Construction and Verification of Software

2022 - 2023

MIEI - Integrated Master in Computer Science and Informatics
Consolidation block

Lecture 2 - Specification and Verification

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based on previous editions by **João Seco** and **Luís Caires**

NOVA SCHOOL OF SCIENCE & TECHNOLOGY

Important Dates

All already on CLIP:

- **Project 1** Deadline — 15/04 (Out 28 March)
- **Project 2** Deadline — 12/05 (Out 25 April)
- **Test 1** — 13/05
- **Project 3** Deadline — 02/06 (Out 16 May)
- **Test 2** — 14/06

Part I
Software Correctness

Software Correctness: What and How

RECAP

- **Key engineering concern**:
  Make sure that the software developed and constructed is “correct”.
- What does this mean?
  - Is it crash-free? (“runtime safety”)
  - Gives the right results? (“functional correctness”)
  - Does it operate effectively? (“resource conformance”)
  - Does it violate user privacy? (“security conformance”)
  - …
- several process and methodological approaches to ensure and validate correctness exist (software engineering course)
- In this course, we cover some techniques to rigorously ensure and validate correctness **during software construction**

Correctness against a specification

RECAP

- Then what does "correct software" mean?
  - Always relative to some given (**our**) specification
- Correct means that software meets **our** specification
  - There is no such thing as the absolute "right specification"
  - But **the spec must not be wrong** !
  - Crafting good / checkable specifications can be challenging.
- It should be "easy" to check what the specification states
  - The spec **must be simple, much simpler than the code**
  - The spec should be **focused**
    - e.g., buffers are not being overrun
    - e.g., never transfer money without logging the source

Checking Specs: Dynamic Verification

RECAP

- By “dynamic verification” we mean that verification is **done at runtime**, while the program executes
- Some successful approaches:
  - **unit testing**
  - **coverage testing**
  - **regression testing**
  - **test generation**
  - **runtime monitoring**
- use runtime monitors to (continuously) check that code do not violate correctness properties
- violations causes exceptional behaviour or halt, so errors are detected after something wrong already occurred (think of a car crash, or a security leak)

Checking Specs: Static Verification

- “Static verification” means verification at **compile time**
- Algorithmic reasoning about what programs do, by analyzing the source code, not by running the code
- Can ensure absence of all errors of a certain well-defined kind (e.g., “no null dereferences”)
- Can also address many complex correctness properties (e.g., functionality, absence of races, security, etc.)
- Does not introduce performance overhead at runtime
- Successful techniques:
    - **type checking**, as performed by the compiler
    - **extended checking**, static checking of assertions
    - **abstract interpretation**, simulates execution on a simpler decidable abstract model of runtime data

Checking Specs: Static vs. Dynamic

- **Dynamic** Verification
  - Unsound
  - Performed at runtime
  - No false positives (usually)
  - Execution overhead

- **Static** Verification
  - Sound
  - Compile time
  - Conservative
(may have false positives)
  - Erasure of verification (no overhead)
  - Each analysis targets a specific of property
  - Complex to design and use

Checking Specs: Static Verification

- Specifications are the essential tool for **abstraction** and **decomposition**.
- For each program we need to know
  - in what conditions it can be used (requires/pre-conditions)
  - what are its effects (effects/ensures/post-conditions)
- **If our reasoning is sound**, the post condition can be assumed after the program's execution, provided that the pre-conditions were met at the beginning.
- We can only know as much as what is stated in the post-condition.

Part II
Specification and Verification

Contract-based Verification

- Axiomatic approaches based on **Hoare Logic** (Pre- and post-conditions)
- If pre-condition holds **and** the program terminates **then** the post-condition holds.
- If all components are verified then all contracts are fulfilled in all cases.
- If a component does not fulfill a contract then no guaranties are given about the system's behavior.

What may specs look like?

- A classical example is the use of "assertions"
  - – You may have used assertions before (POO, AED)?
- A simple and fine-grained spec is the "Hoare triple":

$$\{ \mathbf{A} \}\ \mathrm{P}\ \{ \mathbf{B} \}$$

- **A** and **B** are assertions (conditions on the program state)
- P is the piece of code we want to talk about
- The Hoare triple says:
  - If program P starts in a state satisfying **A**, then, if it terminates, the resulting state satisfies **B**.
  - **A** is called the "pre-condition".
  - **B** is called the "post-condition".

Interface contracts in ADT specs

- ADT specifications (we will detail this later) involve method contracts, expressed as assertions

```
method P(... parameters ...)
requires PRE
ensures POST
modifies MOD
 {
     method code
 }
```

- The method call `P(...)`, whenever started in a state that satisfies PRE, if it terminates, always ends in a state that satisfies POST, and only has effects on MOD

Invariants in ADT specs

- ADT specifications (we will detail this later) may involve representation invariants and abstraction mappings also expressed as assertions

```
class C {

    var v : T
    invariant REPINV
    invariant ABSMAP


    ... methods...
```

- ADT C's implementation relies on a representation type T that satisfies the representation invariant REPINV and maps into the abstract type as specified by ABSMAP

Stack Example : A glimpse of Dafny code

```
class Stack {

  var elements:array<int>;
  var count: int;
  var MAX: int;

  predicate StackInv()
  reads this
  {
    0 < MAX && 0 <= count <= MAX && elements.Length == MAX
  }

  constructor()
    ensures StackInv()
  {
    MAX := 10;
    elements := new int[10];
    count := 0;
  }
```

Stack Example : A glimpse of Dafny code

```
class Stack {
  …
  method push(x:int)
    requires StackInv() && notFull()
    ensures StackInv() && notEmpty()
    modifies elements, `count
  {
    elements[count] := x;
    count := count + 1;
  }

  method pop() returns (x:int)
    requires StackInv() && notEmpty()
    ensures StackInv() && notFull()
    modifies elements, `count
  {
    count := count - 1;
    x := elements[count];
  }
```

```
predicate notFull()
reads `count, `MAX
{ count < MAX }

predicate notEmpty()
reads `count, `MAX
{ count > 0 }
```

Specifications and Program Logics

- Written in a logic used to prove properties of programs
- What kinds of properties are we interested in?
  - Safety properties (partial correctness):
    - If the program terminates (delivers an outcome), then the final state satisfies some property.
  - Liveness properties (total correctness)
  - The program terminates (at least under certain conditions)
- Hoare logic is the “mother of all program logics”: It provides a foundation for most program logics for imperative programming languages.
- Reason of HL success: compositional verification at the level of the programming language constructs.

Dafny

"Dafny is an imperative object-based language with built-in specification constructs. The Dafny static program verifier can be used to verify the functional correctness of programs. The specifications include pre- and postconditions, frame specifications (read and write sets), and termination metrics"

