Knowledge Representation and Reasoning

Solutions to Exercises on Description Logic Ontologies

1 Converting from Description Logics to First-Order Logics

Consider the solutions from the previous class on converting the UML class diagram into description logics. Convert the Description Logic result into first-order logic.

Answer:

$$\forall x. (\exists y. place(x, y) \rightarrow Origin(x))$$
$$\forall x. (\exists y. place(y, x) \rightarrow String(x))$$
$$\forall x. (Origin(x) \rightarrow (\exists y. place(x, y) \wedge \forall y, z. ((place(x, y) \wedge place(x, z)) \rightarrow y = z)))$$
$$\forall x. (\exists y. reference(x, y) \rightarrow PhoneBill(x))$$
$$\forall x. (\exists y. reference(y, x) \rightarrow PhoneCall(x))$$
$$\forall x. (PhoneBill(x) \rightarrow (\exists y. reference(x, y)))$$
$$\forall x. (PhoneCall(x) \rightarrow (\exists y. reference(y, x) \wedge \forall y, z. ((reference(y, x) \wedge reference(z, x)) \rightarrow y = z)))$$
$$\forall x. (\exists y. callO(x, y) \rightarrow Origin(x))$$
$$\forall x. (\exists y. callO(y, x) \rightarrow PhoneCall(x))$$
$$\forall x. (\exists y. fromO(x, y) \rightarrow Origin(x))$$
$$\forall x. (\exists y. fromO(y, x) \rightarrow Phone(x))$$
$$\forall x. (Origin(x) \rightarrow (\exists y. callO(x, y) \wedge \exists y. fromO(x, y) \wedge \forall y, z. ((callO(x, y) \wedge callO(x, z)) \rightarrow y = z) \wedge \forall y, z. ((fromO(x, y) \wedge fromO(x, z)) \rightarrow y = z)))$$
$$\forall x. (PhoneCall(x) \rightarrow (\exists y. callO(y, x) \wedge \forall y, z. ((callO(y, x) \wedge callO(z, x)) \rightarrow y = z)))$$
$$\forall x. (\exists y. callMO(x, y) \rightarrow MobileOrigin(x))$$
$$\forall x. (\exists y. callMO(y, x) \rightarrow MobileCall(x))$$
$$\forall x. (\exists y. fromMO(x, y) \rightarrow MobileOrigin(x))$$
$$\forall x. (\exists y. fromMO(y, x) \rightarrow CellPhone(x))$$
$$\forall x. (MobileOrigin(x) \rightarrow (\exists y. callMO(x, y) \wedge \exists y. fromMO(x, y) \wedge \forall y, z. ((callMO(x, y) \wedge callMO(x, z)) \rightarrow y = z) \wedge \forall y, z. ((fromMO(x, y) \wedge fromMO(x, z)) \rightarrow y = z)))$$
$$\forall x. (MobileOrigin(x) \rightarrow Origin(x))$$
$$\forall x, y. (callMO(x, y) \rightarrow callO(x, y))$$
$$\forall x, y. (fromMO(x, y) \rightarrow fromO(x, y))$$
$$\forall x. (MobileCall(x) \rightarrow PhoneCall(x))$$
$$\forall x. (CellPhone(x) \rightarrow Phone(x))$$
$$\forall x. (FixedPhone(x) \rightarrow (Phone(x) \wedge \neg CellPhone(x)))$$
$$\forall x. (Phone(x) \rightarrow (CellPhone(x) \vee FixedPhone(x)))$$

2 Constructing Models of Ontologies

Consider the following **TBox**:

$$
\begin{aligned}
&Cow \sqsubseteq Vegetarian\\
&MadCow \sqsubseteq Cow \sqcap \exists eat.BrainOfSheep\\
&Sheep \sqsubseteq Animal\\
&Vegetarian \sqsubseteq (\geq 1\ eat) \sqcap \forall eat.\neg(Animal \sqcup PartOfAnimal)\\
&BrainOfSheep \sqsubseteq PartOfAnimal
\end{aligned}
$$

1. Translate the TBox into natural language, and compare with the translation into first-order logic.

$Cow \sqsubseteq Vegetarian$**:** All cows are vegetarians.

$$\forall x.\,(Cow(x) \rightarrow Vegetarian(x))$$

$MadCow \sqsubseteq Cow \sqcap \exists\ eat.BrainOfSheep$**:** All mad cows are cows that eat (some) brain of sheep.

$$\forall x.\,[MadCow(x) \rightarrow (Cow(x) \wedge \exists y.\,(eat(x,y) \wedge BrainOfSheep(y)))]$$

