

1 Introduction to Ontologies

- Ontologies for information management
- Ontologies in information systems
- Issues in ontology-based information management

2 Using Logic for Representing Knowledge

- Language, real world, and mathematical model
- Logic language
- Interpretation of a logical language
- Logical Consequence
- Inference methods

Modeling Information through Ontologies

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Data never sleeps



<https://www.domo.com/learn/data-never-sleeps-8>



THE 4 V'S OF BIG DATA

40 ZETTABYTES
of data will be created by
2020, an increase of 300
times from 2005



6 BILLION PEOPLE
have cell phones
WORLD POPULATION 7 BILLION



Volume

SCALE OF DATA

2.5 QUINTILLION BYTES
of data are created
each day



Most companies in the
U.S. have at least
100 TERABYTES
of data stored



As of 2011, the global size of
data in healthcare was
estimated to be
150 EXABYTES



**30 BILLION
PIECES OF CONTENT**
are shared on facebook
every month



Variety

DIFFERENT
FORMS OF DATA

**4 BILLION +
HOURS OF VIDEO**
are watched on
YouTube each month



4 MILLION TWEETS
are sent per day by about
200 million monthly active
users



The New York Stock
Exchange captures
**1TB OF TRADE
INFORMATION**
during each trading
session



Velocity

ANALYSIS OF
STREAMING DATA

Modern cars have
close to
100 SENSORS
that monitor items such as
fuel level and tire pressure



**1 IN 3 BUSINESS
LEADERS**
don't trust the information
they use to make
decisions



Veracity

UNCERTAINTY
OF DATA

27% OF RESPONDENTS
in one survey were unsure
of how much of data
was inaccurate



Reference : <http://www.ibmbigdatahub.com/infographic/four-vs-big-data>

One key challenge in complex systems today is information management:

- The **volume** of information is enormous.
- Data increases with incredible **velocity**.
- The **variety** of information has increased:
 - structured \rightsquigarrow semi-structured \rightsquigarrow unstructured
 - Information increasingly **distributed** and **heterogeneous**, but nevertheless needs to be accessed in a uniform way.
 - Information human-processable vs. machine-processable
- The underlying data may be of low quality, e.g., incomplete, inconsistent, not *crisp* – assessing **veracity**.

Traditional data management systems not sufficient anymore to fulfill today's information management requirements.

Runs service centers for power plants, each responsible for remote monitoring and diagnosis of many thousands of gas/steam turbines and associated components. When informed about potential problems, diagnosis engineers access a variety of raw and processed data.

- several TB of time-stamped sensor data
- several GB of event data
- data grows at 30GB per day (sensor data rate 1Hz-1kHz)
- over 50 service centers worldwide
- 1,000 service requests per center per year
- 80 % of the time per request used on data gathering

Several efforts come from the database area:

- New kinds of databases are studied, to manage semi-structured (XML), and probabilistic data; also graph databases.
- Information integration one of the major challenges for the future of IT.
 - E.g., the market for information integration software has been growing at a steady rate of +9% since 2007
 - in 2012 the overall market value of such software was \$ 4 billion.

Management of complex kinds of information traditionally the concern of **Knowledge Representation** in AI:

- Research in AI and KR for new insights, solutions, techniques, and technologies (focused on variability and veracity).

However, what has been done in KR needs to be **adapted** and **extended** to address the new challenges coming from today's requirements for information management.

Description Logics [Baader et al., 2003] an important area of KR, studied for the last 30 years; provide the foundations for structured representation of information:

- By grounding the used formalisms in logic, the information is provided with a **formal semantics** (i.e., a meaning).
- The logic-based formalization allows one to provide **automated support** for tasks related to data management, by means of **logic-based inference**.
- **Computational aspects** are of concern, so that **tools** can provide **effective support** for automated reasoning.

Description logics provide the formal foundations for ontology languages.

Definition (Ontology)

An Ontology is a representation scheme that describes a **formal conceptualization** of a domain of interest.

The specification of an ontology usually comprises two distinct levels:

- **Intensional level**: specifies a set of **conceptual elements** and of constraints/axioms describing the conceptual structures of the domain.
- **Extensional level**: specifies a set of **instances** of the conceptual elements described at the intensional level.

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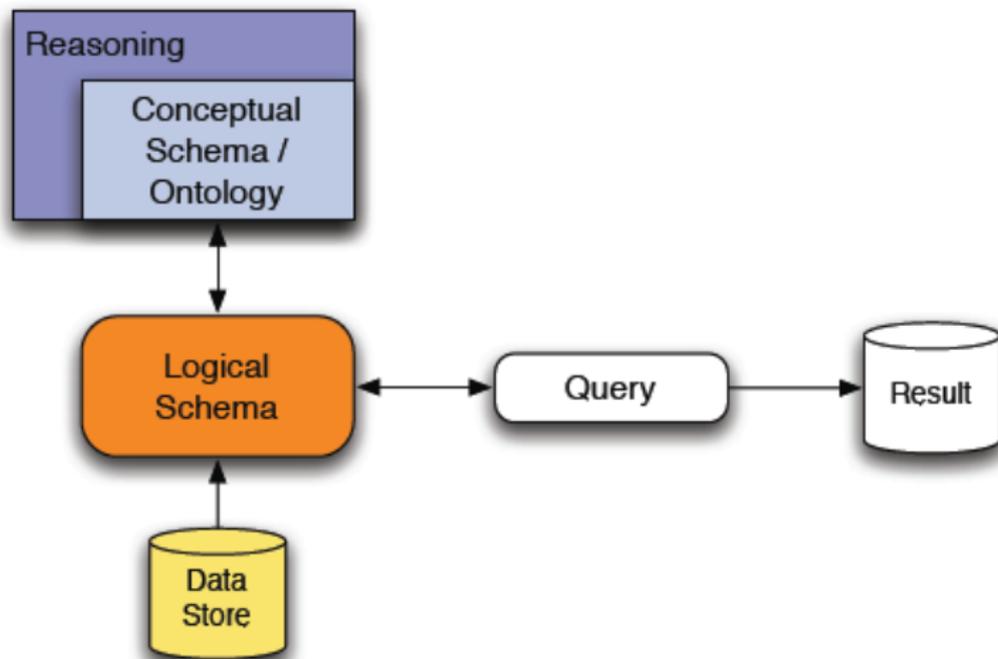
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Intensional information has traditionally played an important role in information systems.

Design phase of the information system:

- 1 From the requirements, a **conceptual schema** of the domain of interest is produced.
- 2 The conceptual schema is used to produce the logical data schema.
- 3 The data are stored according to the logical schema, and queried through it.