Leino, Koenig, 2010

Dafny: An Automatic Program Verifier for Functional Correctness

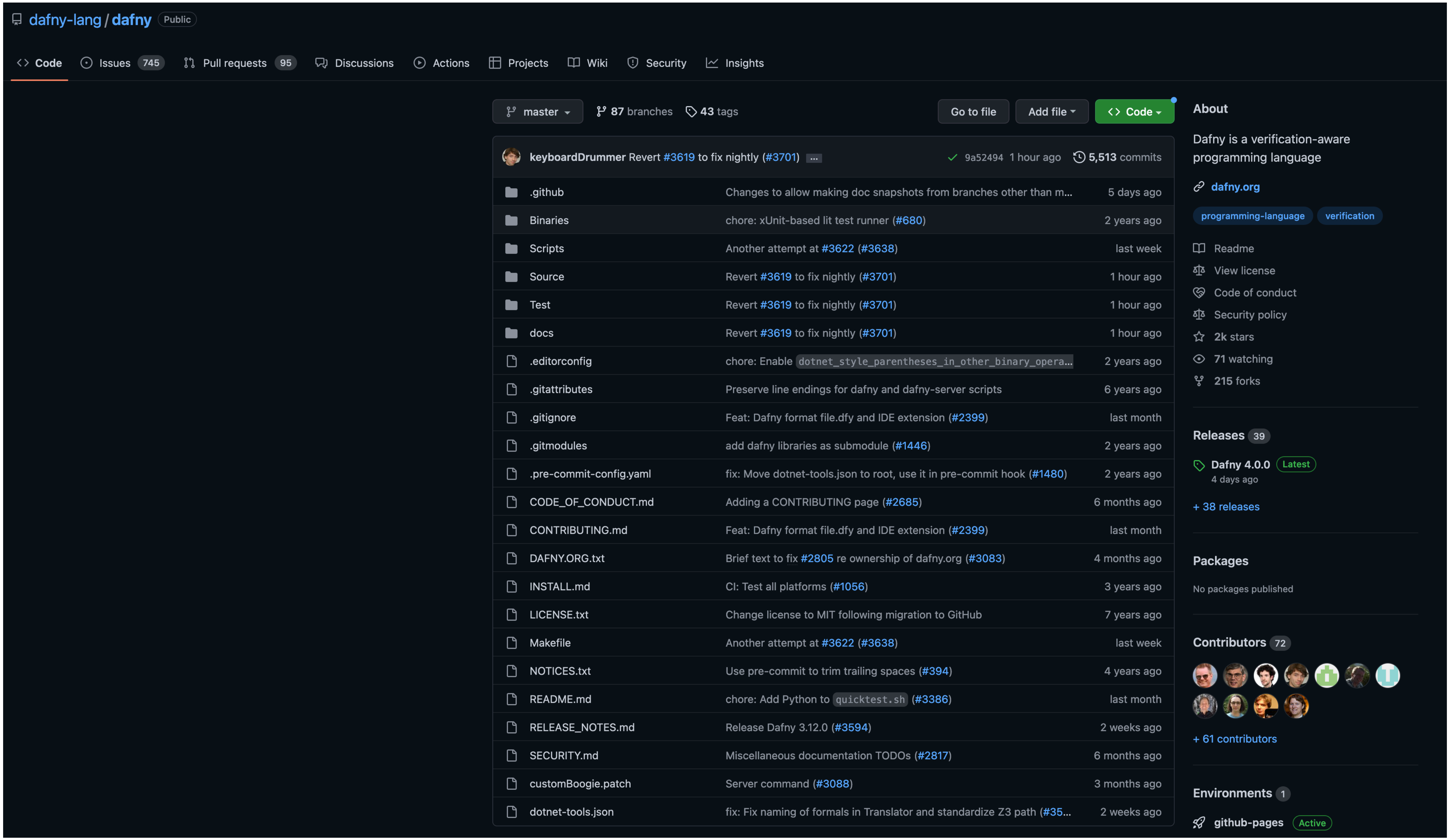
K. Rustan M. Leino
Microsoft Research
leino@microsoft.com

Abstract

Traditionally, the full verification of a program's functional correctness has been obtained with pen and paper or with interactive proof assistants, whereas only reduced verification tasks, such as extended static checking, have enjoyed the automation offered by satisfiability-modulo-theories (SMT) solvers. More recently, powerful SMT solvers and well-designed program verifiers are starting to break that tradition, thus reducing the effort involved in doing full verification.

This paper gives a tour of the language and verifier Dafny, which has been used to verify the functional correctness of a number of challenging pointer-based programs. The paper describes the features incorporated in Dafny, illustrating their use by small examples and giving a taste of how they are coded for an SMT solver. As a larger case study, the paper shows the full functional specification of the Schorr-Waite algorithm in Dafny.

Dafny on GitHub



Dafny in VS Code

Dafny v3.0.5

dafny-lang | 8,636 | ★★★★★ (1)

Dafny for Visual Studio Code

Disable | Uninstall

This extension is enabled globally.

Details | Feature Contributions | Changelog | Runtime Status

Dafny for Visual Studio Code

This extension adds *Dafny 3* support to Visual Studio Code. If you require *Dafny 2* support, consider using the legacy extension. This VSCode plugin requires the Dafny language server (shipped with the Dafny release since v3.1.0). The plugin will install it automatically upon first use.

Features

- **Compile and Run** `.dfy` files.
- **Verification** as one types.
- **Syntax highlighting** thanks to sublime-dafny. See file `LICENSE_sublime-dafny.rst` for license.
- Display **counterexample** for failing proof.
- **IntelliSense** to suggest symbols.
- **Go to definition** to quickly navigate.
- **Hover Information** for symbols.

You can find examples below.

Shortcuts

Dafny in VS Code

```
method max(a:array<int>) returns (m:int)
    requires a.Length > 0
    ensures forall i :: 0 <= i < a.Length ==> m >= a[i]
{
    m := a[0];
    var i := 0;
    while i < a.Length
        invariant 0 <= i <= a.Length
        invariant forall j :: 0 <= j < i ==> m >= a[j]
    {
        if a[i] > m {
            m := a[i];
        }
        i := i+1;
    }
    return m;
}
```

0 0 Verification Succeeded

Dafny Documentation

Dafny Documentation

This site contains links to Dafny documentation.

Project site for releases, issues, installation instructions, and source code

- Quick start material:
  - Dafny Quick Reference
  - Getting started tutorial, focusing mostly on simple imperative programs
  - Cheatsheet: basic Dafny syntax on two pages
- Detailed documents for programmers
  - Dafny Reference Manual
  - Language reference for the Dafny type system, which also describes available expressions for each type
  - Style Guide for Dafny programs
- Dafny Tutorials
  - Introduction to Dafny
  - Value Types
  - Sets
  - Sequences
  - Lemmas and Induction
  - Modules
  - Termination

Part IV
A bit of History

Some bits of history … (extra)

Kick off:

- **“Checking a large routine”**

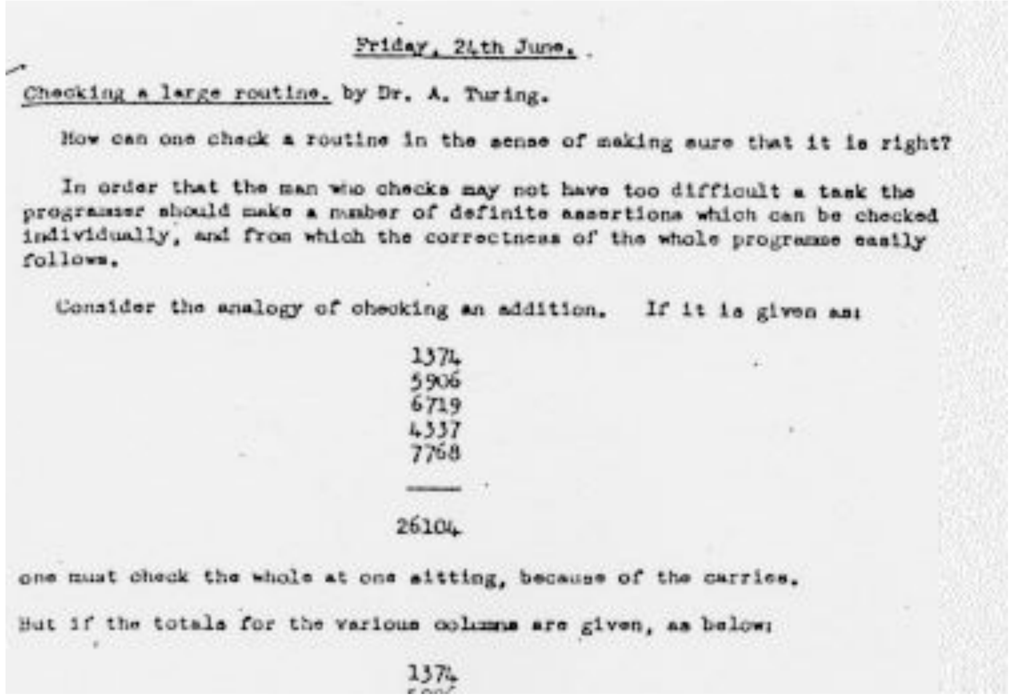
Turing

Kick off:

- **“Checking a large routine”**

“How can one check a routine in the sense of making sure that it is right? In order that the man who checks may not have too difficult a task the programmer should make a number of definite assertions which can be checked individually, and from which the correctness of the whole programme easily follows.”

