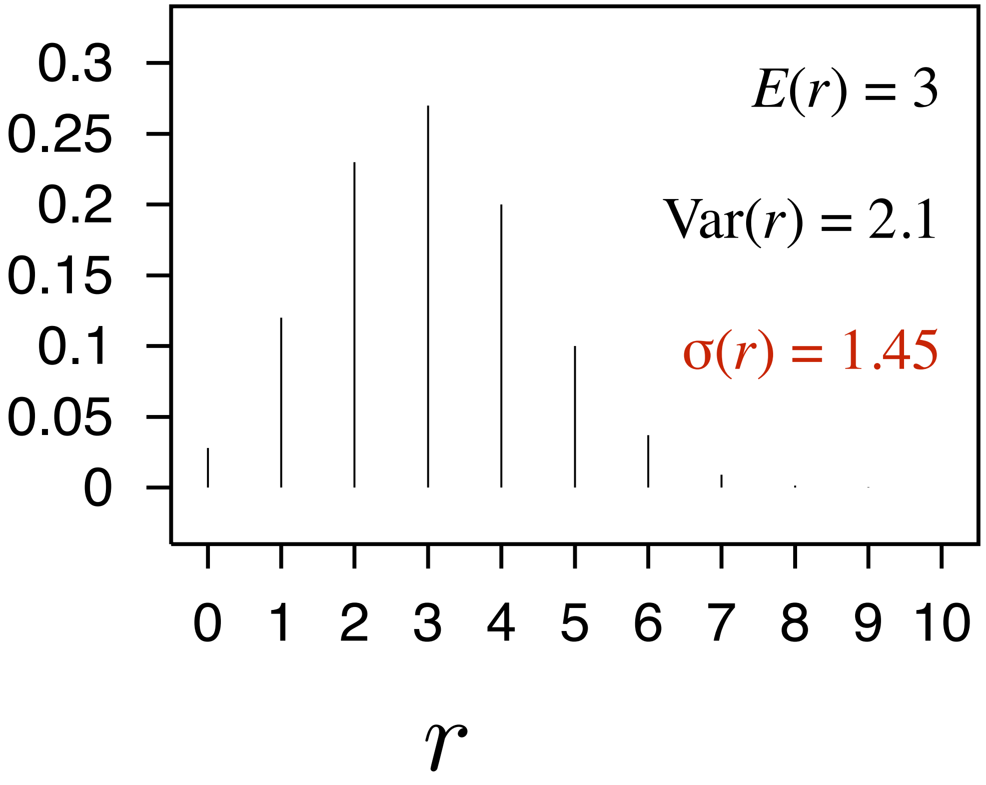
- The Mean

$$E(r) = \sum_{r=0}^{N} r.P(r \mid f, N) = Nf$$

- The Variance

$$Var(r) = E((r - E(r))^2) = Nf(1-f)$$

$f = 0.3, N = 10$



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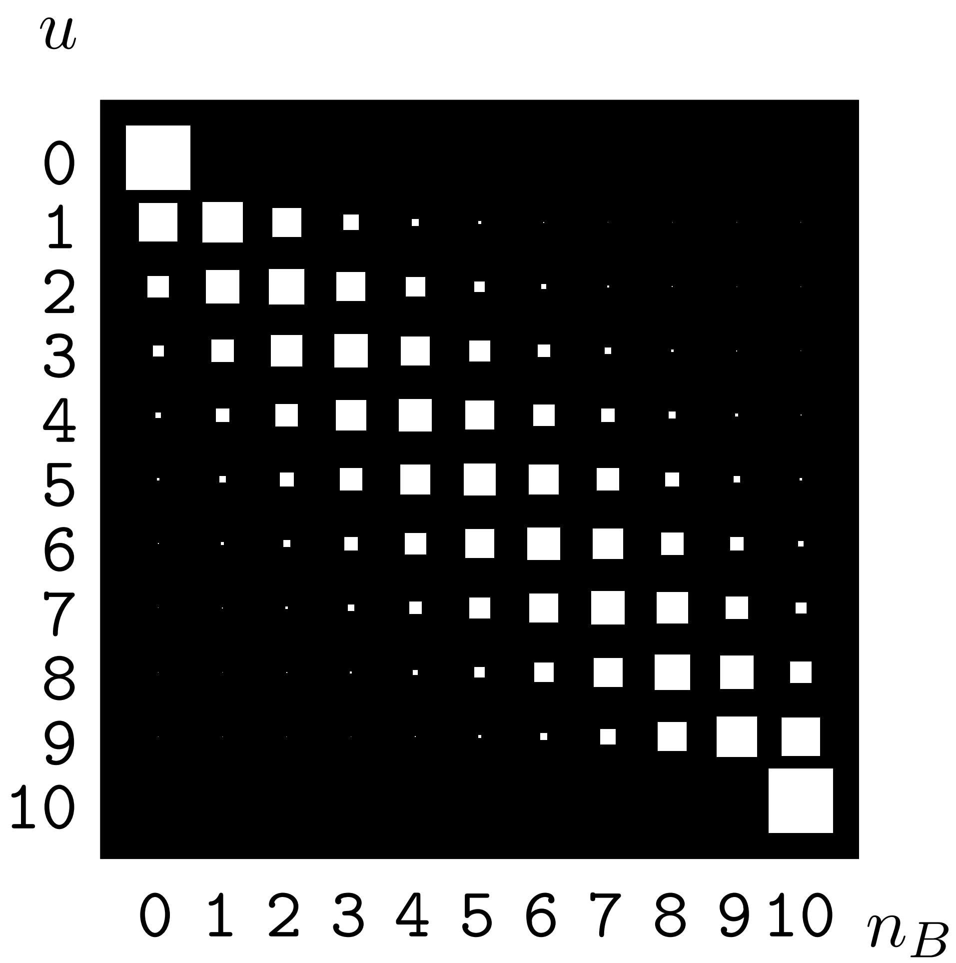

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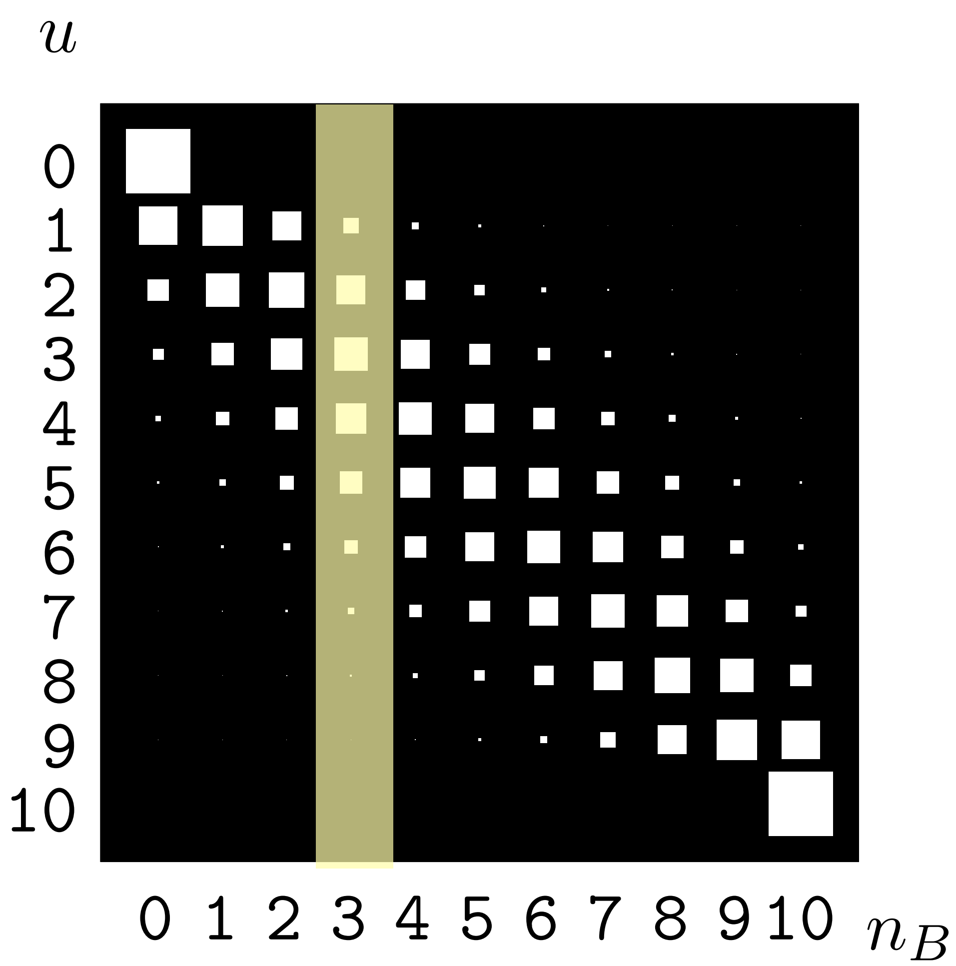