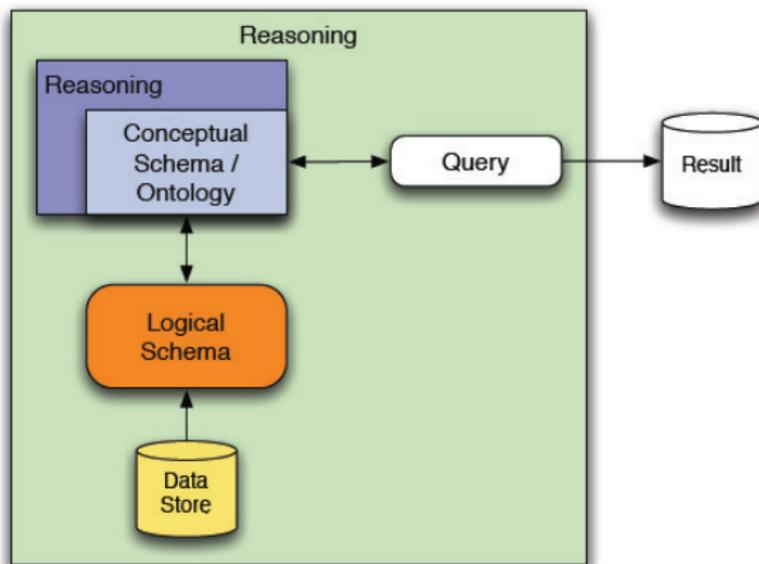
Conceptual schemas used at design-time

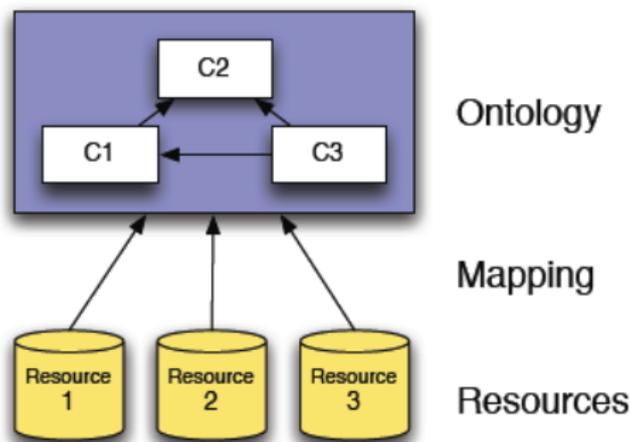


The role of ontologies in information systems goes beyond that of conceptual schemas. Ontologies affect the whole life-cycle of the information system:

- Ontologies, with the associated reasoning capabilities and inference tools, can provide support at design time.
- The use of ontologies can significantly simplify **maintenance** of the information system's data assets.
- The ontology is used also to support the interaction with the information system, i.e., at run-time.

↪ **Reasoning** to take into account the constraints coming from the ontology has to be **done at run-time**.



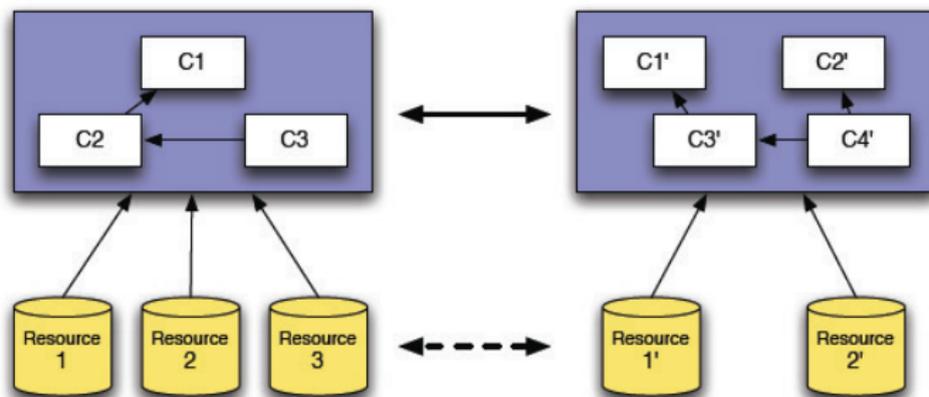


The usage of all system resources (data and services) is done through the domain conceptualization.

Desiderata – achieve **logical transparency** in access to data:

- **Hide** to the user where and how data are stored.
- Present to the user a **conceptual view** of the data.
- Use a **semantically rich formalism** for the conceptual view.

This setting is similar to the one of Data Integration. The difference is that here the ontology provides a rich conceptual description of the data managed by the system.



The cooperation between systems is done at the level of the conceptualization.

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- ① Choice of the formalisms to adopt
- ② Efficiency and scalability
- ③ Tool support

- 1 Which is the right ontology language?
 - many proposals have been made
 - differ in expressive power and in complexity of inference
- 2 Which languages should we use for querying?
 - requirements for querying are different from those for modeling
- 3 How do we connect the ontology to available information sources?
 - mismatch between information in an ontology and data in a data source

In the first part of this course:

- We present and discuss variants of ontology languages, and study their logical and computational properties.

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What is a logic?

- The main objective of a logic: express by means of a **formal language** knowledge about certain phenomena or a certain portion of the world.
- The language of a logic is formal, since it is equipped with:
 - a formal **syntax**: it tells one how to write statements in the logic;
 - a formal **semantics**: it assigns the meaning to these statements.
- Given the formal semantics, one can **reason** over given knowledge, and show which knowledge is a **logical consequence** of the given one.
- A logic often provides a precise set of deterministic rules, called **inference rules**, the basic reasoning steps (correct according to the semantics of the logic).
- By concatenating applications of simple inference rules, one can construct logically correct reasoning chains, which allow one to transform the initial knowledge into some (desired) conclusion.

- Often we want to describe and reason about **real-world** phenomena:
 - A complete description of the real world is clearly impossible.
 - Typically interest in a portion of the world, e.g., a particular physical phenomenon, a social aspect, or modeling rationality of people, ...
- We use sentences of a **language** to describe objects of the real world, their properties, and facts that hold.
 - The language can be:
 - informal (natural lang., graphical lang., icons, ...)
 - formal (logical lang., programming lang., mathematical lang., ...)
 - also mixed languages, i.e., languages with parts that are formal, and others that are informal (e.g., UML class diagrams)
- For a more rigorous description, we provide a **mathematical model**:
 - Is an abstraction of the portion of the real world we are interested in.
 - Represents real world entities in the form of mathematical objects, such as sets, relations, functions, . . .
 - Is not commonly used in everyday communication, but is commonly adopted in science, e.g., to show that a certain argumentation is correct

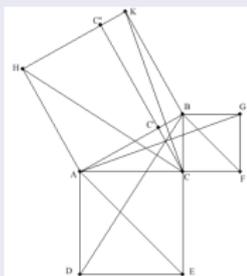
Language

In any right triangle, the area of the square whose side is the hypotenuse (the side opposite the right angle) is equal to the sum of the areas of the squares whose sides are the two legs (the two sides that meet at a right angle).

