Knowledge Representation and Reasoning

Solutions to Exercises on First Order Logic

1 Alpine Club

Formulate the following pieces of knowledge as sentences of first-order logic:

Tony, Mike and John belong to the Alpine Club. Every member of the Alpine Club who is not a skier is a mountain climber. Mountain climbers do not like rain, and anyone who does not like snow is not a skier. Mike dislikes whatever Tony likes, and likes whatever Tony dislikes. Tony likes rain and snow.

Answer: The answer uses the following predicates and constants:

- $Member$: unary predicate meaning a member of the Alpine Club;
- $Skier$: unary predicate meaning a skier;
- $Climber$: unary predicate meaning a climber;
- $Likes$: binary predicate where $Likes\,(x, y)$ means that x likes y;
- constants $tony$, $mike$, $john$, $rain$, $snow$.

In the translation, we name sentences so that it is easy to refer to them later.

- Tony, Mike and John belong to the Alpine Club.

$$S1 : Member\,(tony) \qquad S2 : Member\,(mike) \qquad S3 : Member\,(john)$$

- Every member of the Alpine Club who is not a skier is a mountain climber.

$$S4 : \forall x\,((Member\,(x) \wedge \neg Skier\,(x)) \rightarrow Climber\,(x))$$

- Mountain climbers do not like rain

$$S5 : \forall x\,(Climber\,(x) \rightarrow \neg Likes\,(x, rain))$$

- and anyone who does not like snow is not a skier.

$$S6 : \forall x\,(\neg Likes\,(x, snow) \rightarrow \neg Skier\,(x))$$

- Mike dislikes whatever Tony likes

$$S7 : \forall x\,(Likes\,(tony, x) \rightarrow \neg Likes\,(mike, x))$$

- and likes whatever Tony dislikes.

$$S8 : \forall x\,(\neg Likes\,(tony, x) \rightarrow Likes\,(mike, x))$$

- Tony likes rain and snow.

$$S9 : Likes\,(tony, rain) \qquad S10 : Likes\,(tony, snow)$$

Note that S7 and S8 can be joined in one equivalence $\forall x\,(Likes\,(tony, x) \leftrightarrow \neg Likes\,(mike, x))$ *as* $\forall x\,(\neg Likes\,(tony, x) \rightarrow Likes\,(mike, x))$ *and* $\forall x\,(\neg Likes\,(mike, x) \rightarrow Likes\,(tony, x))$ *are equivalent to* $\forall x\,(\neg Likes\,(tony, x) \vee \neg Likes\,(mike, x))$ *and* $\forall x\,(Likes\,(tony, x) \vee Likes\,(mike, x))$*, respectively. Consider alternatively* $\forall x, y((Likes(tony, x) \wedge \neg Likes(tony, y)) \rightarrow (\neg Likes(mike, x) \wedge Likes(mike, y)))$ *joining the two implications into one. This is, however not equivalent since whenever Tony likes all things (in the domain), then nothing can be said about what Mike likes.*

2 Reduction to CNF

Rewrite all sentences in $KB = \{(p \vee q) \rightarrow r, r \rightarrow s, p\}$ in conjunctive normal form, and present KB in clausal form.

Answer:

- $(p \vee q) \rightarrow r$ is, by definition of $\rightarrow$ equivalent to $\neg(p \vee q) \vee r$
  $\neg(p \vee q) \vee r$ is by de Morgan's law equivalent to $(\neg p \wedge \neg q) \vee r$
  By distributivity, $(\neg p \wedge \neg q) \vee r$ is equivalent to $(\neg p \vee r) \wedge (\neg q \vee r)$
  $(\neg p \vee r) \wedge (\neg q \vee r)$ is in CNF and corresponds to two clauses $[\neg p, r]$ $[\neg q, r]$.
- $r \rightarrow s$ is, by definition of $\rightarrow$ equivalent to $\neg r \vee s$
  $\neg r \vee s$ is in CNF and corresponds to the clause $[\neg r, s]$.

The KB written in clausal form is $KB = \{[\neg p, r], [\neg q, r], [\neg r, s], [p]\}$.

3 Propositional Resolution

a) Show by resolution that the following set of clauses is inconsistent (derive the empty clause from it):

$$[A, B, C], [A, B, \neg C], [A, \neg B, C], [A, \neg B, \neg C]$$
$$[\neg A, B, C], [\neg A, B, \neg C], [\neg A, \neg B, C], [\neg A, \neg B, \neg C]$$

b) Show by resolution that the following sentence is inconsistent:

$$\neg\neg A \wedge (\neg A \vee ((\neg B \vee C) \wedge B)) \wedge \neg C$$

Answer:
a) We can apply resolution as follows:

1. $[A, B]$ from $[A, B, C], [A, B, \neg C]$.
2. $[A, \neg B]$ from $[A, \neg B, C], [A, \neg B, \neg C]$.
3. $[A]$ from 1. and 2.
4. $[\neg A, B]$ from $[\neg A, B, C], [\neg A, B, \neg C]$.
5. $[\neg A, \neg B]$ from $[\neg A, \neg B, C], [\neg A, \neg B, \neg C]$.
6. $[\neg A]$ from 4. and 5.
7. $[]$ from 3. and 6.