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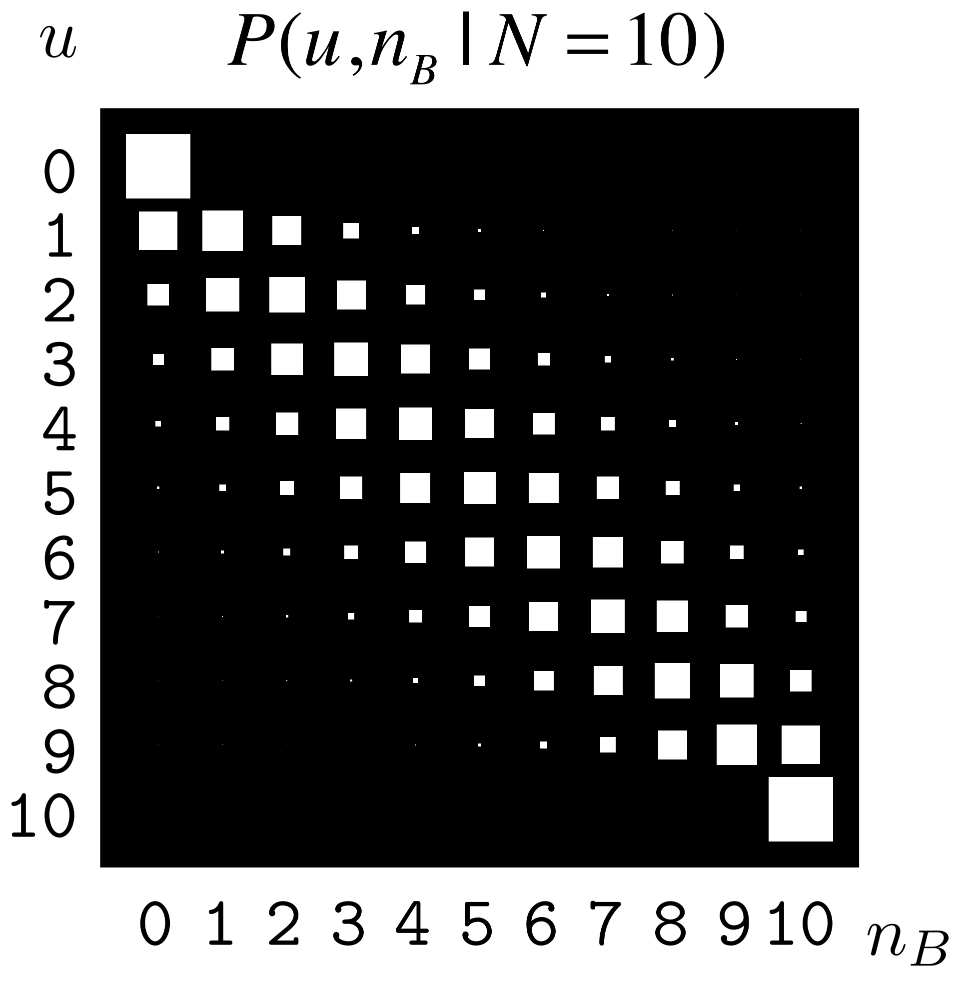

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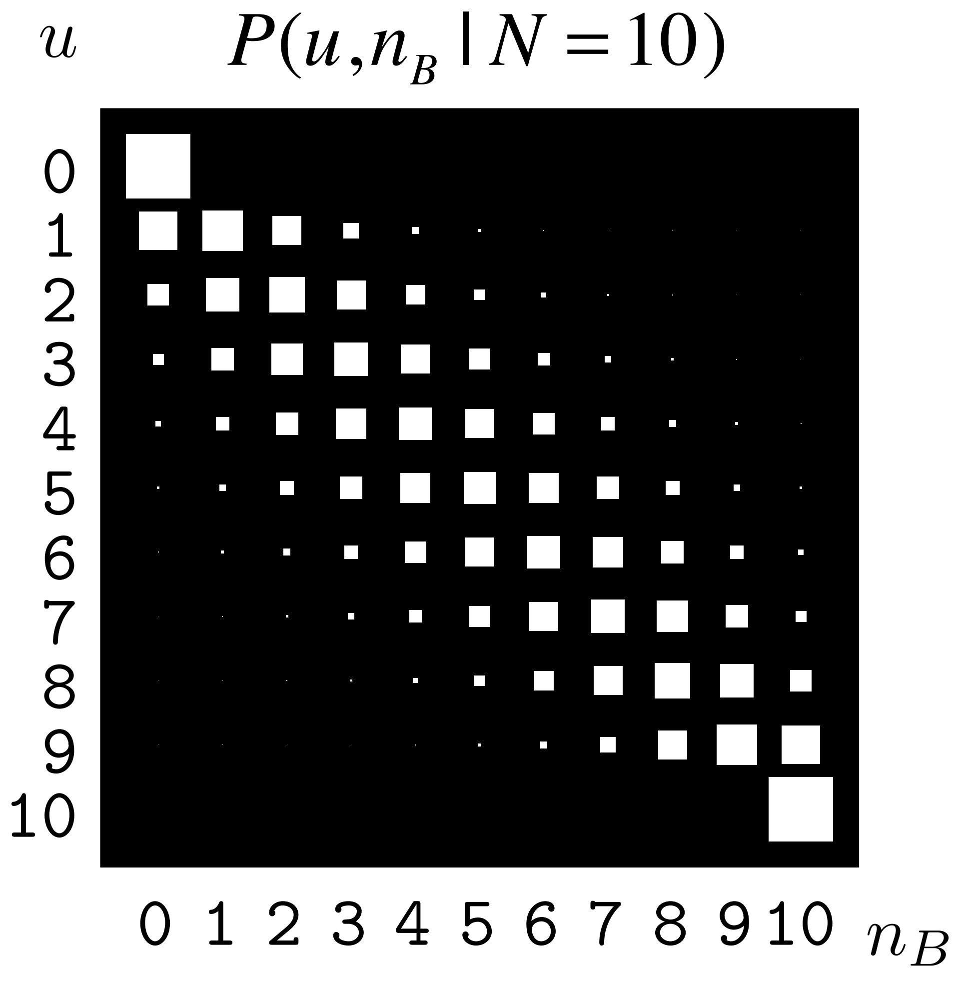