Real world



Math. model



- Facts about euclidean geometry can be expressed in natural language;
- can refer to one or more real world situations (e.g., composition of the forces in free climbing);
- in general, all such different situations can be abstracted in the mathematical structure which is the euclidean geometry.

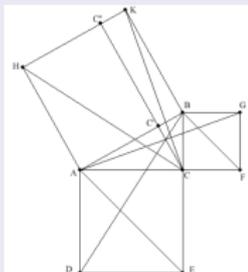
Language

In triangle ABC , if \widehat{BAC} is right, then $\overline{AB}^2 + \overline{AC}^2 = \overline{BC}^2$

Real world



Mathematical model



- The previous example using a language that is “more formal”;
- mixes informal statements (e.g., “if... then...” or “is right”) with formal notation;
- E.g., \widehat{BAC} is an unambiguous and compact way to denote an angle.

Language

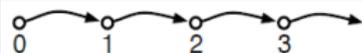
$$x - 2y + 3$$

$$x + y = 0$$

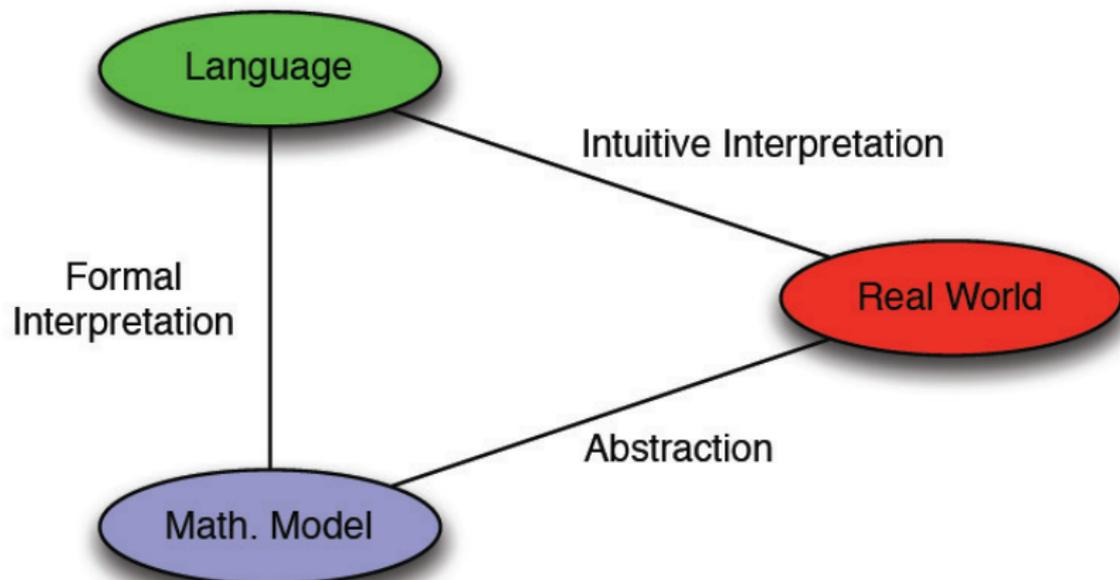
Real world



Mathematical model



- Language purely formal, i.e., the language of arithmetic.
- Abstract language used to represent many situations in the real world (in the primary school, many examples about apples, pears, and how much they cost, which are used by teachers to explain to kids the intuitive meaning of the basic operations on numbers).
- The mathematical model here – the structure of natural numbers.



Connections between language, world, and math. model

Intuitive interpretation (or informal semantics)

When you propose a new language (or when you have to learn a new language), it is important to associate to any element of the language an interpretation in the real world. This is called the intuitive interpretation (or informal semantics).

Formal interpretation (or formal semantics)

Is a function that allows one to transform the elements of the language (i.e., symbols, words, complex sentences,...) into one or more elements of the mathematical structure – formalization of the intuitive interpretation.

Abstraction

Links the real world with its mathematical and abstract representation in a mathematical structure.

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Logic is a special case of the framework we have just seen, where the following important components are defined:

- The language is a **logical language**.
- The formal interpretation allows one to define a notion of **truth**.
- It is possible to define a notion of **logical consequence** between formulas.

- We are given a non-empty set Σ of symbols called **alphabet**.
- A formal language (over Σ) is a subset L of Σ^* , i.e., a set of finite strings of symbols in Σ .
- The elements of L are called **well-formed phrases/formulas**.
- Formal languages can be specified by means of a **grammar**, i.e., a set of formation rules that allow one to build complex well-formed phrases starting from simpler ones.

A language of a logic has the following characteristics:

- **Alphabet:** typically contains basic symbols that are used to indicate the basic (atomic) components of the (part of the) world the logic is supposed to describe. E.g.: individuals, functions, operators, truth-values, propositions,...
- **Grammar:** defines two types of complex elements:
 - **Formulas:** denote propositions, i.e., objects that can assume some truth value (e.g., true, false, true in certain situations, true with probability of 3%, true/false in a period of time,...).
 - **Terms:** denote objects of the world (e.g., cats, dogs, time points, quantities,...).

The alphabet of a logical language is composed of two classes of symbols:

- **Logical constants**, whose formal interpretation is constant and fixed by the logic (e.g., $\wedge, \forall, =, \dots$).
- **Non-logical symbols**, whose formal interpretation is not fixed by the logic, and must be defined by the “user”.

We can make an analogy with (imperative) programming languages:

- Logical constants correspond to reserved words (whose meaning is fixed by the interpreter/compiler).
- Non-logical symbols correspond to the identifiers that are introduced by the programmer for defining functions, variables, procedures, classes, attributes, methods,... (Their meaning is fixed by the programmer.)

The logical constants depend on the logic we are considering:

- **Propositional logic:** \wedge (conjunction), \vee (disjunction), \neg (negation), \supset (implication), \equiv (equivalence), \perp (falsity).

These are usually called **propositional connectives**.

- **Predicate logic:** in addition to the propositional connectives, we have **quantifiers**:
 - universal quantifier \forall , standing for “every object is such that ...”
 - existential quantifier \exists , standing for “there is some object that ...”
- **Modal logic:** in addition to the propositional connectives, we have **modal operators**:
 - \Box standing for “it is necessarily true that...”
 - \Diamond , standing for “it is possibly true that ...”