b) We first need to transform into conjunctive normal form to obtain the clauses:

- $\neg\neg A \wedge (\neg A \vee ((\neg B \vee C) \wedge B)) \wedge \neg C$ is equivalent to
- $A \wedge (\neg A \vee \neg B \vee C) \wedge (\neg A \vee B) \wedge \neg C$ which corresponds to the clauses
- $[A], [\neg A, \neg B, C], [\neg A, B], [\neg C]$; Then:

1. $[\neg B, C]$ from $[A], [\neg A, \neg B, C]$
2. $[B]$ from $[A], [\neg A, B]$
3. $[C]$ from 1. and 2.
4. $[]$ from 3. and $[\neg C]$

Can you find a shorter proof?

4 First-Order Resolution

Determine whether the following sentences are valid using resolution:

a) $\exists x \forall y \forall z \left(\left(P\left(y\right) \rightarrow Q\left(z\right) \right) \rightarrow \left(P\left(x\right) \rightarrow Q\left(x\right) \right) \right)$

b) $\exists x \left(P\left(x\right) \rightarrow \forall y \left(P\left(y\right) \right) \right)$

c) $\neg \exists x \forall y \left(E\left(x,y\right) \leftrightarrow \neg E\left(y,y\right) \right)$

Show by resolution that the following set of clauses is inconsistent.

d) $[P(x), P(f(x))], [\neg P(y), P(f(z))], [\neg P(w), \neg P(f(w))]$

Answer:
To do this we need to check if from the negation of the sentence we can derive an empty clause (a contradiction).
a) First transform the negation into clausal form:

$$\begin{aligned}
&\neg \exists x \forall y \forall z \left(\left(P\left(y\right) \rightarrow Q\left(z\right) \right) \rightarrow \left(P\left(x\right) \rightarrow Q\left(x\right) \right) \right) \\
&\neg \exists x \forall y \forall z \left(\neg \left(\neg P\left(y\right) \vee Q\left(z\right) \right) \vee \left(\neg P\left(x\right) \vee Q\left(x\right) \right) \right) \\
&\forall x \exists y \exists z \neg \left(\neg \left(\neg P\left(y\right) \vee Q\left(z\right) \right) \vee \left(\neg P\left(x\right) \vee Q\left(x\right) \right) \right) \\
&\forall x \exists y \exists z \left(\neg\neg \left(\neg P\left(y\right) \vee Q\left(z\right) \right) \wedge \neg \left(\neg P\left(x\right) \vee Q\left(x\right) \right) \right) \\
&\forall x \exists y \exists z \left(\left(\neg P\left(y\right) \vee Q\left(z\right) \right) \wedge \left(\neg\neg P\left(x\right) \wedge \neg Q\left(x\right) \right) \right) \\
&\forall x \exists y \exists z \left(\left(\neg P\left(y\right) \vee Q\left(z\right) \right) \wedge P\left(x\right) \wedge \neg Q\left(x\right) \right) \\
&\forall x \left(\left(\neg P\left(f\left(x\right)\right) \vee Q\left(g\left(x\right)\right) \right) \wedge P\left(x\right) \wedge \neg Q\left(x\right) \right)
\end{aligned}$$

Clauses:

$$\begin{aligned}
&C1 : \left[\neg P\left(f\left(x\right)\right) \vee Q\left(g\left(x\right)\right) \right] \\
&C2 : \left[P\left(x_1\right) \right] \\
&C3 : \left[\neg Q\left(x_2\right) \right]
\end{aligned}$$

Proof:

1. $\left[Q\left(g\left(x\right)\right) \right]$ from $C1$ and $C2$, $x_1/f\left(x\right)$.
2. $[]$ from (1) and $C3$, $x_2/g\left(x\right)$.

Regarding why renaming of variables in clauses is necessary consider $\forall x(p(a,x) \wedge \neg p(x,b))$. This formula is inconsistent, because it implies both $p(a,b)$ and $\neg p(a,b)$. So we should be able to use resolution to derive the empty clause. The clausal form of the formula is: $\{[p(a,x)], [\neg p(x,b)]\}$ However, we cannot unify $p(a,x)$ and $\neg p(x,b)$, since x would need to be mapped to a and b simultaneously.
b) Transform the negation into clausal form:

$$\begin{aligned}
&\neg \exists x \left(P\left(x\right) \rightarrow \forall y \left(P\left(y\right) \right) \right) \\
&\forall x \neg \left(\neg P\left(x\right) \vee \forall y \left(P\left(y\right) \right) \right) \\
&\forall x \left(P\left(x\right) \wedge \exists y \, \neg P\left(y\right) \right) \\
&\forall x \left(P\left(x\right) \wedge \neg P\left(f\left(x\right)\right) \right)
\end{aligned}$$

Clauses:

$$\begin{aligned}
&C1 : \left[P\left(x\right) \right] \\
&C2 : \left[\neg P\left(f\left(x_1\right)\right) \right]
\end{aligned}$$