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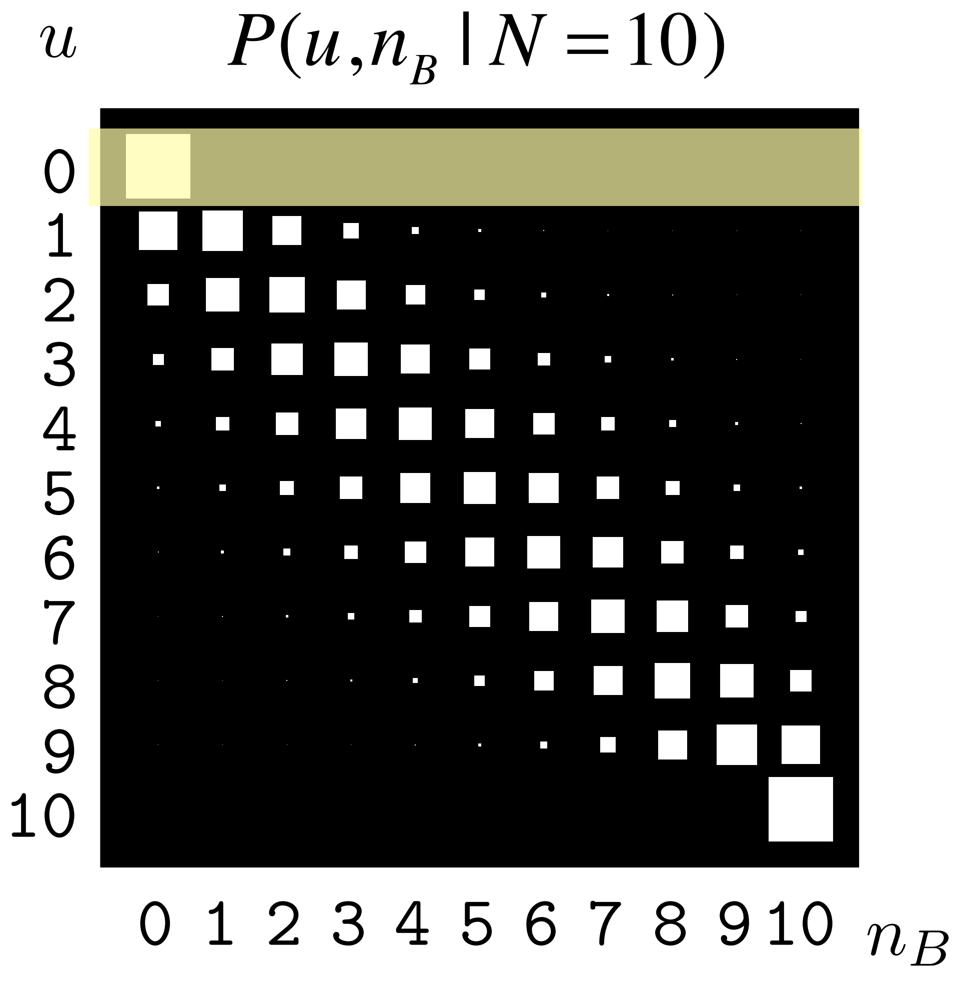

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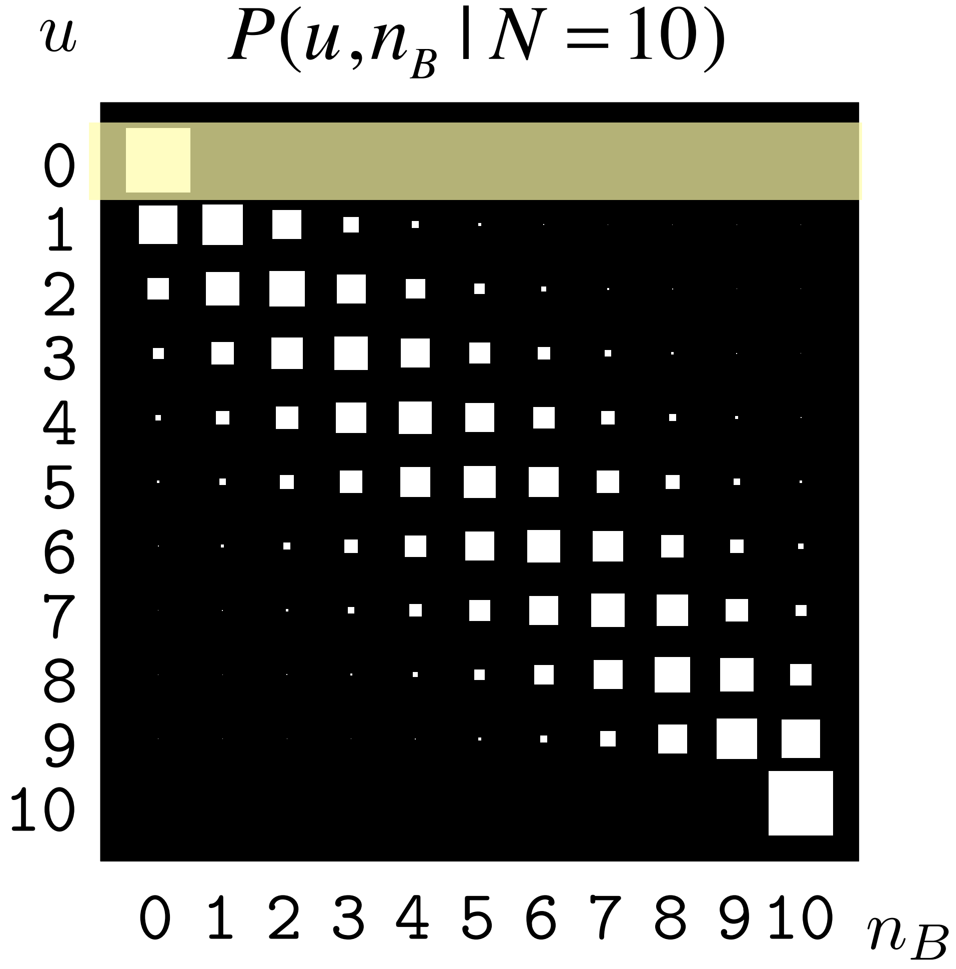

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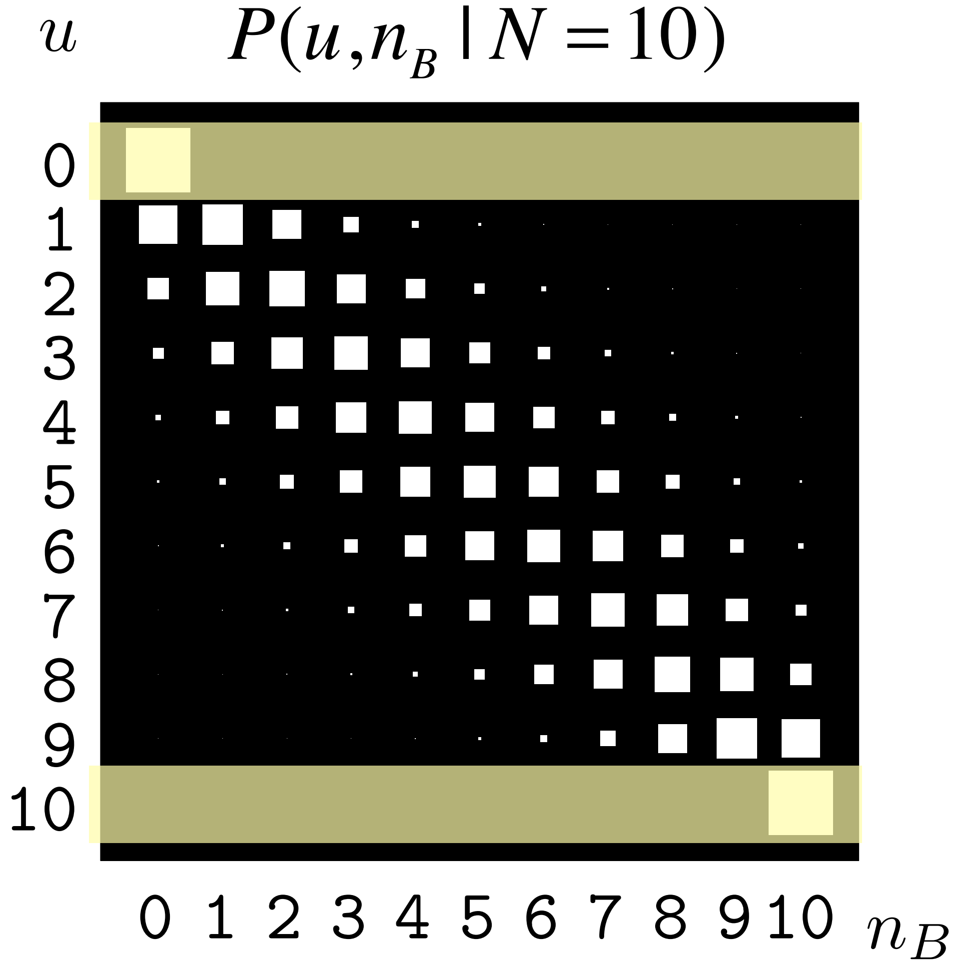

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    - For the urn u, u will be also the number of black balls in that urn,

Inverse Probability Example

$$\boxed{P(u \mid n_B, N) = ?}$$

$$P(u \mid n_B, N) = \frac{P(n_B \mid u, N) P(u)}{P(n_B \mid N)}$$

- $P(u) = 1/11$ for all u since we assume that all urns are equiprobable
- $P(n_B \mid u, N)$ can be calculated as the previous example.
  - For the urn u, u will be also the number of black balls in that urn,
  - $f_u = u/10$ (10 balls in each urn),

$$P(n_B \mid u, N) = \binom{N}{n_B} f_u^{\,n_B} (1 - f_u)^{N-n_B}$$

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Inverse Probability Example

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    - This a marginal from the joint probability $P(u, n_B \mid N)$

Inverse Probability Example

- And how to compute $P(n_B \mid N)$?

$$\boxed{P(u \mid n_B, N) = \frac{P(n_B \mid u, N) P(u)}{P(n_B \mid N)}}$$

$$P(n_B \mid N) = \sum_u P(u, n_B \mid N) = \sum_u P(u) P(n_B \mid u, N)$$

Inverse Probability Example

- And how to compute $P(n_B \mid N)$?