- **Propositional logic:** non-logical symbols are called **propositional variables**, and represent (i.e., have intuitive interpretation) propositions. The proposition associated to each propositional variable is not fixed by the logic.
- **Predicate logic:** there are four families of non-logical symbols:
 - **Variable symbols**, which represent any object.
 - **Constant symbols**, which represent specific objects.
 - **Function symbols**, which represent transformations on objects.
 - **Predicate symbols**, which represent relations between objects.
- **Modal logic:** non-logical symbols are the same as in propositional logic, i.e., propositional variables.

Example of grammar: Language of propositional logic

Grammar of propositional logic

Allows one to define the unique class of phrases, called **formulas** (or well formed formulas), which denote propositions.

<i>Formula</i>	\rightarrow	P	(P is a propositional variable)
		$(Formula \wedge Formula)$	
		$(Formula \vee Formula)$	
		$(Formula \rightarrow Formula)$	
		$(\neg Formula)$	

Example (Well-formed formulas)

$(P \wedge (Q \rightarrow R))$ $((P \rightarrow (Q \rightarrow R)) \vee P)$

There is a sequence of applications of grammar rules generating them.

Exercise: list the rules in each case.

Example (Not well-formed formulas)

$P(Q \rightarrow R)$
 $(P \rightarrow \vee P)$

Grammar of first-order logic

<i>Term</i>	→	x	(x is a variable symbol)
		c	(c is a constant symbol)
		$f(\textit{Term}, \dots, \textit{Term})$	(f is a function symbol)
<i>Formula</i>	→	$P(\textit{Term}, \dots, \textit{Term})$	(P is a predicate symbol)
		$(\textit{Formula} \wedge \textit{Formula})$	
		$(\textit{Formula} \vee \textit{Formula})$	
		$(\textit{Formula} \rightarrow \textit{Formula})$	
		$\neg \textit{Formula}$	
		$\forall x(\textit{Formula})$	(x is a variable symbol)
		$\exists x(\textit{Formula})$	(x is a variable symbol)

Exercise

Give examples of terms and formulas, and of phrases that are neither of the two.

Grammar of the description logic \mathcal{ALC}

Concept	\rightarrow	A (A is a concept symbol)
		$Concept \sqcup Concept$
		$Concept \sqcap Concept$
		$\neg Concept$
		$\exists Role. Concept$
		$\forall Role. Concept$
Role	\rightarrow	R (R is a role symbol)
Individual	\rightarrow	a (a is an individual symbol)
Formula	\rightarrow	$Concept \sqsubseteq Concept$
		$Concept(Individual)$
		$Role(Individual; Individual)$

Example (Concepts and formulas of the DL \mathcal{ALC})

- Concepts: $A \sqcap B$, $A \sqcup \exists R.C$, $\forall S.(C \sqcup \forall R.D) \sqcap \neg A$
- Formulas: $A \sqsubseteq B$, $A \sqsubseteq \exists R.B$, $A(a)$, $R(a,b)$, $\exists R.C(a)$

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While, non-logical symbols do not have a fixed formal interpretation, they usually have a fixed intuitive interpretation. Consider for instance:

Type	Symbol	Intuitive interpretation
propositional variable	<i>rain</i>	it is raining
constant symbol	<i>MobyDick</i>	the whale of a novel by Melville
function symbol	<i>color(x)</i>	the color of the object x
predicate symbol	<i>Friends(x, y)</i>	x and y are friends

Intuitive interpretation of the non-logical symbols does not affect the logic itself.

- Changing the intuitive interpretation does not affect the properties that will be proved in the logic.
- Replacing these logical symbols with less evocative ones, like r , M , $c(x)$, $F(x, y)$ will not affect the logic.

Interpretation of complex formulas

The intuitive interpretation of complex formulas is done by combining the intuitive interpretations of the components of the formulas.

Example

Consider the propositional formula:

$$(raining \vee snowing) \rightarrow \neg go_to_the_beach$$

If the intuitive interpretations of the symbols are:

symbol	intuitive meaning
<i>raining</i>	it is raining
<i>snowing</i>	it is snowing
<i>go_to_the_beach</i>	we go to the beach
\vee	either ... or ...
\rightarrow	if ... then ...
\neg	it is not the case that ...

then the above formula intuitively represent the proposition:

- if (it is raining or it is snowing), then it is not the case that (we go to the beach)

- **Class of models:** The models in which a logic is *formally interpreted* are the members of a class of algebraic structures, each of which is an abstract representation of the relevant aspects of the (portion of the) world we want to formalize with this logic.
- **Models** represent only the components and aspects of the world which are relevant to a certain analysis, and abstract away from irrelevant facts.
Example: if we are interested in the average temperature of each day, we can represent time with the natural numbers and use a function that associates to each natural number a floating point number (the average temperature of the day corresponding to the point).
- **Applicability of a model:** Since the real world is complex, in the construction of the formal model, we usually do **simplifying assumptions** that bound the usability of the logic to the cases in which these assumptions are verified.
Example: if we take integers as formal model of time, then this model is not applicable to represent continuous change.
- Each **model represents** a single possible (or impossible) state of the world. The class of models of a logic will represent all the (im)possible states of the world.

- Given a structure S and a logical language L , the **formal interpretation** in S of L is a function that associates an element of S to any non-logical symbol of the alphabet.
- The formal interpretation in the algebraic structure is the parallel counterpart (or better, the formalization) of the intuitive interpretation in the real world.
- The formal interpretation is specified only for the non-logical symbols.
- The formal interpretation of the logical symbols is fixed by the logic.
- The formal interpretation of a complex expression e , obtained as a combination of the sub-expressions e_1, \dots, e_n , is uniquely determined as a function of the formal interpretation of the sub-components e_1, \dots, e_n .

- The goal of logic is the formalization of what is true/false in a particular world. The particular world is formalized by a structure, also called an **interpretation**.
- The main objective of the formal interpretation is that it allows to define when a **formula is true in an interpretation**.
- Every logic therefore defines the **satisfiability relation** (denoted by \models) between interpretations and formulas.
- If \mathcal{I} is an interpretation and φ a formula, then

$$\mathcal{I} \models \varphi$$

stands for the fact that \mathcal{I} **satisfies** φ , or equivalently that φ **is true in** \mathcal{I} .

- An interpretation \mathcal{M} such that $\mathcal{M} \models \varphi$ is called a **model** of φ .