$Sheep \sqsubseteq Animal$**:** All sheep are animals.

$$\forall x.\,(Sheep(x) \supset Animal(x))$$

$Vegetarian \sqsubseteq (\geq 1\ eat) \sqcap \forall eat.\neg(Animal \sqcup PartOfAnimal)$**:** All vegetarians eat something, but never anything which is an animal or part of an animal.

$$\forall x.\,[Vegetarian(x) \rightarrow \exists y.\,(eat(x,y)) \wedge \forall y.\,(eat(x,y) \rightarrow \neg(Animal(y) \vee PartOfAnimal(y)))]$$

$BrainOfSheep \sqsubseteq PartOfAnimal$**:** All brains of sheep are parts of animals.

$$\forall x.\,(BrainOfSheep(x) \rightarrow PartOfAnimal(x))$$

2. Construct a model for the ontology $\mathcal{O}_1 = (\mathbf{TBox}, \{Cow(mimosa)\})$.

Answer: A model is $\mathcal{I} = (\Delta^{\mathcal{I}}, .^{\mathcal{I}})$ (others exist), where the domain is $\Delta^{\mathcal{I}} = \{m, e\}$ and the interpretation mapping is:

$$
\begin{aligned}
mimosa^{\mathcal{I}} &= m\\
Cow^{\mathcal{I}} &= \{m\}\\
MadCow^{\mathcal{I}} &= \{\}\\
Sheep^{\mathcal{I}} &= \{\}\\
BrainOfSheep^{\mathcal{I}} &= \{\}\\
Animal^{\mathcal{I}} &= \{\}\\
PartOfAnimal^{\mathcal{I}} &= \{\}\\
Vegetarian^{\mathcal{I}} &= \{m\}\\
eat^{\mathcal{I}} &= \{(m,e)\}
\end{aligned}
$$

All assertions must be satisfied, i.e. $\mathcal{I} \models \mathcal{O}_1$ iff $\mathcal{I} \models \mathbf{TBox}$ and $\mathcal{I} \models Cow(mimosa)$:

$\mathcal{I} \models Cow \sqsubseteq Vegetarian$ iff $Cow^{\mathcal{I}} \subseteq Vegetarian^{\mathcal{I}}$ iff $\{m\} \subseteq \{m\}$
$\mathcal{I} \models MadCow \sqsubseteq Cow \sqcap \exists\ eat.BrainOfSheep$ iff $MadCow^{\mathcal{I}} \subseteq Cow^{\mathcal{I}} \cap (\exists\ eat.BrainOfSheep)^{\mathcal{I}}$ iff $\{\} \subseteq \{\}$
$\mathcal{I} \models Sheep \sqsubseteq Animal$ iff $Sheep^{\mathcal{I}} \subseteq Animal^{\mathcal{I}}$ iff $\{\} \subseteq \{\}$
$\mathcal{I} \models Vegetarian \sqsubseteq (\geq 1\ eat) \sqcap \forall eat.\neg(Animal \sqcup PartOfAnimal)$
$\mathcal{I} \models BrainOfSheep \sqsubseteq PartOfAnimal$ iff $BrainOfSheep^{\mathcal{I}} \subseteq PartOfAnimal^{\mathcal{I}}$ iff $\{\} \subseteq \{\}$

$\mathcal{I} \models Cow(mimosa)$ iff $mimosa^{\mathcal{I}} \in Cow^{\mathcal{I}}$ iff $m \in \{m\}$

3. Show that there is no model for the ontology $\mathcal{O}_2 = (\mathbf{TBox}, \{MadCow(mimosa)\})$.

   We will show that it is impossible to construct an interpretation $\mathcal{I}$ that satisfies $\mathcal{O}_2$.
   So suppose there is an interpretation that models $\mathcal{O}_2$.
   Since we have to satisfy assertion $MadCow(mimosa)$, there is an individual m in the domain of $\mathcal{I}$ such that $mimosa^{\mathcal{I}} = m$ and $mimosa^{\mathcal{I}} \in MadCow^{\mathcal{I}}$, i.e., $m \in MadCow^{\mathcal{I}}$.
   Since every $MadCow$ is a Cow, $m \in Cow^{\mathcal{I}}$ holds, and furthermore $m \in Vegetarian^{\mathcal{I}}$. Moreover, every $MadCow$ eats at least some brain of sheep (let's denote this brain by b, and thus $b \in BrainOfSheep^{\mathcal{I}}$ and $(m, b) \in eat^{\mathcal{I}}$. In addition, $b \in PartOfAnimal^{\mathcal{I}}$. But then, since $m \in Vegetarian^{\mathcal{I}}$, we also require that $m \in (\forall eat.\neg(Animal \sqcup PartOfAnimal))^{\mathcal{I}}$. Since $(m, b) \in eat^{\mathcal{I}}$, $b \in (\neg(Animal \sqcup PartOfAnimal))^{\mathcal{I}}$, i.e., $b \notin (Animal \sqcup PartOfAnimal)^{\mathcal{I}}$, and in particular $b \notin PartOfAnimal^{\mathcal{I}}$. We derive a contradiction.