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- Since …

Inverse Probability Example

- And how to compute $P(n_B | N)$?

$$\boxed{P(u | n_B, N) = \frac{P(n_B | u, N) P(u)}{P(n_B | N)}}$$

$$P(n_B | N) = \sum_u P(u, n_B | N) = \sum_u P(u) P(n_B | u, N)$$

- Since …

$$P(n_B | u, N) = \binom{N}{n_B} f_u^{\,n_B} (1 - f_u)^{N - n_B}$$

Inverse Probability Example

- And how to compute $P(n_B \mid N)$?

$$\boxed{P(u \mid n_B, N) = \frac{P(n_B \mid u, N) P(u)}{P(n_B \mid N)}}$$

$$P(n_B \mid N) = \sum_u P(u, n_B \mid N) = \sum_u P(u) P(n_B \mid u, N)$$

- Since …

$$P(n_B \mid u, N) = \binom{N}{n_B} f_u^{\,n_B} (1 - f_u)^{N - n_B}$$

$$P(u) = \tfrac{1}{11}$$

Inverse Probability Example

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- And how to compute $P(n_B \mid N)$?

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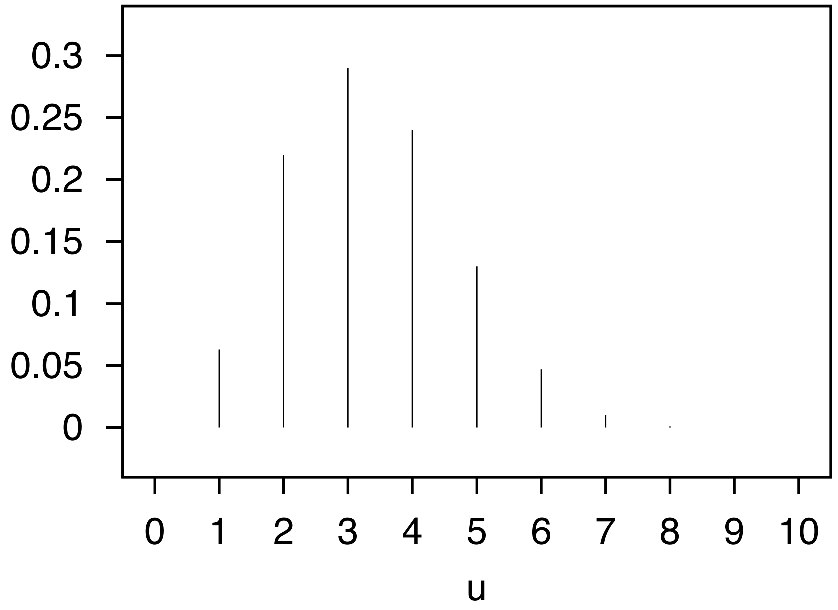
Inverse Probability Example

$$P(u \mid n_B, N) = \frac{1}{P(n_B \mid N)} \frac{1}{11} \binom{N}{n_B} f_u^{\,n_B} (1 - f_u)^{N - n_B}$$

- For $n_B = 3$ (and $N = 10$) the normalizing constant, the the marginal probability of $n_{B,}$

  $P(n_B \mid N) = 0.083$

- The posteriori probability $P(u \mid n_B = 3, N = 10)$



u	$P(u \mid n_B = 3, N)$
0	0
1	0.063
2	0.22
3	0.29
4	0.24
5	0.13
6	0.047
7	0.0099
8	0.00086
9	0.0000096
10	0

Terminology of inverse Probability

$$P(x \mid y, \boldsymbol{H}) = \frac{P(y \mid x, \boldsymbol{H}) P(x \mid \boldsymbol{H})}{P(y \mid \boldsymbol{H})}$$

Terminology of inverse Probability

$\boldsymbol{H}$ the overall hypothesis

$$P(x \mid y, \boldsymbol{H}) = \frac{P(y \mid x, \boldsymbol{H}) P(x \mid \boldsymbol{H})}{P(y \mid \boldsymbol{H})}$$

Terminology of inverse Probability

- $P(x \mid \boldsymbol{H})$ or just $P(x)$ - ***prior* probability of *x***.

H the overall hypothesis

prior probability of x

$$P(x \mid y,\boldsymbol{H}) = \frac{P(y \mid x,\boldsymbol{H})P(x \mid \boldsymbol{H})}{P(y \mid \boldsymbol{H})}$$

Terminology of inverse Probability

$\boldsymbol{H}$ the overall hypothesis

- $P(x \mid \boldsymbol{H})$ or just $P(x)$ - ***prior* probability of $\boldsymbol{x}$**.
- $P(y \mid x, \boldsymbol{H})$ or just $P(y \mid x)$ - ***likelihood* of $\boldsymbol{x}$**.
  - For a fixed x, $P(y \mid x, \boldsymbol{H})$ defines a probability over y.
  - For a fixed y, $P(y \mid x, \boldsymbol{H})$ defines the ***likelihood*** of x.

$$P(x \mid y, \boldsymbol{H}) = \frac{\overbrace{P(y \mid x, \boldsymbol{H})}^{\textit{likelihood of } x}\,\overbrace{P(x \mid \boldsymbol{H})}^{\textit{prior} \text{ probability of } x}}{P(y \mid \boldsymbol{H})}$$

Terminology of inverse Probability

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- $P(x \mid y, \boldsymbol{H})$ or just $P(x \mid y)$ - ***posterior* probability of $\boldsymbol{x}$ given $\boldsymbol{y}$**.

likelihood of x — *prior* probability of x

$$P(x \mid y, \boldsymbol{H}) = \frac{P(y \mid x, \boldsymbol{H})\,P(x \mid \boldsymbol{H})}{P(y \mid \boldsymbol{H})}$$

posterior probability of x

Terminology of inverse Probability

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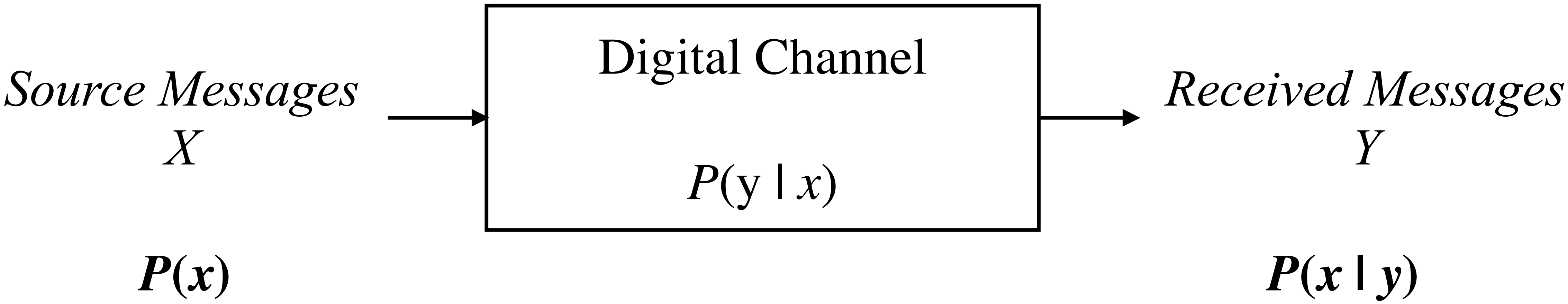
Terminology of inverse Probability