Example

Consider again the propositional formula:

$$(raining \vee snowing) \rightarrow \neg go_to_the_beach$$

In propositional logic:

- We only need to interpret propositional variables
- map them to true or false
- represented as those mapped to true
- e.g., $\{raining\}$ and $\{go_to_the_beach\}$ are models of the given formula
- $\{snowing, go_to_the_beach\}$ is not

On the basis of truth in an interpretation (\models) the following notions are defined in any logic:

- φ is **satisfiable** if it has model, i.e., if there is a structure \mathcal{M} such that $\mathcal{M} \models \varphi$
- φ is **unsatisfiable** if it is not satisfiable, i.e., it has no models.
- φ is **valid**, denoted $\models \varphi$, if it is true in all interpretations.

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Logical consequence (or implication)

- The notion of **logical consequence** (or implication) is defined on the basis of the notion of truth in an interpretation.
- Intuitively, a formula φ is a logical consequence of a set of formulas (sometimes called assumptions) Γ (denoted $\Gamma \models \varphi$) if such a formula is true under this set of assumptions.
- Formally, $\Gamma \models \varphi$ holds when:

For all interpretations \mathcal{I} , if $\mathcal{I} \models \Gamma$, then $\mathcal{I} \models \varphi$.

In words: φ is true in all the possible situations in which all the formulas in Γ are true.

- Notice that the two relations, “truth in a model” and “logical consequence” are denoted by the same symbol \models (this should remind you that they are tightly connected).

Difference between \models and implication (\rightarrow)

At first glance, \models looks like implication (usually denoted by \rightarrow or \supset). In most of the cases, they represent the same relation between formulas.

Similarity

- For instance, in propositional logic (but not only) the fact that φ is a logical consequence of the singleton set $\{\psi\}$, i.e., $\{\psi\} \models \varphi$, can be encoded in the formula $\psi \rightarrow \varphi$.
- Similarly, the fact that φ is a logical consequence of the set of formulas $\{\varphi_1, \dots, \varphi_n\}$ i.e., $\{\varphi_1, \dots, \varphi_n\} \models \varphi$ can be encoded by the formula $\varphi_1 \wedge \dots \wedge \varphi_n \rightarrow \varphi$

Difference

- When $\Gamma = \{\gamma_1, \gamma_2, \dots\}$ is an infinite set of formulas, the fact that φ is a logical consequence of Γ cannot be represented with a formula $\gamma_1 \wedge \gamma_2 \wedge \dots \rightarrow \varphi$ because this would be infinite, and in logic all the formulas are finite. (Actually there are logics, called infinitary logics, where formulas can have infinite size.)

Exercise

Show that if $\Gamma = \emptyset$, then $\Gamma \models \varphi \iff \varphi$ is valid.

Solution

(\implies) Since Γ is empty, every interpretation \mathcal{I} satisfies all the formulas in Γ . Therefore, if $\Gamma \models \varphi$, then every interpretation \mathcal{I} must satisfy φ , hence φ is valid.

(\impliedby) If φ is valid, then every \mathcal{I} is such that $\mathcal{I} \models \varphi$. Hence, whatever Γ is (in particular, when $\Gamma = \emptyset$), every model of Γ is also a model of φ , and so $\Gamma \models \varphi$.

Exercise

Show that if φ is unsatisfiable then $\{\varphi\} \models \psi$ for every formula ψ .

Solution

If φ is unsatisfiable then it has no model, which implies that each interpretation that satisfies φ (namely, none) satisfies also ψ , independently from ψ .

Exercise

Show that the following properties hold for the logical consequence relation defined above:

Reflexivity: $\Gamma \cup \{\varphi\} \models \varphi$

Monotonicity: $\Gamma \models \varphi$ implies that $\Gamma \cup \Sigma \models \varphi$

Cut: $\Gamma \models \varphi$ and $\Sigma \cup \{\varphi\} \models \psi$ implies that $\Gamma \cup \Sigma \models \psi$

Solution

Reflexivity: If \mathcal{I} satisfies all the formulas in $\Gamma \cup \{\varphi\}$ then it satisfies also φ , and therefore $\Gamma \cup \{\varphi\} \models \varphi$.

Monotonicity: Let \mathcal{I} be an interpretation that satisfies all the formulas in $\Gamma \cup \Sigma$. Then it satisfies all the formulas in Γ , and if $\Gamma \models \varphi$, then $\mathcal{I} \models \varphi$. Therefore, we can conclude that $\Gamma \cup \Sigma \models \varphi$.

Cut: Let \mathcal{I} be an interpretation that satisfies all the formulas in $\Gamma \cup \Sigma$. Then it satisfies all the formulas in Γ , and if $\Gamma \models \varphi$, then $\mathcal{I} \models \varphi$. This implies that \mathcal{I} satisfies all the formulas in $\Sigma \cup \{\varphi\}$. Then, since $\Sigma \cup \{\varphi\} \models \psi$, we have that $\mathcal{I} \models \psi$. Therefore we can conclude that $\Gamma \cup \Sigma \models \psi$.

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Problem

Does there exist an algorithm that checks if a formula φ is a logical consequence of a set of formulas Γ ?

Solution 1: If Γ is finite and the set of models of the logic is finite, then it is possible to directly apply the definition by checking for every interpretation \mathcal{I} , that if $\mathcal{I} \models \Gamma$ then, $\mathcal{I} \models \varphi$.

Solution 2: If Γ is infinite or the set of models is infinite, then Solution 1 is not applicable as it would run forever.

An alternative solution could be to generate, starting from Γ , all its logical consequences by applying a set of rules.

- **Propositional logic:** The method based on **truth tables** can be used to check logical consequence by enumerating all the interpretations of Γ and φ and checking if every time all the formulas in Γ are true then φ is also true. This is possible because, when Γ is finite then there are a finite number of interpretations.
- **First-order logic:** A first-order language in general has an infinite number of interpretations. Therefore, to check logical consequence, it is not possible to apply a method that enumerates all the possible interpretations, as in truth tables.
- **Modal logic:** presents the same problem as first-order logic. In general for a set of formulas Γ , there is an infinite number of interpretations, which implies that a method that enumerates all the interpretations is not effective.

- An alternative method for determining if a formula is a logical consequence of a set of formulas is based on **inference rules**.
- An inference rule is a rewriting rule that takes a set of formulas and transforms it into another set of formulas.
- The following are examples of **inference rules**.

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi}$$

$$\frac{\varphi \quad \psi}{\varphi \rightarrow \psi}$$

$$\frac{\forall x.\varphi(x)}{\varphi(c)}$$

$$\frac{\exists x.\varphi(x)}{\varphi(d)}$$

- Differently from truth tables, which apply a brute force exhaustive analysis not interpretable by humans, the deductive method simulates human argumentation and provides also an understandable explanation (i.e., a **deduction**) of the reason why a formula is a logical consequence of a set of formulas.