Alan Turing, 24th June 1949



Friday, 24th June.

Checking a large routine. by Dr. A. Turing.

How can one check a routine in the sense of making sure that it is right?

In order that the man who checks may not have too difficult a task the programmer should make a number of definite assertions which can be checked individually, and from which the correctness of the whole programme easily follows.

Consider the analogy of checking an addition. If it is given as:

1374
5906
6719
4337
7768

26104

one must check the whole at one sitting, because of the carries.

But if the totals for the various columns are given, as below:

1374

Assertions

Second boost:

– **Floyd's Assertion Method**

Robert Floyd's, "Assigning Meanings to Programs," opened the field of program verification. His basic idea was to attach so-called "tags" in the form of logical assertions to individual program statements or branches that would define the effects of the program based on a formal semantic definition of the programming language.

R. Floyd, MFCS, June 1967

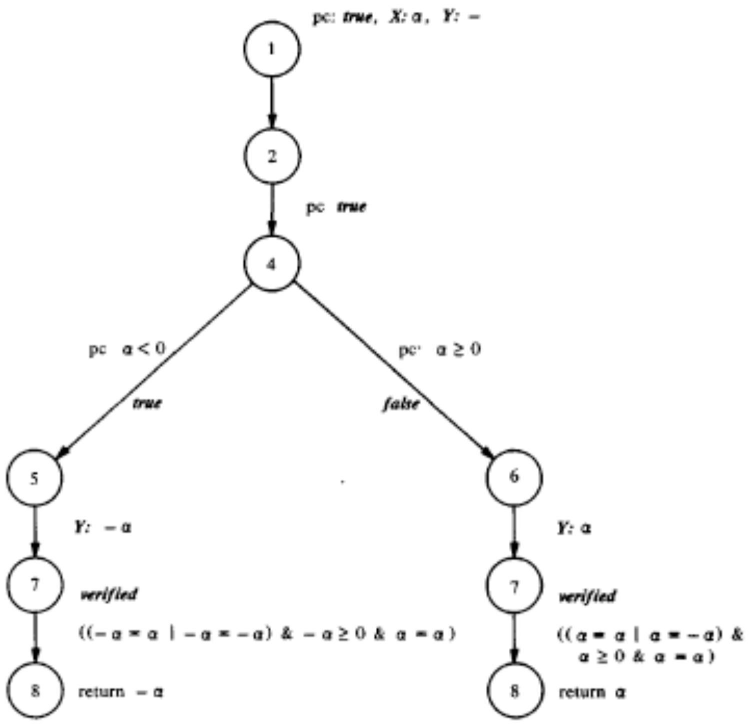


FIGURE 3. Symbolic execution tree for procedure *ABSOLUTE*.

Assertions

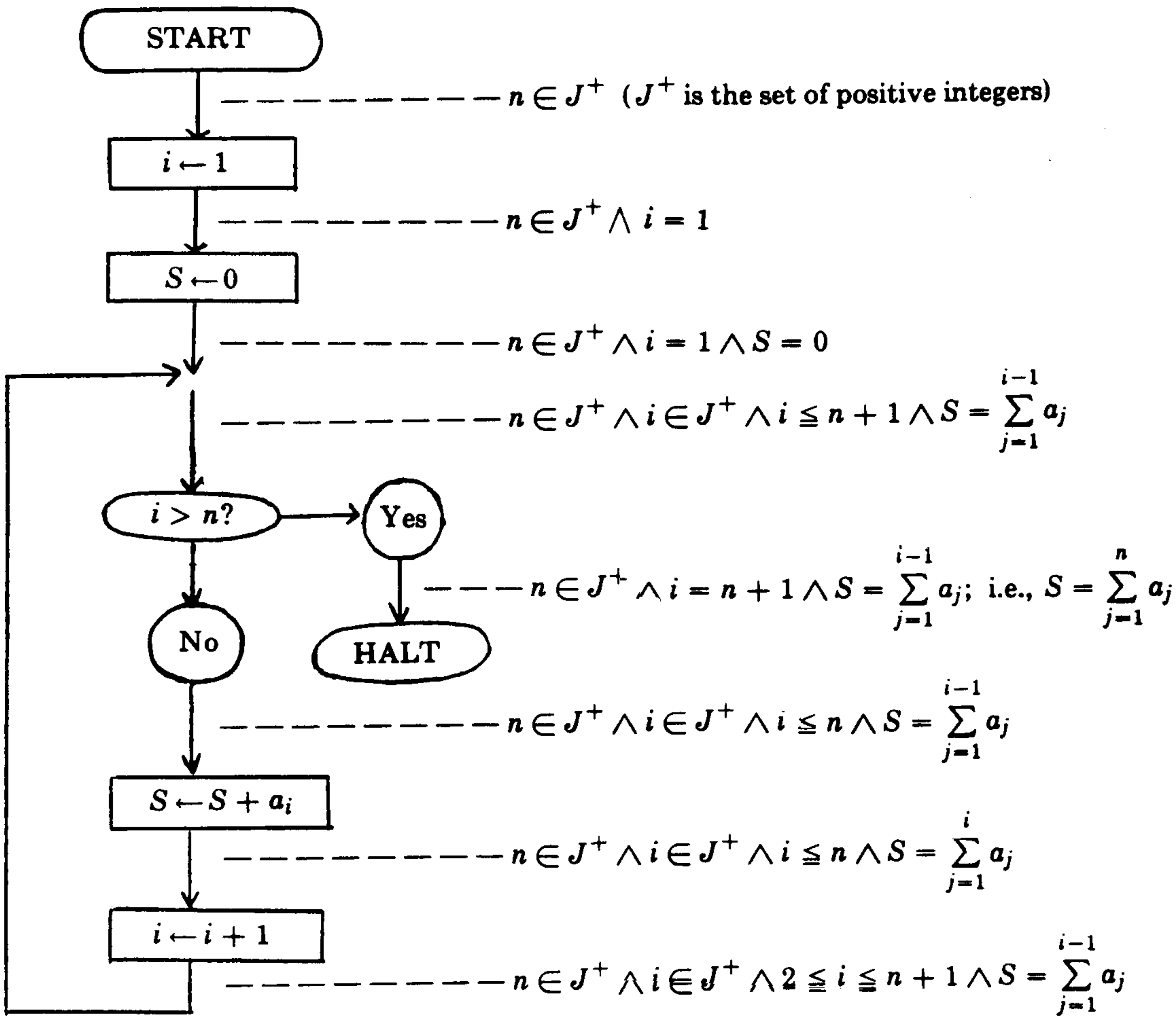


FIGURE 1. Flowchart of program to compute $S = \sum_{j=1}^{n} a_j$ $(n \geqq 0)$

Language Based Program Specs

Lift Off:

- **Hoare Logic**

"Computer Programming is an exact science in that all the properties of a program and all consequences of executing it in any given environment can, in principle, be found out from the text of the program itself by means of purely deductive reasoning."

Tony Hoare, CACM 1969

AXIOM 1: ASSIGNMENT AXIOM

$$\{p[t/x]\}\ x := t\ \{p\}.$$

RULE 2: COMPOSITION RULE

$$\frac{\{p\}\ S_1\ \{r\},\ \{r\}\ S_2\ \{q\}}{\{p\}\ S_1;\ S_2\ \{q\}}.$$

RULE 3: **if-then-else** RULE

$$\frac{\{p \wedge e\}\ S_1\ \{q\},\ \{p \wedge \neg e\}\ S_2\ \{q\}}{\{p\}\ \mathbf{if}\ e\ \mathbf{then}\ S_1\ \mathbf{else}\ S_2\ \mathbf{fi}\ \{q\}}$$

RULE 4: **while** RULE

$$\frac{\{p \wedge e\}\ S\ \{p\}}{\{p\}\ \mathbf{while}\ e\ \mathbf{do}\ S\ \mathbf{od}\ \{p \wedge \neg e\}}$$

The Weakest Precondition

Programming Languages

T.A. Standish
Editor

Guarded Commands, Nondeterminacy and Formal Derivation of Programs

Edsger W. Dijkstra
Burroughs Corporation

So-called "guarded commands" are introduced as a building block for alternative and repetitive constructs that allow nondeterministic program components for which at least the activity evoked, but possibly even the final state, is not necessarily uniquely determined by the initial state. For the formal derivation of programs expressed in terms of these constructs, a calculus will be be shown.