Source Messages

X

Digital Channel

$P(y \mid x)$

Received Messages

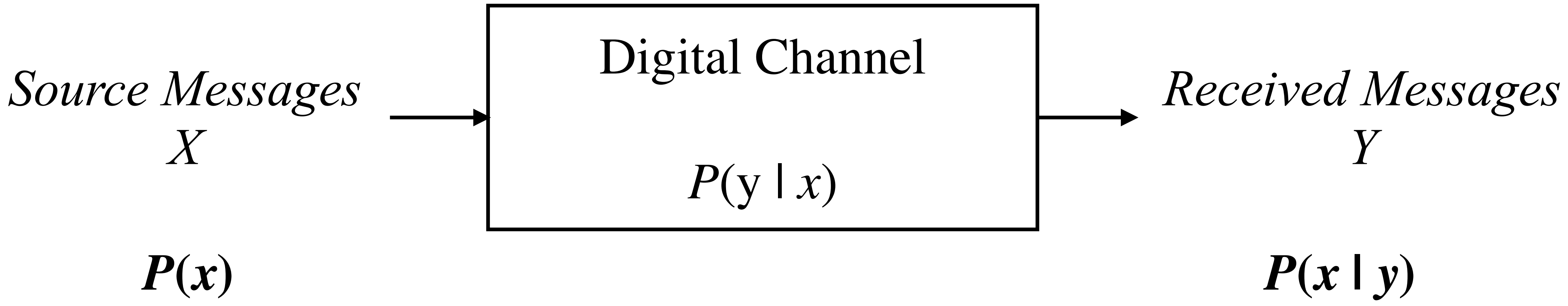
Y

$\boldsymbol{P(x)}$

$\boldsymbol{P(x \mid y)}$

Terminology of inverse Probability

H the overall hypothesis



Source Messages
X

Digital Channel

$P(y \mid x)$

Received Messages
Y

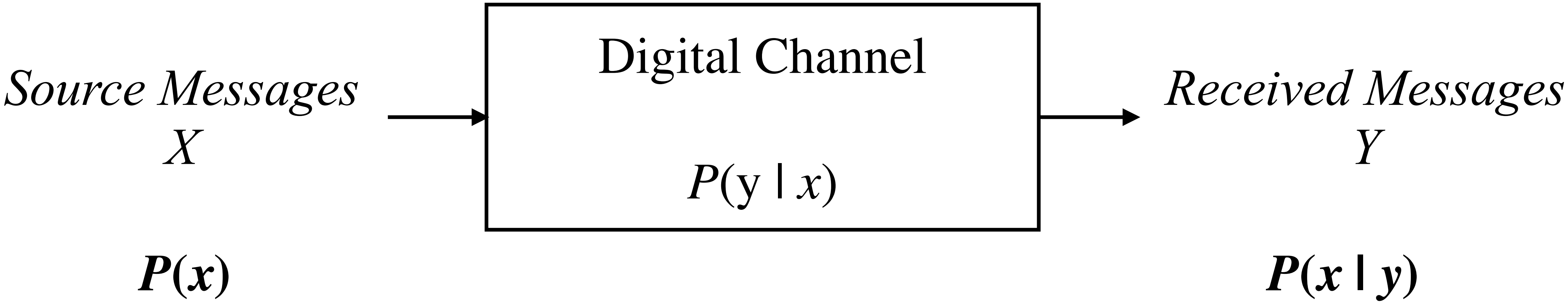
$\boldsymbol{P(x)}$

$\boldsymbol{P(x \mid y)}$

Terminology of inverse Probability

- $P(x \mid \boldsymbol{H})$ or just $P(x)$ - ***prior* probability of *x***.

H the overall hypothesis



Source Messages
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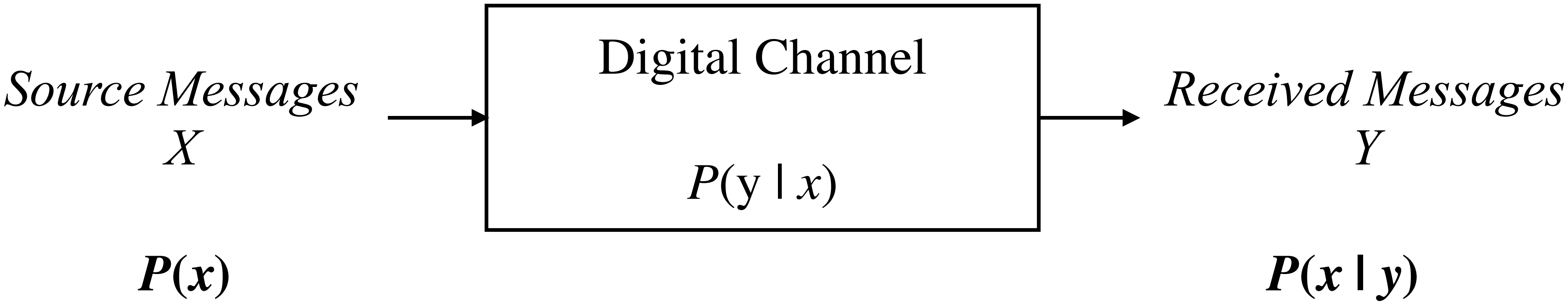

Source Messages
X

$\boldsymbol{P(x)}$

Digital Channel

$P(\mathrm{y} \mid x)$

Received Messages
Y

$\boldsymbol{P(x \mid y)}$

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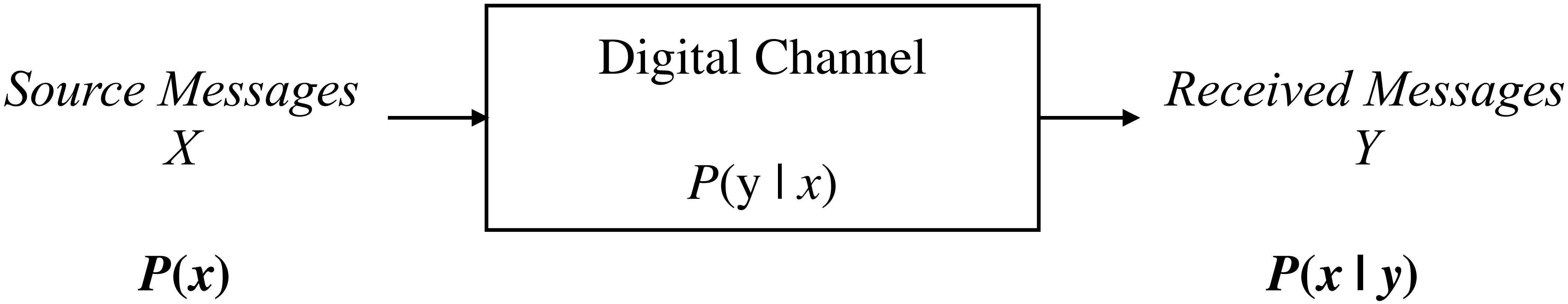

Source Messages
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Received Messages
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$\boldsymbol{P(x \mid y)}$

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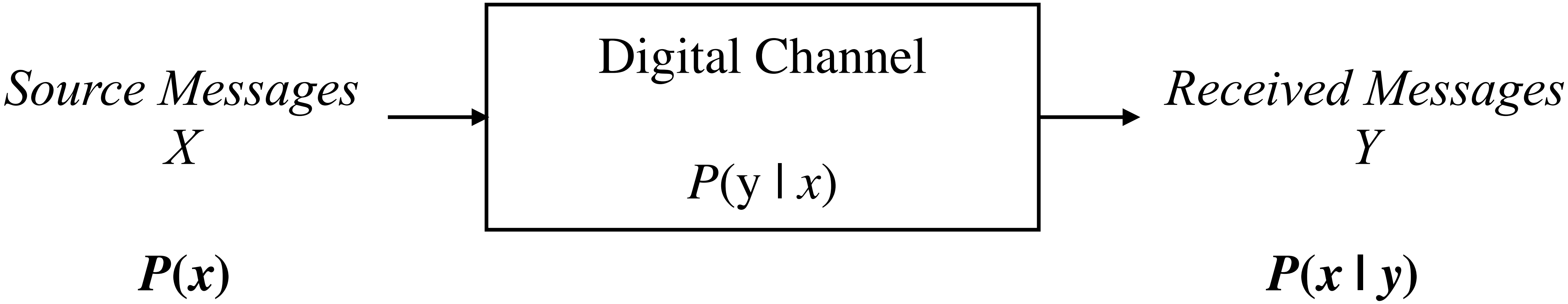