Inference rules to check logical consequence - Example

Let $\Gamma = \{p \rightarrow q, \neg p \rightarrow r, q \vee r \rightarrow s\}$.

The following is a deduction (an explanation of) the fact that s is a logical consequence of Γ , i.e., that $\Gamma \models s$, which uses the following inference rules:

$$\frac{\varphi \rightarrow \psi \quad \neg\varphi \rightarrow \vartheta}{\psi \vee \vartheta} \quad (*)$$

$$\frac{\varphi \quad \varphi \rightarrow \psi}{\psi} \quad (**)$$

Example of deduction

- | | | |
|-----|--------------------------|----------------------------------|
| (1) | $p \rightarrow q$ | Belongs to Γ . |
| (2) | $\neg p \rightarrow r$ | Belongs to Γ . |
| (3) | $q \vee r$ | By applying (*) to (1) and (2). |
| (4) | $q \vee r \rightarrow s$ | Belongs to Γ . |
| (5) | s | By applying (**) to (3) and (4). |

In a Hilbert-style deduction system, a formal deduction is a **finite sequence of formulas**

$$\begin{array}{c} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \vdots \\ \varphi_n \end{array}$$

where each φ_i

- is either an **axiom**, or
- it is derived from previous formulas $\varphi_{j_1}, \dots, \varphi_{j_k}$ with $j_1, \dots, j_k < i$, by applying the **inference rule**

$$\frac{\varphi_{j_1}, \dots, \varphi_{j_k}}{\varphi_i}$$

Hilbert axioms for classical propositional logic

Axioms

- A1** $\varphi \rightarrow (\psi \rightarrow \varphi)$
A2 $(\varphi \rightarrow (\psi \rightarrow \theta)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \theta))$
A3 $(\neg\psi \rightarrow \neg\varphi) \rightarrow ((\neg\psi \rightarrow \varphi) \rightarrow \psi)$

Inference rule(s)

$$\frac{\varphi \quad \varphi \rightarrow \psi}{\psi} \text{ (MP)}$$

Example (Proof of $A \rightarrow A$)

1. **A1** : $A \rightarrow ((A \rightarrow A) \rightarrow A)$
2. **A2** : $(A \rightarrow ((A \rightarrow A) \rightarrow A)) \rightarrow ((A \rightarrow (A \rightarrow A)) \rightarrow (A \rightarrow A))$
3. **MP**(1, 2) : $(A \rightarrow (A \rightarrow A)) \rightarrow (A \rightarrow A)$
4. **A1** : $(A \rightarrow (A \rightarrow A))$
5. **MP**(4, 3) : $A \rightarrow A$

Reasoning by refutation is based on the principle of “**Reductio ad absurdum**”.

Reductio ad absurdum

In order to show that a proposition φ is true, we assume that it is false (i.e., that $\neg\varphi$ holds) and try to infer a contradictory statement, such as $A \wedge \neg A$ (usually denoted by \perp , i.e., the false statement).

Reasoning by refutation is one of the most important principles for building **automated decision procedures**. This is mainly due to the fact that, proving a formula φ corresponds to the reduction of $\neg\varphi$ to \perp .

Propositional resolution is the most simple example of reasoning via refutation. The procedure can be described as follows:

Definition (Propositional resolution)

INPUT: a propositional formula φ

OUTPUT: $\models \varphi$ or $\not\models \varphi$

- 1 Convert $\neg\varphi$ to conjunctive normal form, i.e., to a set C of formulas (called clauses) of the form

$$p_1 \vee \cdots \vee p_k \vee \neg p_{k+1} \vee \cdots \vee \neg p_n$$

also represented as $[p_1 \dots, p_k, \neg p_{k+1}, \dots, \neg p_n]$ that is logically equivalent to φ .

- 2 Apply exhaustively the following inference rule

$$\frac{c \vee p \quad \neg p \vee c'}{c \vee c'} \quad (\text{Resolution})$$

and add $c \vee c'$ to C

- 3 if C contains the empty clause, then return $\models \varphi$, otherwise return $\not\models \varphi$

In order to show that $\models \varphi$ (i.e., that φ is valid) we search for a model of $\neg\varphi$, i.e., we show that $\neg\varphi$ is satisfiable.

If we are not able to find such a model, then we can conclude that there is no model of $\neg\varphi$, i.e., that all the models satisfy φ , which is: that φ is valid.

There are two basic methods of searching for a model for φ :

SAT-based decision procedures

- This method incrementally builds a model.
- At every stage it defines a “partial model” μ_i and does an early/lazy check if φ can be true in some extension of μ_i .
- At each point the algorithm has to decide how to extend μ_i to μ_{i+1} , until it constructs a full model for φ .

Tableaux-based decision procedures

- This method builds the model of φ via a “top-down” approach.
- I.e., φ is decomposed in its sub-formulas $\varphi_1, \dots, \varphi_n$ and the algorithm recursively builds n models M_1, \dots, M_n for them.
- The model M of φ is obtained by a suitable combination of M_1, \dots, M_n

We illustrate a SAT-based decision procedure on a propositional logic example.

To find a model for $(p \vee q) \wedge \neg p$, we proceed as follows:

partial model	lazy evaluation	result of lazy evaluation
$\mu_0 = \{p = true\}$	$(true \vee q) \wedge \neg true$	<i>false</i> (<i>backtrack</i>)
$\mu_1 = \{p = false\}$	$(false \vee q) \wedge \neg false$	<i>q</i> (<i>continue</i>)
$\mu_2 = \{p = false,$ $q = true\}$	$(false \vee true) \wedge \neg false$	<i>true</i> (<i>success!</i>)

Let R be an inference method, and let \vdash_R denote the corresponding inference relation.

Definition (Soundness of an inference method)

An inference method R is **sound** if

$$\begin{aligned} \vdash_R \varphi &\implies \models \varphi \\ \Gamma \vdash_R \varphi &\implies \Gamma \models \varphi \quad (\text{strongly sound}) \end{aligned}$$

Definition (Completeness of an inference method)

An inference method R is **complete** if

$$\begin{aligned} \models \varphi &\implies \vdash_R \varphi \\ \Gamma \models \varphi &\implies \Gamma \vdash_R \varphi \quad (\text{strongly complete}) \end{aligned}$$

Clausal form as before, but atom is $P(t_1, t_2, \dots, t_n)$, where t_i may contain variables

Interpretation as before, but variables are understood *universally*

- Example: $\{[P(x), \neg R(a, f(b, x))], [Q(x, y)]\}$
 - interpreted as
- $\forall x \forall y \{ [R(a, f(b, x)) \supset P(x)] \wedge Q(x, y) \}$

Substitutions: $\theta = \{\nu_1/t_1, \nu_2/t_2, \dots, \nu_n/t_n\}$

Notation: If ρ is a literal and θ is a substitution, then $\rho\theta$ is the result of applying substitution θ to ρ (and similarly, $c\theta$ where c is a clause)

- Example: $\theta = \{x/a, y/g(x, b, z)\}$
- $P(x, z, f(x, y))\theta = P(a, z, f(a, g(x, b, z)))$

A literal is ground if it contains no variables.