Key Words and Phrases: programming languages, sequencing primitives, program semantics, programming language semantics, nondeterminacy, case-construction, repetition, termination, correctness proof, derivation of programs, programming methodology

CR Categories: 4.20, 4.22

1. Introduction

In Section 2, two statements, an alternative construct and a repetitive construct, are introduced, together with an intuitive (mechanistic) definition of their semantics. The basic building block for both of them is the so-called "guarded command," a statement list prefixed by a boolean expression: only when this boolean expression is initially true, is the statement list eligible for execution. The potential nondeterminacy allows us to map otherwise (trivially) different programs on the same program text, a circumstance that seems largely responsible for the fact that programs can now be derived in a manner more systematic than before.

In Section 3, after a prelude defining the notation, a formal definition of the semantics of the two constructs is given, together with two theorems for each of the constructs (without proof).

In Section 4, it is shown how, based upon the above, a formal calculus for the derivation of programs can be founded. We would like to stress that we do not present "an algorithm" for the derivation of programs: we have used the term "a calculus" for a formal discipline—a set of rules—such that, if applied successfully: (1) it will have derived a correct program; and (2) it will tell us that we have reached such a goal. (We use the term as in "integral calculus.")

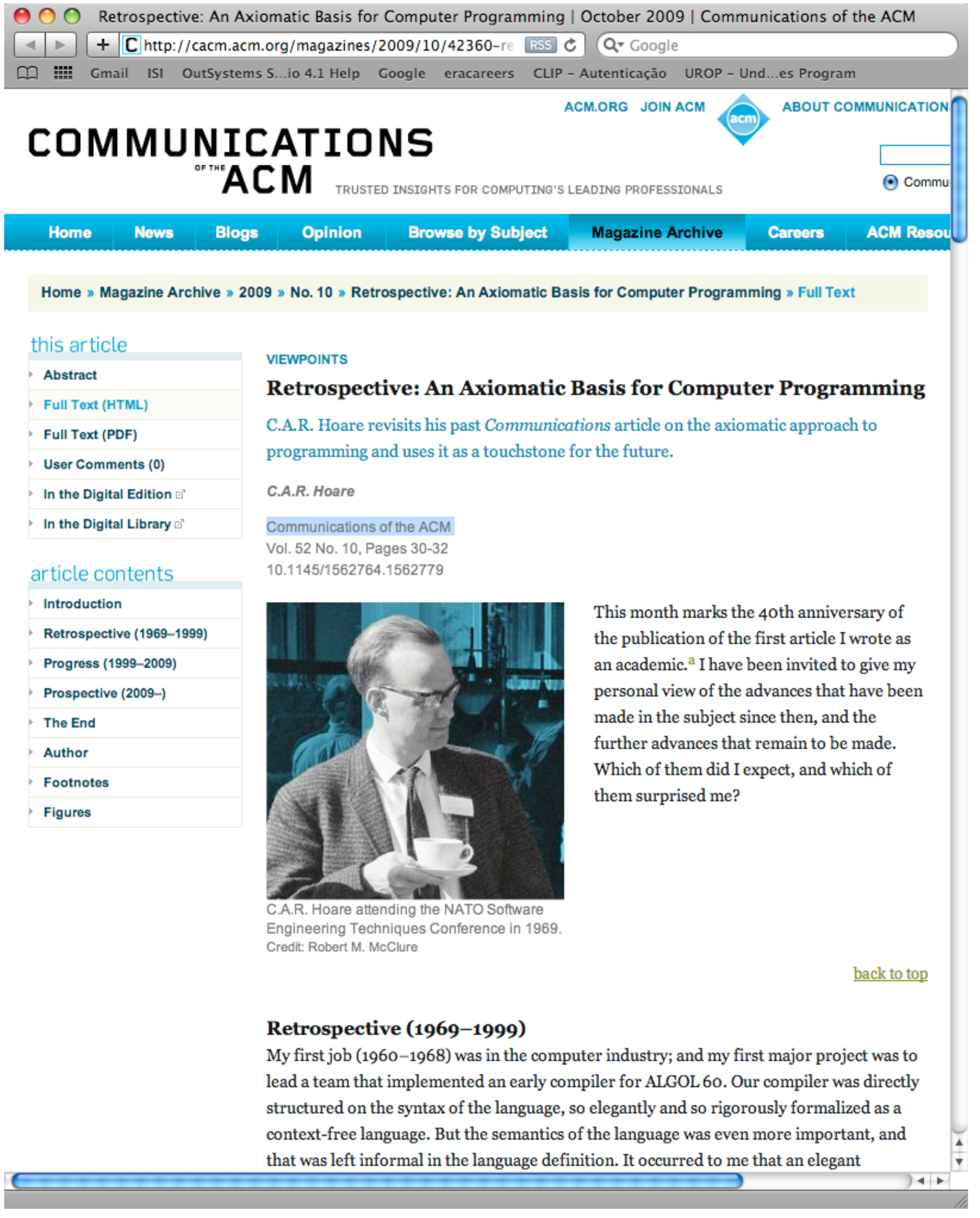
Closer to the present

Still relevant

– **Hoare Logic**

“ The axiomatic method gives an objective criterion of the quality of a programming language, and the ease with which programmers could use it. The latest response comes from hardware designers, who are using axioms in anger to define the properties of modern multicore chips with weak memory consistency.”

Tony Hoare, CACM 2009



Extended Static Checking

Spec#

Spec# is an extension of the object-oriented language C#. It extends the type system to include non-null types and checked exceptions. It provides method contracts in the form of pre- and postconditions as well as object invariants.

Barnett, Leino, Schulte, 2004

The Spec# Programming System: An Overview

Mike Barnett, K. Rustan M. Leino, and Wolfram Schulte

Microsoft Research, Redmond, WA, USA
`{mbarnett,leino,schulte}@microsoft.com`

Manuscript KRML 136, 12 October 2004. To appear in CASSIS 2004 proceedings.

Abstract. The Spec# programming system is a new attempt at a more cost effective way to develop and maintain high-quality software. This paper describes the goals and architecture of the Spec# programming system, consisting of the object-oriented Spec# programming language, the Spec# compiler, and the Boogie static program verifier. The language includes constructs for writing specifications that capture programmer intentions about how methods and data are to be used, the compiler emits run-time checks to enforce these specifications, and the verifier can check the consistency between a program and its specifications.