Source Messages
X

Digital Channel

$P(y \mid x)$

Received Messages
Y

$\boldsymbol{P(x)}$

$\boldsymbol{P(x \mid y)}$

Terminology of inverse Probability

- **In general** *Θ* represents the **unknown parameters**
- *D* represents the **data**
- ***H*** the overall hypothesis

$$P(\theta \mid D, \boldsymbol{H}) = \frac{P(D \mid \theta, \boldsymbol{H}) P(\theta \mid \boldsymbol{H})}{P(D \mid \boldsymbol{H})}$$

$$posterior = \frac{likelihood \times prior}{evidence}$$

Terminology of inverse Probability: an example

- Variables (*a* - Jo's state of the health; *b* - Result of the test)

Initial information

$P(a = 1)$	$P(a = 0)$
0.01	**0.99**

$P(b \mid a)$	$a = 1$	$a = 0$
$b = 1$	**0.95**	**0.05**
$b = 0$	**0.05**	**0.95**

Terminology of inverse Probability: an example

- Variables (*a* - Jo's state of the health; *b* - Result of the test)

Initial information

$P(a = 1)$	$P(a = 0)$
0.01	**0.99**

→ ***prior*** **probability of *a***

$P(b \mid a)$	$a = 1$	$a = 0$
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$b = 0$	**0.05**	**0.95**

Terminology of inverse Probability: an example

- Variables (*a* - Jo's state of the health; *b* - Result of the test)

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→ ***prior* probability of *a***

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$b = 1$	**0.95**	**0.05**
$b = 0$	**0.05**	**0.95**

→ ***likelihood* of *a given b* = 1**

Terminology of inverse Probability: an example

- Variables (a - Jo's state of the health; b - Result of the test)

Initial information

$P(a = 1)$	$P(a = 0)$
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$P(b \mid a)$	$a = 1$	$a = 0$
$b = 1$	**0.95**	**0.05**
$b = 0$	**0.05**	**0.95**

→ ***likelihood* of a *given* $b = 1$**

→ ***likelihood* of a *given* $b = 0$**

Terminology of inverse Probability: an example

- Variables (a - Jo's state of the health; b - Result of the test)

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$P(a = 1)$	$P(a = 0)$
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→ ***prior* probability of a**

$P(b \mid a)$	$a = 1$	$a = 0$
$b = 1$	**0.95**	**0.05**
$b = 0$	**0.05**	**0.95**

→ ***likelihood* of a *given* $b = 1$**

→ ***likelihood* of a *given* $b = 0$**

↓

***probability* of b *given* $a = 0$**

Terminology of inverse Probability: an example

- Variables (a - Jo's state of the health; b - Result of the test)

Initial information

$P(a = 1)$	$P(a = 0)$
0.01	**0.99**

prior **probability of** ***a***

$P(b \mid a)$	$a = 1$	$a = 0$
$b = 1$	**0.95**	**0.05**
$b = 0$	**0.05**	**0.95**

likelihood **of *a given b* = 1**

likelihood **of *a given b* = 0**

probability **of *b given a* = 0**

probability **of *b given a* = 1**

Terminology of inverse Probability: an example

- Variables (a - Jo’s state of the health; b - Result of the test)

Initial information

$P(a = 1)$	$P(a = 0)$
0.01	**0.99**

→ *prior* **probability of** a

$P(b \mid a)$	$a = 1$	$a = 0$
$b = 1$	**0.95**	**0.05**
$b = 0$	**0.05**	**0.95**

→ *likelihood* **of** a *given* $b = 1$

→ *likelihood* **of** a *given* $b = 0$

$$P(a, b) = P(b \mid a)\ P(a)$$

$P(a, \mathrm{b})$	$a = 1$	$a = 0$
$b = 1$	**0.0095**	**0.0495**
$b = 0$	**0.0005**	**0.9405**

$$P(b) = \sum_{a} P(a,b)$$

$P(b = 1)$	**0.059**
$P(b = 0)$	**0.941**

Terminology of inverse Probability: an example

- Variables (*a* - Jo’s state of the health; *b* - Result of the test)

Initial information

$P(a = 1)$	$P(a = 0)$
0.01	**0.99**

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→ ***likelihood* of *a given b* = 1**

→ ***likelihood* of *a given b* = 0**

$$P(a, b) = P(b \mid a)\, P(a)$$

$P(a, \mathrm{b})$	$a = 1$	$a = 0$
$b = 1$	**0.0095**	**0.0495**
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joint distribution

$$P(b) = \sum_a P(a,b)$$

$P(b = 1)$	**0.059**
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Terminology of inverse Probability: an example

- Variables (*a* - Jo’s state of the health; *b* - Result of the test)

Initial information

P(*a* = 1)	*P*(*a* = 0)
0.01	**0.99**

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P(*b* | *a*)	*a* = 1	*a* = 0
b = 1	**0.95**	**0.05**
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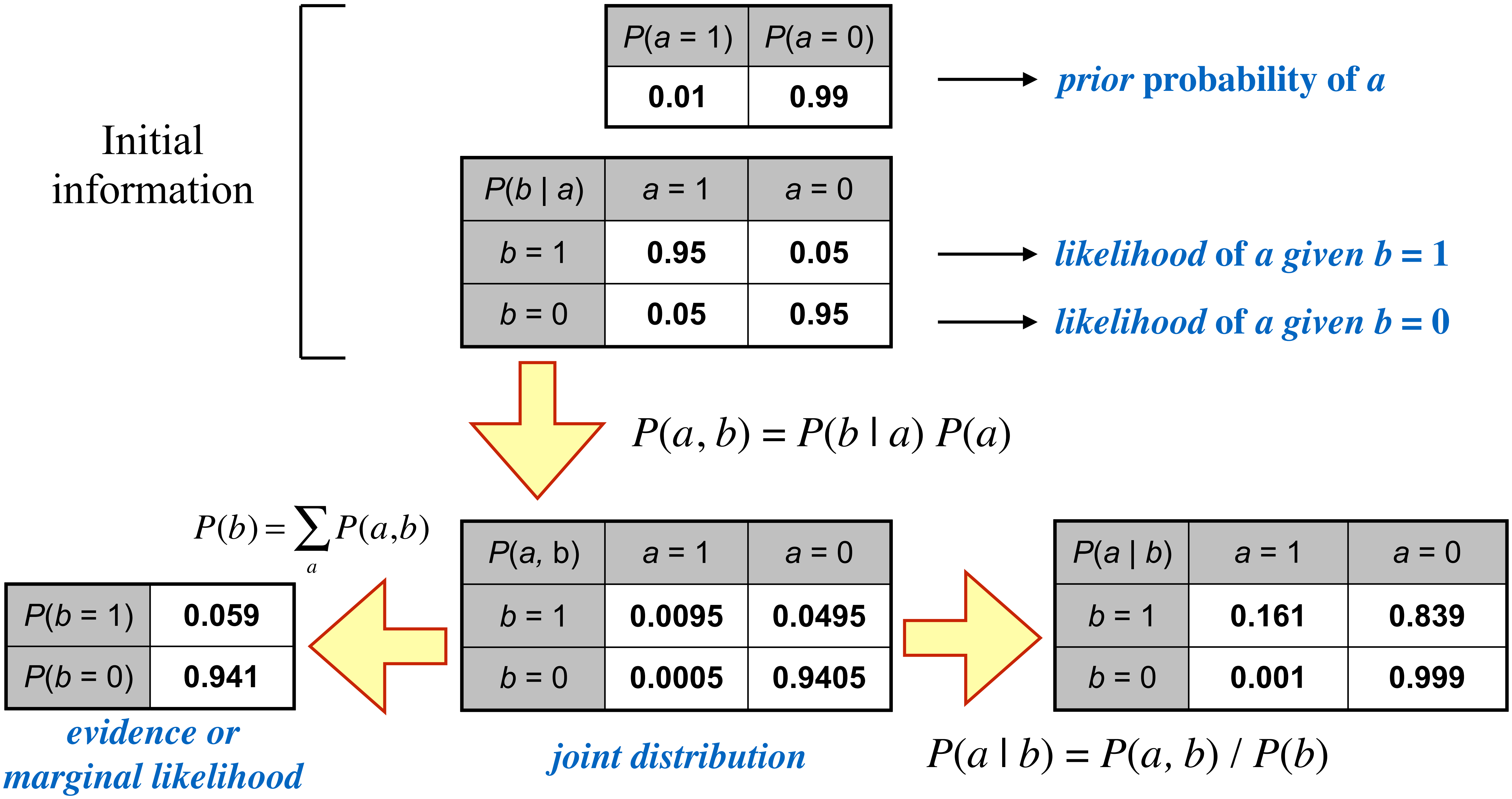
evidence or marginal likelihood