A literal ρ is an instance of ρ' , if for some θ , $\rho = \rho'\theta$

Resolution will generalize to handling variables

But, to convert well-formed formulas to CNF, we need additional steps:

- 1 eliminate \supset and \equiv
- 2 push \neg inward using also $\neg\forall x.\alpha \Rightarrow \exists x.\neg\alpha$ etc.
- 3 standardize variables: each quantifier gets its own variable
 - e.g. $\exists x[P(x)] \wedge Q(x) \Rightarrow \exists z[P(z)] \wedge Q(x)$ where z is a new variable
- 4 eliminate all existentials (*discussed later*)
- 5 move universals to the front using $(\forall x\alpha) \wedge \beta \Rightarrow \forall x(\alpha \wedge \beta)$
 - where β does not use x (due to step 3)
- 6 distribute \wedge over \vee

Get universally quantified conjunction of disjunctions of literals

- then drop all the quantifiers...

Main idea: a literal (with variables) stands for all its instances; so allow all such inferences

- Given $[P(x, a), \neg Q(x)]$ and $[\neg P(b, y), \neg R(b, f(y))]$, we want to infer $[\neg Q(b), \neg R(b, f(a))]$ among others
 - since $[P(x, a), \neg Q(x)]$ has $[P(b, a), \neg Q(b)]$ and $[\neg P(b, y), \neg R(b, f(y))]$ has $[\neg P(b, a), \neg R(b, f(a))]$

Resolution:

- Given clauses $\{\rho_1\} \cup C_1$ and $\{\overline{\rho_2}\} \cup C_2$
- Rename variables, so that they are distinct in the two clauses.
- For any θ such that $\rho_1\theta = \rho_2\theta$, infer $(C_1 \cup C_2)\theta$.
 - We say that ρ_1 unifies with ρ_2 and that θ is a unifier of the two literals

Resolution derivation: as before

Theorem: $S \rightarrow \square$ iff $S \models \square$ iff S is unsatisfiable

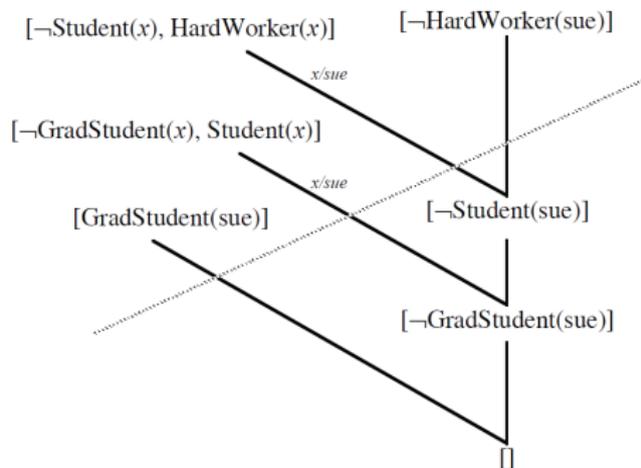
- Note: There are pathological examples where a slightly more general definition of Resolution is required. We ignore them for now...

Example 3

?
 $KB \models \text{HardWorker}(sue)$

KB

$\forall x (\text{GradStudent}(x) \supset \text{Student}(x))$ $\forall x (\text{Student}(x) \supset \text{HardWorker}(x))$ $\text{GradStudent}(sue)$



- Label each step with the unifier
- Point to relevant literals in clauses

KB: $Plus(\text{zero}, x, x)$
 $Plus(x, y, z) \supset Plus(\text{succ}(x), y, \text{succ}(z))$
 Q: $\exists u Plus(2, 3, u)$

For readability, we use

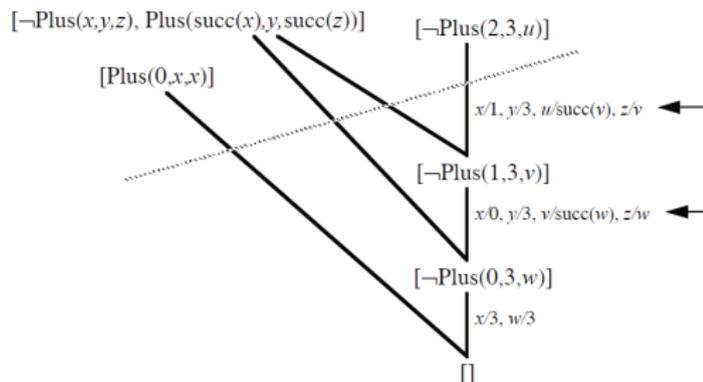
- 0 for *zero*,
- 1 for *succ(zero)*,
- 2 for *succ(succ(zero))*
- etc.

Can find the answer in the derivation

- $u/\text{succ}(\text{succ}(3))$

that is: $u/5$

Can also derive $Plus(2, 3, 5)$



In first-order logic, we have the possibility of deriving $\exists xP(x)$ without being able to derive $P(t)$ for any t .

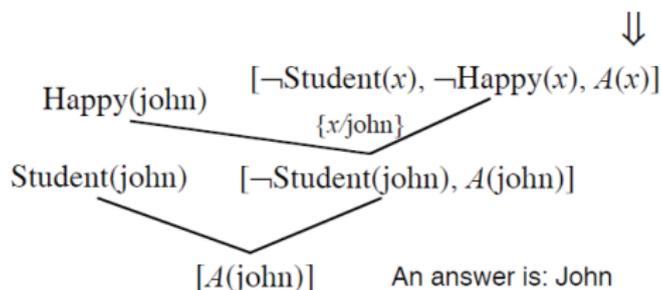
- e.g. the three-blocks problem
 - $\exists x\exists y[On(x, y) \wedge Green(x) \wedge \neg Green(y)]$
 - but cannot derive which block is which

Solution: answer-extraction process

- replace query $\exists xP(x)$ by $\exists x[P(x) \wedge \neg A(x)]$
 - where A is a new predicate symbol called the answer predicate
- instead of deriving \square , derive any clause containing just the answer predicate
- can always convert to and from a derivation of \square