Proof:

1. $[]$ from $C1$ and $C2$, $x/f\left(x_1\right)$

c) Transform the negation into clausal form:

$$\neg\neg\exists x\forall y\left(E\left(x,y\right)\leftrightarrow\neg E\left(y,y\right)\right)$$
$$\exists x\forall y\left(\left(\neg E\left(x,y\right)\vee\neg E\left(y,y\right)\right)\wedge\left(E\left(x,y\right)\vee E\left(y,y\right)\right)\right)$$
$$\forall y\left(\left(\neg E\left(a,y\right)\vee\neg E\left(y,y\right)\right)\wedge\left(E\left(a,y\right)\vee E\left(y,y\right)\right)\right)$$

Clauses:

$$C1:\left[\neg E\left(a,y\right),\neg E\left(y,y\right)\right]$$
$$C2:\left[E\left(a,y_1\right),E\left(y_1,y_1\right)\right]$$

Proof:

1. $[\neg E\left(a,a\right)]$ factorization $C1$, y/a
2. $[E\left(a,a\right)]$ factorization $C2$, y_1/a
3. [] from 1. and 2.

Alternatively, we may use a generalization of the resolution rule, which allows resolving more than one unified (identical) atom per clause.
Proof:

1. [] from 1. and 2., $y/a, y_1/a$

d) We can apply resolution directly (and there are several possible solutions).
Clauses:

$$C1:[P(x),P(f(x))]$$
$$C2:[\neg P(y),P(f(z))]$$
$$C3:[\neg P(w),\neg P(f(w))]$$

Proof:

1. $[\neg P(y),\neg P(w)]$ resolution $C2$ and $C3$, z/w
2. $[\neg P(y)]$ factorization $C1$, w/y
3. $[P(f(x))]$ resolution $C1$ and 2., y/x
4. [] from 2. and 3., $y/f(x)$

An example of an alternative proof without factorization follows:

Alternative Proof:

1. $[\neg P(y),\neg P(w)]$ resolution $C2$ and $C3$, z/w
2. $[P(f(x))]$ resolution $C1$ and 1., y/x and w/x
3. [] from 1. and 2., $y/f(x)$ and $w/f(x)$

5 Alpine Club and First-Order Resolution

As a follow-up to the Alpine Club Exercise, use resolution to prove that there exists a member of the Alpine club who is a climber but not a skier. Can you determine his name?

Answer:
Translation into first-order logic as given in the solution for Exercise 1 (S1 – S10) together with:

$$S11 : \exists x \left(Member\left(x\right) \wedge Climber\left(x\right) \wedge \neg Skier\left(x\right) \right)$$

Now in clausal form (with S11 negated):

$$\begin{aligned} C1 &: \left[Member\left(tony\right)\right] \\ C2 &: \left[Member\left(mike\right)\right] \\ C3 &: \left[Member\left(john\right)\right] \\ C4 &: \left[\neg Member\left(x\right), Skier\left(x\right), Climber\left(x\right)\right] \\ C5 &: \left[\neg Climber\left(x_1\right), \neg Likes\left(x_1, rain\right)\right] \\ C6 &: \left[Likes\left(x_2, snow\right), \neg Skier\left(x_2\right)\right] \\ C7 &: \left[\neg Likes\left(tony, x_3\right), \neg Likes\left(mike, x_3\right)\right] \\ C8 &: \left[Likes\left(tony, x_4\right), Likes\left(mike, x_4\right)\right] \\ C9 &: \left[Likes\left(tony, rain\right)\right] \\ C10 &: \left[Likes\left(tony, snow\right)\right] \\ C11 &: \left[\neg Member\left(x_5\right), \neg Climber\left(x_5\right), Skier\left(x_5\right)\right] \end{aligned}$$

Prove that, together, $C1 - C11$ are inconsistent:

1. $[\neg Likes\left(mike, snow\right)]$ from $C10$ and $C7$, $x_3/snow$
2. $[\neg Skier\left(mike\right)]$ from 1. and $C6$, $x_2/mike$
3. $[\neg Member\left(mike\right), Climber\left(mike\right)]$ from 2. and $C4$, $x/mike$
4. $[Climber\left(mike\right)]$ from 3. and $C2$
5. $[\neg Member\left(mike\right), Skier\left(mike\right)]$ from (4) and $C11$, $x_5/mike$
6. $[Skier\left(mike\right)]$ from 5. and $C2$
7. [] from 6. and 2.

To determine his name, we can change S11 to:

$$S11 : \exists x \left(Member\left(x\right) \wedge Climber\left(x\right) \wedge \neg Skier\left(x\right) \wedge \neg A(x) \right)$$

This results in $C11 = [\neg Member\left(x_5\right), \neg Climber\left(x_5\right), Skier\left(x_5\right), A(x_5)]$.
In step 5, we thus add $A(mike)$, which then occurs in the clauses of steps 6. and 7., thus providing the answer.
Can you find an even shorter proof (there is one with 5 resolution steps)?