Terminology of inverse Probability: an example

- Variables (*a* - Jo's state of the health; *b* - Result of the test)



Initial information

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$b = 0$	**0.05**	**0.95**

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→ ***likelihood* of *a given b* = 0**

$$P(a, b) = P(b \mid a)\, P(a)$$

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joint distribution

$$P(b) = \sum_a P(a,b)$$

$P(b = 1)$	**0.059**
$P(b = 0)$	**0.941**

evidence or marginal likelihood

$P(a \mid b)$	$a = 1$	$a = 0$
$b = 1$	**0.161**	**0.839**
$b = 0$	**0.001**	**0.999**

$$P(a \mid b) = P(a, b) \,/\, P(b)$$

Terminology of inverse Probability: an example

- Variables (*a* - Jo's state of the health; *b* - Result of the test)

Initial information

$P(a = 1)$	$P(a = 0)$
0.01	**0.99**

→ ***prior* probability of *a***

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$b = 1$	**0.95**	**0.05**
$b = 0$	**0.05**	**0.95**

→ ***likelihood* of *a given b* = 1**

→ ***likelihood* of *a given b* = 0**

$$P(a, b) = P(b \mid a)\, P(a)$$

$$P(b) = \sum_a P(a,b)$$

$P(b = 1)$	**0.059**
$P(b = 0)$	**0.941**

evidence or marginal likelihood

$P(a, \mathrm{b})$	$a = 1$	$a = 0$
$b = 1$	**0.0095**	**0.0495**
$b = 0$	**0.0005**	**0.9405**

joint distribution

posterior* probability of *a

$P(a \mid b)$	$a = 1$	$a = 0$
$b = 1$	**0.161**	**0.839**
$b = 0$	**0.001**	**0.999**

$$P(a \mid b) = P(a, b) \,/\, P(b)$$

Terminology of inverse Probability: an example

- Variables (a - Jo’s state of the health; b - Result of the test)

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$P(a = 1)$	$P(a = 0)$
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→ ***prior* probability of *a***

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$$P(a, b) = P(b \mid a)\, P(a)$$

$P(a, \mathrm{b})$	$a = 1$	$a = 0$
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$b = 0$	**0.0005**	**0.9405**

joint distribution

$$P(b) = \sum_a P(a,b)$$

$P(b = 1)$	**0.95**
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evidence or marginal likelihood

***posterior* probability of *a* given *b* = 1**

$P(a \mid b)$	$a = 1$	$a = 0$
$b = 1$	**0.161**	**0.839**
$b = 0$	**0.001**	**0.999**

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$b = 0$	**0.05**	**0.95**

→ ***likelihood*** **of *a given b* = 1**

→ ***likelihood*** **of *a given b* = 0**

$$P(a,b) = P(b \mid a)\,P(a)$$

$P(a, \mathrm{b})$	$a = 1$	$a = 0$
$b = 1$	**0.0095**	**0.0495**
$b = 0$	**0.0005**	**0.9405**

joint distribution

$$P(b) = \sum_a P(a,b)$$

$P(b=1)$	**0.95**
$P(b=0)$	**0.05**

evidence or marginal likelihood

posterior **probability of *a* given *b* = 1**

$P(a \mid b)$	$a = 1$	$a = 0$
$b = 1$	**0.161**	**0.839**
$b = 0$	**0.001**	**0.999**

posterior **probability of *a* given *b* = 0**

Inverse probability and **prediction**

- Assuming again that Bill has observed $n_B = 3$ blacks in $N = 10$ draws, let Fred draw another ball from the **same urn**. **What is the probability that the next drawn ball is a black?**

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$$P(ball_{N+1} = black \mid u, n_B, N) = f_u = {}^{u}\!/_{10}$$

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$$P(ball_{N+1} = black \mid n_B, N) = \sum_u f_u P(u \mid n_B, N)$$

- Since the values of $P(u \mid n_B, N)$ were previously calculated,

$$P(ball_{N+1} = black \mid n_B, N) = 0.333$$

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Inverse probability and prediction

u
0
1
2
3
4
5
6
7
8
9
10

Inverse probability and prediction

u	$P(u \mid n_B, N)$
0	0
1	0,063
2	0,22
3	0,29
4	0,24
5	0,13
6	0,047
7	0,0099
8	0,00086
9	0,0000096
10	0

Inverse probability and prediction

$$P(ball_{N+1} = black \mid u, n_B, N) = f_u = {}^{1}\!/_{10}$$

u	$P(u \mid n_B, N)$	$f_u = u/10$
0	0	0
1	0,063	0,1
2	0,22	0,2
3	0,29	0,3
4	0,24	0,4
5	0,13	0,5
6	0,047	0,6
7	0,0099	0,7
8	0,00086	0,8
9	0,0000096	0,9
10	0	1

Inverse probability and prediction

u	$P(u \mid n_B, N)$	$f_u = u/10$	$P(.)$
0	0	0	0
1	0,063	0,1	0,0063
2	0,22	0,2	0,044
3	0,29	0,3	0,087
4	0,24	0,4	0,096
5	0,13	0,5	0,065
6	0,047	0,6	0,0282
7	0,0099	0,7	0,00693
8	0,00086	0,8	0,000688
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$$P(ball_{N+1} = black \mid u, n_B, N) = f_u = {}^{1}\!/_{10}$$

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Inverse probability and prediction

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$$P(ball_{N+1} = black \,|\, n_B, N) = \sum_u f_u P(u \,|\, n_B, N)$$

Inverse probability and prediction vs the classical way

$$P(ball_{N+1} = black \mid u, n_B, N) = f_u = {}^1\!/_{10}$$

$$P(ball_{N+1} = black, u, n_B, N) =$$
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Selecting the most plausible hypothesis

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Selecting the most plausible hypothesis

$\boldsymbol{u = 3}$

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$$P(ball_{N+1} = black \mid n_B, N) = \sum_u f_u P(u \mid n_B, N)$$

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Selecting the most plausible hypothesis

$$u = 3$$

then making the predictions assuming that hypothesis to be true

$$P = 0.3$$

$$P(ball_{N+1} = black \mid n_B, N) = 0.333$$

$$P(ball_{N+1} = black \mid n_B, N) = \sum_u f_u P(u \mid n_B, N)$$

The likelihood principle

The likelihood principle: given a generative model for data d given parameters $\boldsymbol{\theta}$, $P(d \,|\, \boldsymbol{\theta})$, and having observed a particular outcome d_1, all inferences and predictions should depend only on the function $P(d_1 \,|\, \boldsymbol{\theta})$.

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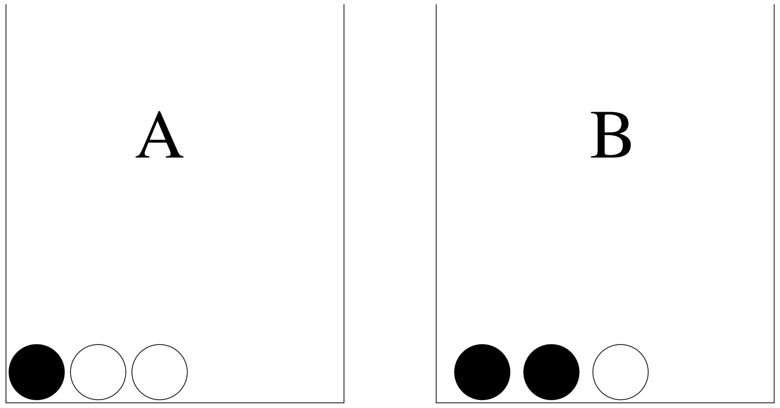
The likelihood principle

- Urn A contains three balls: one black, and two white; urn B contains three balls: two black, and one white.