$KB : Student(john)$
 $Student(jane)$
 $Happy(john)$

$Q :$
 $\exists x[Student(x) \wedge Happy(x)]$



Disjunctive answers

KB :

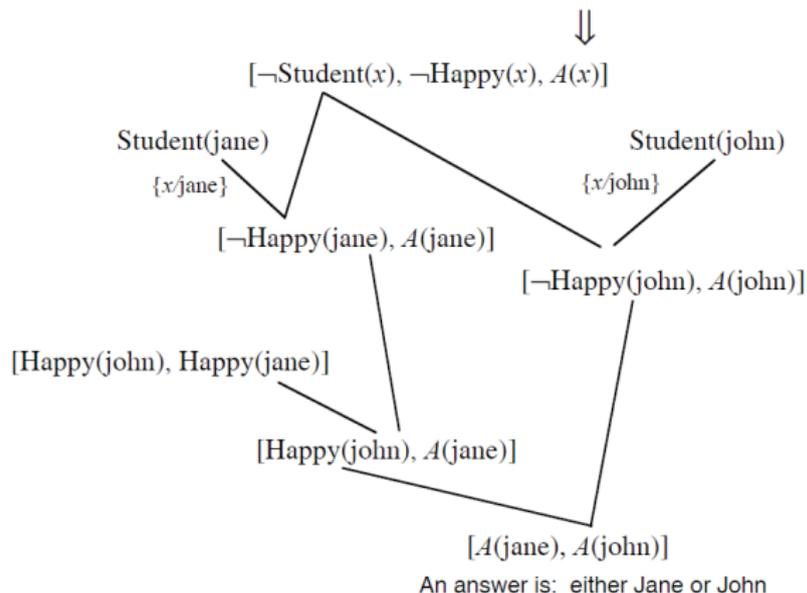
- $Student(john)$
- $Student(jane)$
- $Happy(john) \vee Happy(jane)$

Query :

- $\exists x[Student(x) \wedge Happy(x)]$

Note:

- can have variables in answer
- need to watch for Skolem symbols... (next)



So far, converting well-formed formulas to CNF ignored existentials

- e.g. $\exists x \forall y \exists z P(x, y, z)$

Idea: names for individuals claimed to exist, called Skolem constant and function symbols

- there exists an x , call it a
- for each y , there is a z , call it $f(y)$
 - get $\forall y P(a, y, f(y))$

replace $\forall x_1 (\dots \forall x_2 (\dots \forall x_n (\dots \exists y [\dots y \dots] \dots) \dots) \dots)$ by
 $\forall x_1 (\dots \forall x_2 (\dots \forall x_n (\dots [\dots f(x_1, x_2, \dots, x_n) \dots] \dots) \dots) \dots)$

- f is a new function symbol that appears nowhere else

Skolemization does not preserve equivalence

- e.g. $\not\equiv \exists x P(x) \equiv P(a)$

But it does preserve satisfiability

- α is satisfiable iff α' is satisfiable (where α' is the result of Skolemization) sufficient for resolution!

Show that $\exists x \forall y R(x, y) \models \forall y \exists x R(x, y)$

- show $\{\exists x \forall y R(x, y), \neg \forall y \exists x R(x, y)\}$ unsatisfiable
 - $\exists x \forall y R(x, y) \Rightarrow \forall y R(a, y)$
 - $\neg \forall y \exists x R(x, y) \Rightarrow \exists y \forall x \neg R(x, y) \Rightarrow \forall x \neg R(x, b)$
- then $\{[R(a, y)], [\neg R(x, b)]\} \rightarrow []$ with $\{x/a, y/b\}$.

Show that $\forall y \exists x R(x, y) \not\models \exists x \forall y R(x, y)$

- show $\{\forall x \exists x R(x, y), \neg \exists x \forall y R(x, y)\}$ unsatisfiable
 - $\forall y \exists x R(x, y) \Rightarrow \forall y R(f(y), y)$
 - $\neg \exists x \forall y R(x, y) \Rightarrow \forall x \exists y \neg R(x, y) \Rightarrow \forall x \neg R(x, g(x))$
- then get $\{[R(f(y), y)], [\neg R(x, g(x))]\}$
 - where the two literals do not unify

Note: important to get dependence of variables correct

- $R(f(y), y)$ vs. $R(a, y)$ above

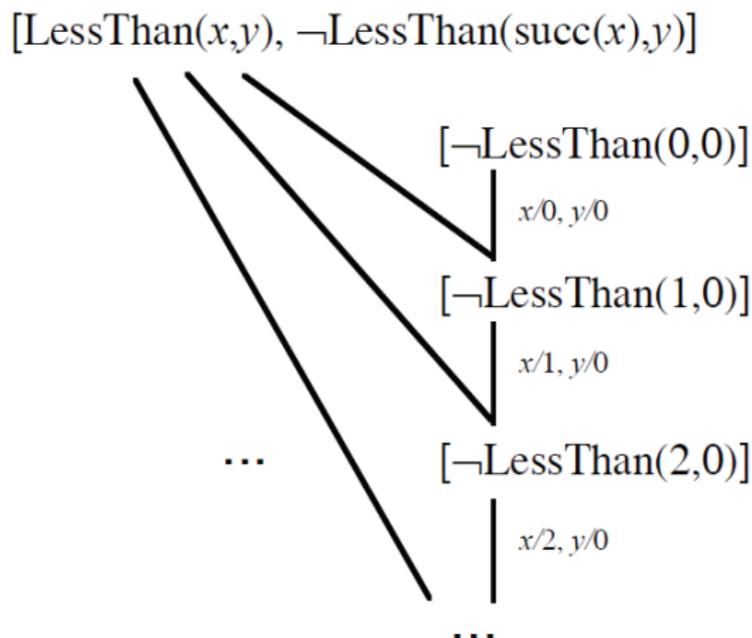
KB :

- $LessThan(succ(x), y) \supset LessThan(x, y)$

Query :

- $LessThan(0, 0)$

Should fail since $KB \not\models Q$



- Infinite branch of resolvents cannot use a simple depth-first procedure to search for \square

Is there a way to detect when this happens?

No! first-order logic is very powerful

- can be used as a full programming language
- as there is no way to detect in general when a program is looping

There can be no procedure that does this:

- process *Clauses*
- If *Clauses* are unsatisfiable
 - then return YES
 - else return NO

However: Resolution is (refutation-)complete

- some branch will contain \square , for unsatisfiable clauses

So breadth-first search guaranteed to find \square

- search may not terminate on satisfiable clauses