Dafny

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Leino, Koenig, 2010

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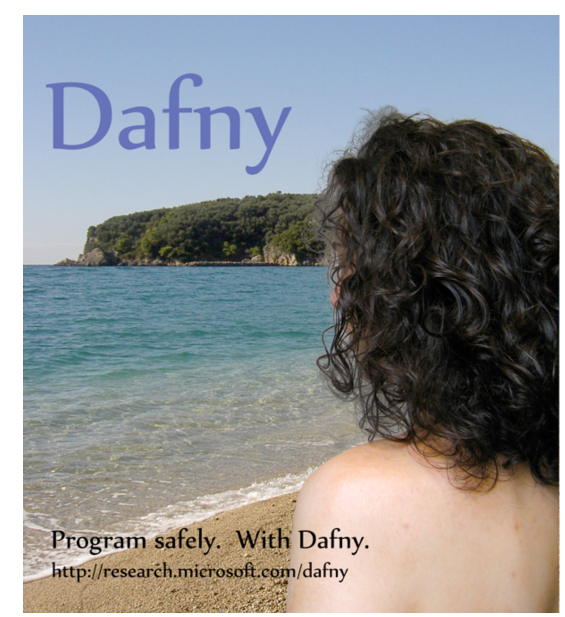
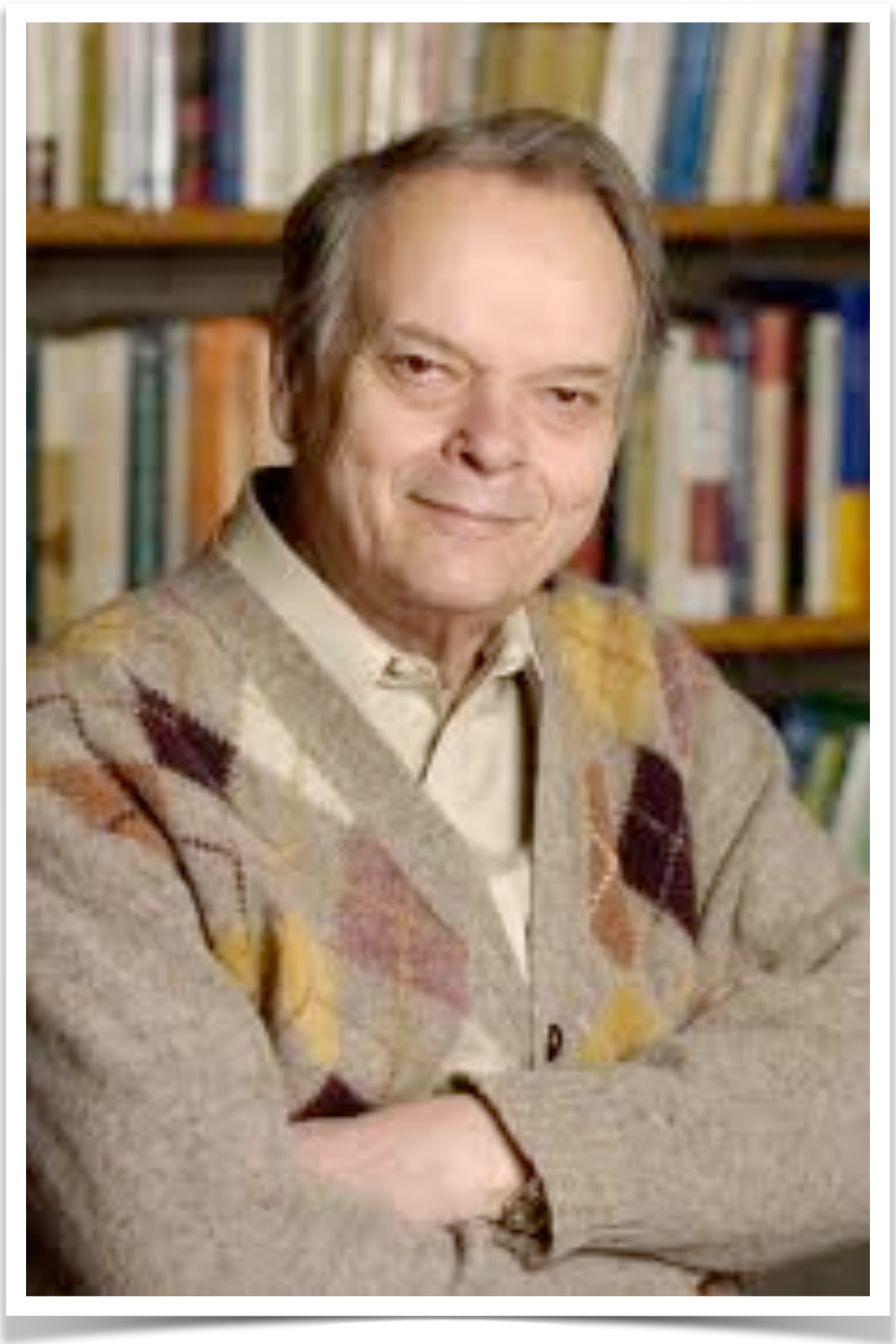


Separation Logic



John C. Reynolds

$$\frac{s, h \models P * (P -\!\!* Q)}{s, h \models Q} \qquad \frac{\{P\}\ C\ \{Q\}}{\{P * R\}\ C\ \{Q * R\}}\ \mathsf{mod}(C) \cap \mathsf{fv}(R) = \emptyset$$

Peter O'Hearn

Verifast

Verifast

VeriFast is a verifier for single-threaded and multithreaded C and Java programs annotated with preconditions and postconditions written in separation logic.

Jacobs, Smans, Piessens, 2010

NB: separation logic is a spec language for talking about programs that allocate memory and use references

```
public void broadcast_message(String message) throws IOException
    //@ requires room(this) &*& message != null;
    //@ ensures room(this);
{
    //@ open room(this);
    //@ assert foreach(?members0, _);
    List membersList = this.members;
    Iterator iter = membersList.iterator();
    boolean hasNext = iter.hasNext();
    //@ length_nonnegative(members0);
    while (hasNext)
        /*@
        invariant
            foreach<Member>(?members, @member) &*& iter(iter, membersList, members, ?i)
            &*& hasNext == (i < length(members)) &*& 0 <= i &*& i <= length(members);
        @*/
    {
        Object o = iter.next();
        Member member = (Member)o;
        //@ mem_nth(i, members);
        //@ foreach_remove<Member>(member, members);
        //@ open member(member);
        Writer writer = member.writer;
        writer.write(message);
        writer.write("\r\n");
        writer.flush();
        //@ close member(member);
        //@ foreach_unremove<Member>(member, members);
        hasNext = iter.hasNext();
    }
    //@ iter_dispose(iter);
    //@ close room(this);
}
```



Part V
Hoare Logic

Basic Program Specs (Hoare Logic)

C.A.R. HOARE
United Kingdom – **1980**

For his fundamental contributions to the definition and design of programming languages.

Hoare Logic (1969)

An Axiomatic Basis for Computer Programming

C. A. R. Hoare
The Queen's University of Belfast, Northern Ireland*

In this paper an attempt is made to explore the logical foundations of computer programming by use of techniques which were first applied in the study of geometry and have later been extended to other branches of mathematics. This involves the elucidation of sets of axioms and rules of inference which can be used in proofs of the properties of computer programs. Examples are given of such axioms and rules, and a formal proof of a simple theorem is displayed. Finally, it is argued that important advantages, both theoretical and practical, may follow from a pursuance of these topics.

KEY WORDS AND PHRASES: axiomatic method, theory of programming' proofs of programs, formal language definition, programming language design, machine-independent programming, program documentation
CR CATEGORY: 4.0, 4.21, 4.22, 5.20, 5.21, 5.23, 5.24

of axioms it is possible to deduce such simple theorems as:

$$x = x + y \times 0$$

$$y \leqslant r \supset r + y \times q = (r - y) + y \times (1 + q)$$

The proof of the second of these is:

A5 $(r - y) + y \times (1 + q)$

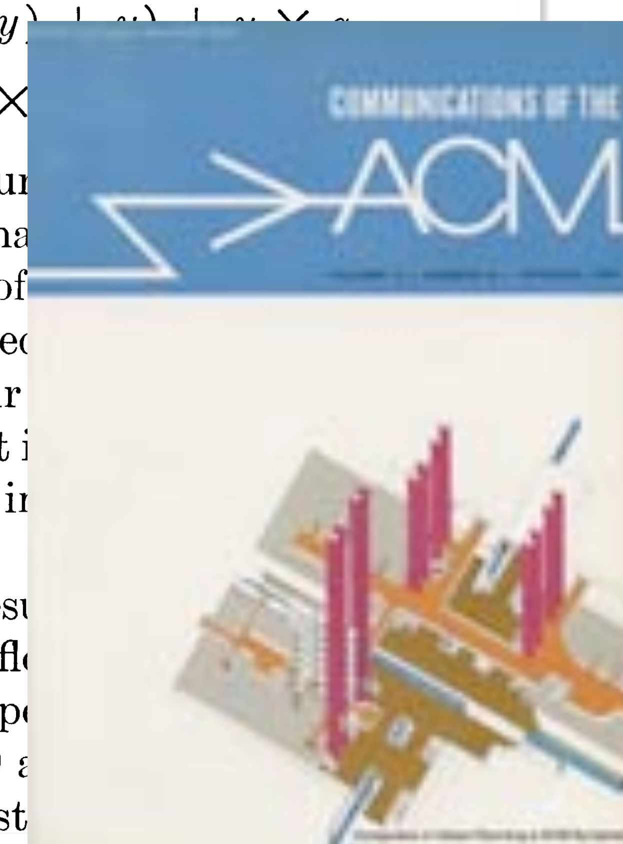
$= (r - y) + (y \times 1 + y \times q)$

A9 $= (r - y) + (y + y \times q)$

A3 $= ((r - y$

A6 $= r + y \times$

The axioms A1 to A9 are, of cou
tional infinite set of integers in ma
they are also true of the finite sets of
manipulated by computers provide
fined to *nonnegative* numbers. Their
of the size of the set; furthermore, it i
of the choice of technique applied in
flow"; for example:
(1) Strict interpretation: the resu
operation does not exist; when overfl
ing program never completes its op
this case, the equalities of A1 to A9 a
that both sides exist or fail to exist



Simple Programming Language

E	$::=$	Expressions	
		num	Integer
	$\mid$	x	Variable
	$\mid$	$E + E \mid \ldots$	Integer operators
	$\mid$	$E < E \mid \ldots$	Relational operators
	$\mid$	$E \texttt{ and } E \ldots$	Boolean operators
P	$::=$		Programs
		`skip`	No op
	$\mid$	$x := E$	Assignment
	$\mid$	$P; P$	Sequential Composition
	$\mid$	`if` E `then` P `else` P	Conditional
	$\mid$	`while` E `do` P	Iteration

States and State Transformers

- An **imperative** program is a **state transformer**.
It transforms an **initial** state into a **target** state.
- What is a program state? An assignment of values to state variables:

$$\sigma = \{x \mapsto 1, y \mapsto 2, z \mapsto 3\}$$

- An imperative program transforms states into states

$$P \triangleq x := y + x; z := z - x$$

- If P is executed in state σ it yields state σ' where

$$\sigma' = \{x \mapsto 3, y \mapsto 2, z \mapsto 0\}$$

- We may say that P transforms σ into σ'
- P is only defined on states σ where vars(P) ⊆ dom(σ)

States and Assertions

- A (safety) property is a set of (safe) states
- Essentially an assertion is a boolean expression that only depends on observing program (state) variables
- An assertion is just a **pure observation**, it is either true or false, its evaluation **does not change** the state.
- In general, one may use all the expressiveness of (first order) logic in assertions (e.g. quantifiers, etc…).
- The assertion language is part of the specification language, separate from the programming language.
- In some cases, assertions may be expressed using the programming language itself (e.g. a subset of Dafny).

Program Proofs in Hoare Logic

- A program proof in Hoare logic adds assertions between program statements, making sure that all Hoare triples are satisfied/valid.
- For example, consider the code snippet

```
if (x > y) {
    z := x
} else {
    z := y
}
```

Program Proofs in Hoare Logic

- A Hoare Logic “proof” may look like

```
{ true }
if (x > y) {
      { (x > y)  }
      z := x;
      { (x > y) && (z == x) }
}
else {
      { (x <= y) }
      z := y;
      { (x <= y) && (z == y) }
}
{ (x>y) && (z == x) || (x<=y) && (z == y)}
{ z == max(x,y) }
```

In Dafny

```
function max(x:int, y:int):int { if x > y then x else y }

method maxImp(x:int, y:int) returns (z:int)
  ensures z == max(x,y)
{
  assert true;
  if (x>y) {
    assert x > y ;
    z := x;
    assert z > y && z == x;
    assert z >= y && z == x;
  } else {
    assert x <= y;
    z := y ;
    assert z >= x && z == y;
  }
  assert (z >= y && z == x) || (z >= x && z == y);
  assert z == max(x,y);
}
```

Interlude
Verification of Functions

Functions as Specifications

- We will often use functions in the specification of (imperative) methods.
- Functions must be **pure** (i.e, no state).
- Functions must **provably terminate** (aka total).
- Pure total functions ≅ Mathematical functions.

```
function max(x:int, y:int):int { if x > y then x else y }

method maxImp(x:int, y:int) returns (z:int)
  ensures z == max(x,y)
{…}
```

- Dafny proves, for all `x` and `y`, `z` is equal to `max(x,y)`
- What is the relationship between `max` and "math" max?

Functions as Specifications

- We can prove that `max` is actually equivalent to mathematical max (**adequacy**).
- Since Dafny functions are both **pure** and **total**, we can reason about them using relatively simple techniques:
    - Calculation by evaluation (e.g. `max(2,3)` ↪ `3`)
    - Mathematical induction
    - Structural induction
- Reasoning about functions without totality is much harder (and out of scope of this course).
- Reasoning about functions that manipulate state is as hard as reasoning about imperative code.

Function Evaluation

- Identify function calls with their definitions, substituting args for formal parameters (**referential transparency**).
- Same as how it works in mathematics:
    - If $f(x) = x^2 - x$ then $f(5)$ is definitionally the same as 20.
    - `max(2,3)` is definitionally the same as `3`.

- In a general purpose lang., functions might not terminate and so we distinguish:
    - Evaluation ($e \hookrightarrow v$) — e evaluates to value v (no "calculation" left).
    - Reduction ($e \Longrightarrow e'$) — e computes to expr. e' (some "calculation" may remain).
- In our setting we need not make this distinction.

Proving functional adequacy

- Assume that values (e.g., numbers, booleans) and ops. (e.g., +,-,*, …) map directly onto their mathematical counterparts.
- For non-recursive functions, proceed by **calculation**.
- For all n , m integers, `max(n,m)` = n if n > m; `max(n,m)` = m otherwise.
- Calculation: `max(n,m) = if n > m then n else m`
- Case #1: `n>m` ↪ `true , max(n,m) = n` ✔
- Case #2: `n>m` ↪ `false, max(n,m) = m` ✔

Proving functional adequacy

- For recursive functions, we use **induction**.
- Functions on natural numbers —**mathematical induction**:
  - Proof technique for statements of the form $\forall n \,.\, P(n)$.
  - Prove base case $P(0)$
  - Prove inductive case $\forall k \,.\, P(k) \longrightarrow P(k+1)$
  - We can construct an argument for any n by "iterating" the inductive case up from the base case.
- Many equivalent variations exist:
  - Base case as $P(k)$ for some specific k ($\forall n \geq k \,.\, P(n)$)
  - **Strong induction**, inductive case as $\forall k \,.\, (\forall n \leq k \,.\, P(n)) \longrightarrow P(k+1)$
  - …

Proving functional adequacy

- Lets prove the following function is equivalent to +:

```
function slowAdd(a:nat, b:int) : int {
  if (a==0) then b else 1+slowAdd(a-1,b)
}
```

- $\forall n, m$. slowAdd(n, m) = $n + m$
- By induction on n

Proving functional adequacy

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}
```

- $\forall n, m\,.\;$ `slowAdd`(n, m) `=` $n + m$
- By induction on n
  - Base case: `slowAdd`$(0,m)$ $\hookrightarrow$ m ✔

Proving functional adequacy

- Lets prove the following function is equivalent to +:

```
function slowAdd(a:nat, b:int) : int {
  if (a==0) then b else 1+slowAdd(a-1,b)
}
```

- $\forall n, m \,.\, \texttt{slowAdd}(n, m) = n + m$
- By induction on n
  - Base case: $\texttt{slowAdd}(0,m) \hookrightarrow m$
  - Inductive case:
    - Assume $\forall m \,.\, \texttt{slowAdd}(k, m) \hookrightarrow k + m$, for some k.
    - Prove $\texttt{slowAdd}(k + 1,m) \hookrightarrow k + 1 + m$

Proving functional adequacy

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  - Base case: `slowAdd`$(0,m) \hookrightarrow m$ ✔
  - Inductive case:
    - Assume $\forall m$. `slowAdd`$(k, m) \hookrightarrow k + m$, for some k.
    - Prove `slowAdd`$(k + 1,m) \hookrightarrow k + 1 + m$
    - `slowAdd`$(k + 1,m)$ = `1+slowAdd`(k, m)

Proving functional adequacy

- Lets prove the following function is equivalent to +:

```
function slowAdd(a:nat, b:int) : int {
  if (a==0) then b else 1+slowAdd(a-1,b)
}
```

- $\forall n, m\,.\ \texttt{slowAdd}(n, m)\ =\ n + m$
- By induction on n
  - Base case: $\texttt{slowAdd}(0,m) \hookrightarrow m$
  - Inductive case:
    - Assume $\forall m\,.\texttt{slowAdd}(k, m) \hookrightarrow k + m$, for some k.
    - Prove $\forall m'\,.\texttt{slowAdd}(k + 1,m') \hookrightarrow k + 1 + m'$
    - $\texttt{slowAdd}(k + 1,m')\ =\ \texttt{1+slowAdd}(k, m')$
    - By i.h. $= 1 + k + m' = k + 1 + m'$

Generalizing the inductive hypothesis

- Sometimes the “obvious” statement is not general enough:

```
function factAcc(n:nat, a:int) : int {
  if (n==0) then a else factAcc(n-1,n*a)
}
```

- $\forall n$. factAcc(n,1) = $n!$
- By induction on n
  - Base case: factAcc(0,1) $\hookrightarrow$ 1 ✅
  - Inductive case:
    - Assume factAcc(k,1) $\hookrightarrow$ $k!$, for some k.
    - Prove factAcc($k+1$,1) $\hookrightarrow$ $(k+1)!$
    - factAcc($k+1$,1) = factAcc(k, $k+1$)
    - We are stuck…

Generalizing the inductive hypothesis

```
function factAcc(n:nat, a:int) : int {
  if (n==0) then a else factAcc(n-1,n*a)
}
```

- Instead, prove $\forall n, m\,.\ \texttt{factAcc}(n, m) = m \times n!$
- By induction on n
  - Base case: $\forall m\,.\,\texttt{factAcc}(0,m) \hookrightarrow m = m \times 0!$
  - Inductive case:
    - Assume $\forall m\,.\,\texttt{factAcc}(k, m) \hookrightarrow m \times k!$, for some k.
    - Prove $\forall m'\,.\,\texttt{factAcc}(k+1,m') \hookrightarrow m' \times (k+1)!$
    - $\texttt{factAcc}(k+1,m') = \texttt{factAcc}(k, m' \times (k+1))$
    - by i.h. ($m = m' \times (k+1)$):
      $= m' \times (k+1) \times k! = m' \times (k+1)!$

Generalizing the inductive hypothesis

- Related to inventing “good” loop invariants (later!).
- Sometimes this isn’t enough:

```
function fib(n:nat) : int {
  if n==0 then 0
    else if n==1 then 1
      else fib(n-1)+fib(n-2)
}
```

- Use **strong induction**… (Exercise!)

Structural Induction

- Functions over inductive structures (lists, trees, etc.)

```
datatype List<T> = Nil | Cons(head:T, tail:List<T>)

function length<T>(l:List<T>) : int {
    match l
    case Nil => 0
    case Cons(_,xs) => 1+length<T>(xs)
}
```

- We can prove adequacy also using (a form of) induction.
- Generalization of mathematical induction called **structural** induction:
    - Show property holds for `Nil`
    - Assume it holds for some `l’ : List<T>`, show property holds for `Cons(n,l’)`, for any `n`.

Structural Induction

- Functions over inductive structures (lists, trees, etc.)

```
datatype List<T> = Nil | Cons(head:T, tail:List<T>)

function length<T>(l:List<T>) : int {
    match l
    case Nil => 0
    case Cons(_,xs) => 1+length<T>(xs)
}
```

- Show $\forall l\,.\,\mathtt{length} < \mathtt{T} > (l) = |\,l\,|$ (where $|\,l\,|$ is the length of l).
- By structural induction on l
- Case $l = \mathtt{Nil}$: $\mathtt{length} < \mathtt{T} > (l) = 0$ ✓
- Case $l = \mathtt{Cons}(n, l')$ (for some n and l')
  - Assume $\mathtt{length} < \mathtt{T} > (l') = |\,l'\,|$ (i.h.)
  - $\mathtt{length} < \mathtt{T} > (\mathtt{Cons}(n, l')) = 1 + \mathtt{length} < \mathtt{T} > (l')$
  - by i.h. = $1 + |\,l'\,| = |\,\mathtt{Cons}(n, l')\,|$ ✓

Part V (redux)
Hoare Logic

Program Proofs in Hoare Logic

- A Hoare Logic "proof" may look like

```
{ true }
if (x > y) {
    { (x > y)  }
    z := x;
    { (x > y) && (z == x) }
}
else {
    { (x <= y) }
    z := y;
    { (x <= y) && (z == y) }
}
{ (x>y) && (z == x) || (x<=y) && (z == y)}
{ z == max(x,y) }
```

Example: Rule for Sequence

- A sequence defines a dependency on the effects of both program statements.

$$\frac{\{A\}\ P\ \{B\} \quad \{B\}\ Q\ \{C\}}{\{A\}\ P;Q\ \{C\}}$$

- If $\{A\}\ P\ \{B\}$ and $\{B\}\ Q\ \{C\}$ then $\{A\}\ P;Q\ \{C\}$

Rules of Hoare Logic (general form)

- The inference rules of Hoare logic are used to derive (valid) Hoare triples given some already derived Hoare triples

$$\frac{\{A_1\}\ P_1\ \{B_1\}\ \ldots\ \{A_n\}\ P_n\ \{B_n\}}{\{A\}\ C(P1, \ldots, P_n)\ \{B\}}$$

- What is nice here:
  - the program in the conclusion contains the subprograms $P_1, \ldots, P_n$ as components
  - we derive properties of the composite from the properties of its parts (compositionality)
  - pretty much the same as with a type system

“Structural” Proof Rules

- Basic logic proof systems operate on assertions, e.g.

$$\frac{A \quad A \implies B}{B} \qquad \frac{A \quad B}{A \wedge B} \qquad \frac{A}{A \vee B} \qquad \frac{B}{A \vee B}$$

- Hoare logic proof system operates on Hoare triples, e.g.

$$\frac{\{A\}\ P\ \{B\} \quad \{B\}\ Q\ \{C\}}{\{A\}\ P;Q\ \{C\}}$$

One rule for each PL construct

AXIOM 1: ASSIGNMENT AXIOM

$$\{p[t/x]\}\ x := t\ \{p\}.$$

RULE 2: COMPOSITION RULE

$$\frac{\{p\}\ S_1\ \{r\},\ \{r\}\ S_2\ \{q\}}{\{p\}\ S_1;\ S_2\ \{q\}}.$$

RULE 3: if-then-else RULE

$$\frac{\{p \wedge e\}\ S_1\ \{q\},\ \{p \wedge \neg e\}\ S_2\ \{q\}}{\{p\}\ \textbf{if}\ e\ \textbf{then}\ S_1\ \textbf{else}\ S_2\ \textbf{fi}\ \{q\}}$$

RULE 4: while RULE

$$\frac{\{p \wedge e\}\ S\ \{p\}}{\{p\}\ \textbf{while}\ e\ \textbf{do}\ S\ \textbf{od}\ \{p \wedge \neg e\}}$$

- A cool idea:
  - Programmers can use the rules informally to mentally check their code
  - Tools exist that automate most of the process
  - Lets go through each rule, one by one…

Simple Programming Language

RECAP

E	$::=$	Expressions	
		num	Integer
	$\mid$	x	Variable
	$\mid$	$E + E \mid \ldots$	Integer operators
	$\mid$	$E < E \mid \ldots$	Relational operators
	$\mid$	E `and` $E \ldots$	Boolean operators
P	$::=$		Programs
		`skip`	No op
	$\mid$	$x := E$	Assignment
	$\mid$	$P; P$	Sequential Composition
	$\mid$	`if` E `then` P `else` P	Conditional
	$\mid$	`while` E `do` P	Iteration

Rule for Skip

$$\{A\}\ \texttt{skip}\ \{A\}$$

Rule for Skip

$$\{A\}\ \texttt{skip}\ \{A\}$$