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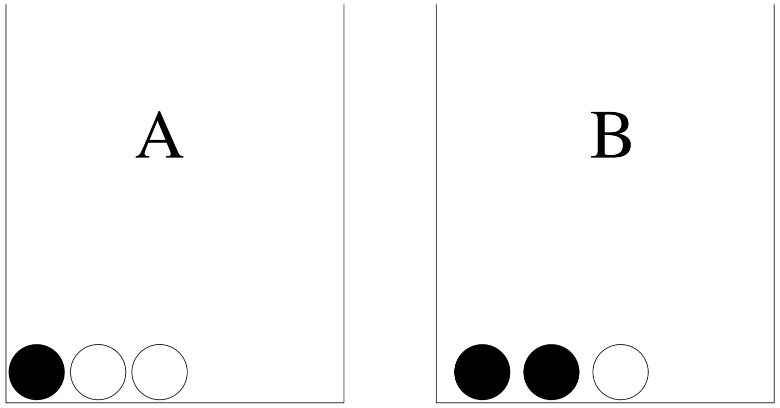
- One of the urns is selected at random and one ball is drawn. The ball is black.



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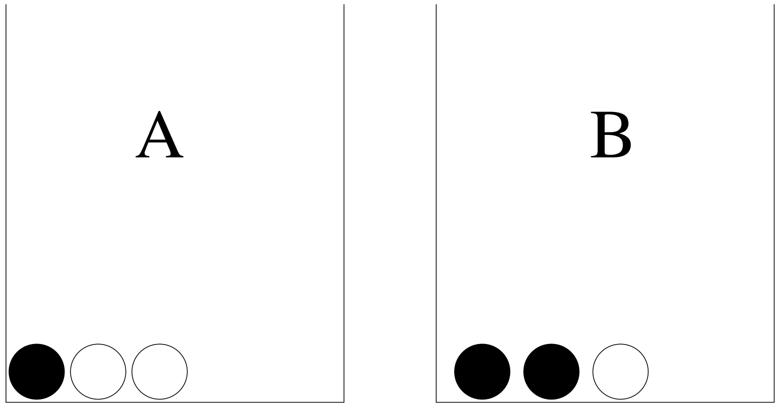

$$\boxed{P(u = A \mid b = bl) = ?}$$

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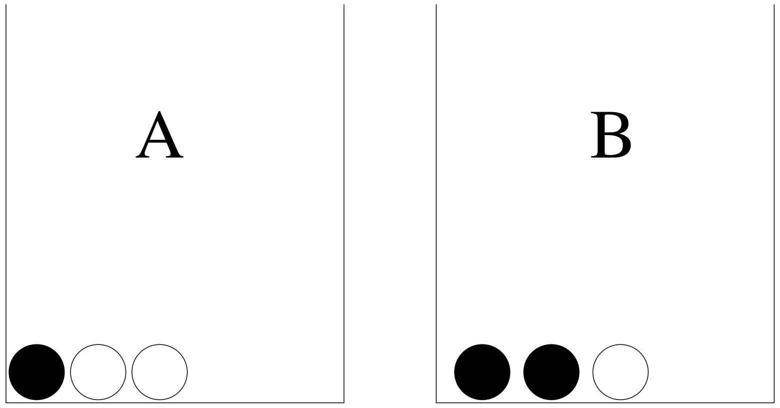

- One of the urns is selected at random and one ball is drawn. The ball is black.
- **What is the probability that the selected urn is urn A**?
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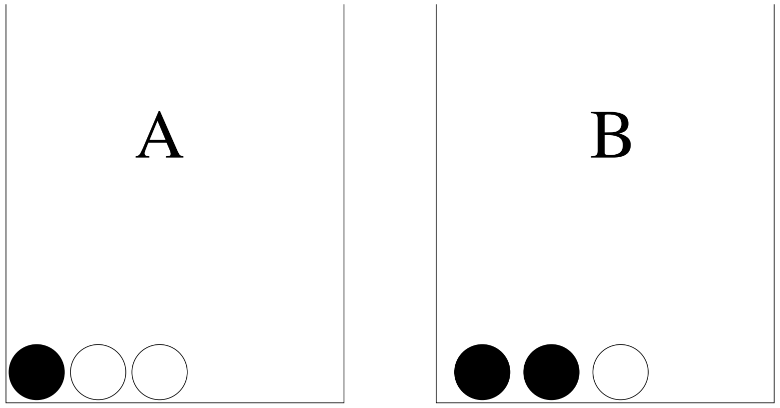

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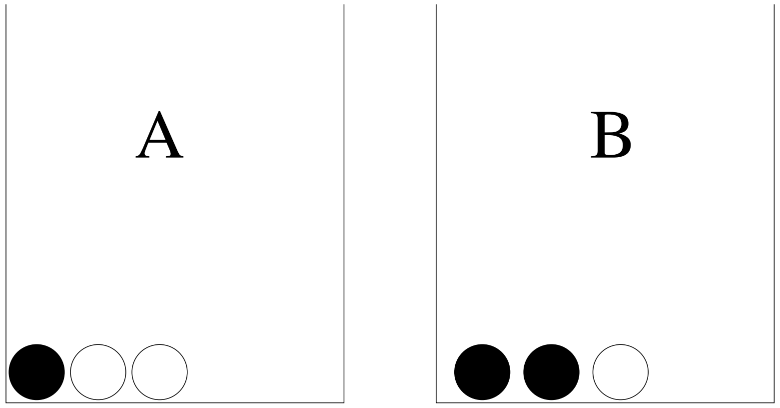

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  - $P(b$ = bl | u = A) = 1/3; P(b = wh | u = A) = 2/3

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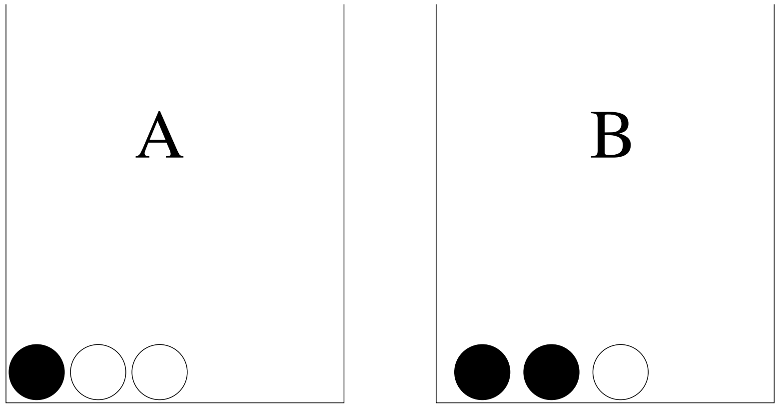

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  - $P(b = bl \mid u = B) = 2/3$; $P(b = wh \mid u = A) = 1/3$

$$\boxed{P(u = A \mid b = bl) = ?}$$

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  - $P(b = \mathrm{bl} \mid u = \mathrm{B}) = 2/3$; $\mathrm{P}(b = \mathrm{wh} \mid u = \mathrm{A}) = 1/3$

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- u = {A, B}; $P(u = \mathrm{A}) = P(u = \mathrm{B}) = 1/2$
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  - $P(b = \mathrm{bl} \mid u = \mathrm{A}) = 1/3$; $\mathrm{P}(b = \mathrm{wh} \mid u = \mathrm{A}) = 2/3$
  - $P(b = \mathrm{bl} \mid u = \mathrm{B}) = 2/3$; $\mathrm{P}(b = \mathrm{wh} \mid u = \mathrm{A}) = 1/3$

how the probability of the **data that happened** varies with the hypothesis

$$P(u = A \mid b = bl) = \frac{P(u = A, b = bl)}{P(b = bl)} = \frac{P(b = bl \mid u = A)P(u = A)}{P(b = bl)}$$

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- One of the urns is selected at random and one ball is drawn.

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Further Reading and Summary

Q&A

Further Reading

- **Recommend Readings**
  - Information Theory, Inference, and Learning Algorithms from David MacKay, 2015, pages 22 - 32.
- **Supplemental readings:**

What you should know

- Joint Probability; Marginal Probability; Condicional Probability.
- Product and Sum rules.
- Bayes' Theorem
- Statistical Independence
- The two common but different interpretations for the meaning of Probabilities
- Forward Probabilities and Inverse Probabilities
- Terminology of inverse Probability: prior probability; likelihood; likelihood; evidence
- The likelihood principle
- To address Inverse probability problems

Further Reading and Summary

Q&A