```
if x < 0
  { x := -x; }
```

```
if x < 0
  { x := -x; }
else
  skip
```

Rule for Sequence

RECAP

- A sequence defines a dependency on the effects of both program statements.

$$\frac{\{A\}\ P\ \{B\} \quad \{B\}\ Q\ \{C\}}{\{A\}\ P;Q\ \{C\}}$$

Rule for Conditional

$$\frac{\{A \land E\}\ P\ \{B\} \quad \{A \land \neg E\}\ Q\ \{B\}}{\{A\}\ \texttt{if}\ E\ \texttt{then}\ P\ \texttt{else}\ Q\ \{B\}}$$

Rule for Deduction

$$\frac{A' \implies A \quad \{A\}\ P\ \{B\} \quad B \implies B'}{\{A'\}\ P\ \{B'\}}$$

- A ⇒ B means “A logically implies B”
- We prove A ⇒ B using the principles of first order logic, plus basic properties of the domain data types, e.g. properties of integers, arrays, etc.

Rule for Assignment

$$\{A[^E/_x]\}\ x := E\ \{A\}$$

- $A[^E/_x]$ means:
  - Result of replacing all free occurrences of variable x in assertion A by the expression E
- For this rule to be sound, we require E to be an expression without side effects (a **pure** expression)

Rule for Assignment

$$\{A[^E/_x]\}\ x := E\ \{A\}$$

- We can think of A as a condition where “x” appears in some places. A is a condition dependent on “x”.
- The assignment x := E changes the value of x to E, but leaves everything else unchanged
- So everything that could be said of E in the precondition, can be said of x in the postcondition, since the value of x after the assignment is E
- Example: { x + 1 > 0 } x := x + 1 { x > 0 }

Rule for Assignment

$$\{A[^E/_x]\}\ x := E\ \{A\}$$

- Example, let's check { x > -1 } x := x + 1 { x > 0 }

  { (x+1 > 0) } x := x+1 { x > 0 }   by the := Rule

  that is, { (x > 0)[x+1/x] }   x := (x+1)   { x > 0 }

  { x > -1 } x := x + 1 { x > 0 }   by deduction

Rule for Assignment

$$\{A[^E/_x]\}\ x := E\ \{A\}$$

- Trick: if x does not appear in E or A.

  We can always write { A && E == E } x := E { x == E }
  So, if x does not occur in E, A the triple

{ A } x := E { A && x == E }

is always valid

Rule for Assignment

$$\{A[^E/_x]\}\ x := E\ \{A\}$$

- Exercises. Derive:
  - { y > 0 } x := y { x > 0 && y == x }
  - { x == y } x := 2*x { y == x div 2 }
  - { P(y) && Q(z) } (here P and Q are any properties)

    x := y ; y := z; z:= x

    { P(z) && Q(y) }

Example

- Consider the program

  P ≜ if (x>y) then z := x else z := y

- We can (mechanically) check the triple

  { true } P { z == max(x,y) }

Example

- Consider the program

  P ≜ if (x>y) then z := x else z := y

- We can (mechanically) check the triple

  { true } P { z == max(x,y) }

  { x == max(x,y) } z := x { z == max(x,y) }
  { x > y } z := x { z == max(x,y) }

  { y == max(x,y) } z := y { z == max(x,y) }
  { y >= x } z := y { z == max(x,y) }

Procedures and method calls

$$
\begin{array}{lcll}
E & ::= & \text{Expressions} & \\
 & | & \ldots & \\
 & & & \\
S & ::= & \text{Statements} & \\
 & | & \ldots & \\
 & | & x := m(E_1, \ldots, E_n) & \text{Call + Assignment} \\
 & & & \\
D & ::= & \text{Declarations} & \\
 & | & \texttt{method}\ m(x_1, \ldots, x_n)\ \texttt{returns}\ (r) & \\
 & & \texttt{requires}\ \mathit{Pre}(x_1, \ldots, x_n) & \\
 & & \texttt{ensures}\ \mathit{Post}(x_1, \ldots, x_n, r) & \\
 & & \{S\} & \\
 & & & \\
P & ::= & \text{Program} & \\
 & | & \overline{D} &
\end{array}
$$

Procedures and method calls

- Declarations annotated with pre- and post-conditions.
- Method calls built into a form of assignment.
- A program P is a set of method declarations.
- Each method decl. is validated, assuming its pre-condition and establishing its post-condition:

$$\frac{\{Pre(x_1,\ldots,x_n)\}\,S\,\{Post(x_1,\ldots,x_n,r)\}}{\begin{array}{l}\texttt{method}\ m(x_1,\ldots,x_n)\ \texttt{returns}\ (r)\\ \texttt{requires}\ Pre(x_1,\ldots,x_n)\\ \texttt{ensures}\ Post(x_1,\ldots,x_n,r)\ \{S\}\end{array}}$$

Procedures and method calls

- Method calls built into a form of assignment:

$$\begin{array}{l}\texttt{method}\ m(x_1,\ldots,x_n)\ \texttt{returns}\ (r) \\ \texttt{requires}\ Pre(x_1,\ldots,x_n) \\ \texttt{ensures}\ Post(x_1,\ldots,x_n,r)\ \{S\}\end{array} \in P$$

$$\frac{A \Rightarrow Pre(E_1,\ldots,E_n) \qquad Post(E_1,\ldots,E_n,r) \Rightarrow B[^r/_x]}{\{A\}\, x := m(E_1,\ldots,E_n)\,\{B\}}$$

Procedures and method calls

$$\begin{array}{ll} \texttt{method}\ m(x_1,\ldots,x_n)\ \texttt{returns}\ (r) & \\ \texttt{requires}\ Pre(x_1,\ldots,x_n) & \in P \\ \texttt{ensures}\ Post(x_1,\ldots,x_n,r)\ \{S\} & \end{array}$$

$$\frac{A \Rightarrow Pre(E_1,\ldots,E_n) \qquad Post(E_1,\ldots,E_n,r) \Rightarrow B[^r/_x]}{\{A\}\, x := m(E_1,\ldots,E_n)\, \{B\}}$$

- Instantiated method pre-condition must follow from A
- Instantiated method post-condition must imply B
- Calls are **opaque**! We only know what's in the post-condition.
- Verification with method calls is **modular**.

Procedures and method calls

```
method maxImp(x:int,y:int) returns (r:int)
  ensures r >= x && r >= y
{
  if x > y { r := x; } else { r := y; }
  return r;
}

method Main() {
  var a := -10;
  var b := 23;
  var c := maxImp(a,b);
  assert (c == b);          ⊗ assertion violation  Verifier


}
```

Procedures and method calls

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```

Procedures and method calls

```
method maxImp(x:int,y:int) returns (r:int)
  ensures (x>y ==> r == x) && (x <= y ==> r == y)
{
  if x > y { r := x; } else { r := y; }
  return r;
}


method Main() {
  var a := -10;
  var b := 23;
  var c := maxImp(a,b);
  assert (c == b);

}
```

Next Week:

- Hoare Logic (continuation)
- Loops and Loop invariants
- Verification of ADTs