3 Knowledge Representation in $\mathcal{ALC}$

Express the following sentences in terms of the description logic $\mathcal{ALC}$.

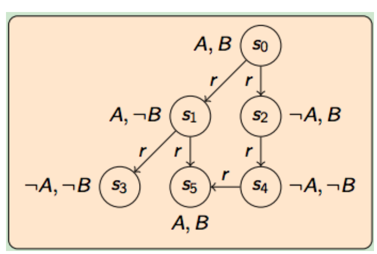
1. All employees are humans.
2. A mother is a female who has a child.
3. A parent is a mother or a father.
4. A grandmother is a mother who has a child who is a parent.
5. Only humans have children that are humans.

Answer:

1. $Employee \sqsubseteq Human$
2. $Mother \equiv Female \sqcap \exists hasChild.\top$
3. $Parent \equiv Mother \sqcup Father$
4. $Grandmother \equiv Mother \sqcap \exists hasChild.Parent$
5. $\exists hasChild.Human \sqsubseteq Human$

4 Semantics of $\mathcal{ALC}$

Let $\mathcal{I}$ be the following $\mathcal{ALC}$ interpretation on the domain $\Delta^{\mathcal{I}} = \{s_0, s_1, ..., s_5\}$.



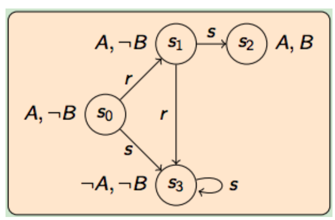
Determine the interpretation of the following concepts:

Answer:

1. $\top^{\mathcal{I}} = \{s_0, s_1, \ldots, s_5\}$.
2. $\bot^{\mathcal{I}} = \emptyset$.
3. $A^{\mathcal{I}} = \{s_0, s_1, s_5\}$.
4. $B^{\mathcal{I}} = \{s_0, s_2, s_5\}$.
5. $(A \sqcap B)^{\mathcal{I}} = \{s_0, s_5\}$.
6. $(A \sqcup B)^{\mathcal{I}} = \{s_0, s_1, s_2, s_5\}$.
7. $(\neg A)^{\mathcal{I}} = \{s_2, s_3, s_4\}$.
8. $(\exists r.A)^{\mathcal{I}} = \{s_0, s_1, s_4\}$.
9. $(\forall r.\neg B)^{\mathcal{I}} = \{s_2, s_3, s_5\}$.
10. $(\forall r.(A \sqcup B))^{\mathcal{I}} = \{s_0, s_3, s_4, s_5\}$.

5 Semantics of $\mathcal{ALC}$

Let $\mathcal{I}$ be the following $\mathcal{ALC}$ interpretation on the domain $\Delta^{\mathcal{I}} = \{s_0, s_1, ..., s_3\}$.



Determine the interpretation of the following concepts:

Answer:

1. $(A \sqcup B)^{\mathcal{I}} = \{s_0, s_1, s_2\}$.
2. $(\exists s.\neg A)^{\mathcal{I}} = \{s_0, s_3\}$.
3. $(\forall s.A)^{\mathcal{I}} = \{s_1, s_2\}$.
4. $(\exists s.\exists s.\exists s.\exists s.A)^{\mathcal{I}} = \emptyset$.
5. $(\neg\exists r.(\neg A \sqcup \neg B))^{\mathcal{I}} = \{s_2, s_3\}$.
6. $(\exists s.(A \sqcup \forall s.\neg B) \sqcup \neg\forall r.\exists r.(A \sqcup \neg A))^{\mathcal{I}} = \{s_0, s_1, s_3\}$.

6 (Un)Satisfiability and Validity of $\mathcal{ALC}$

For each of the following formulas, indicate if it is valid, satisfiable or unsatisfiable. If it is not valid, provide a model that falsifies it:

1. $\forall r.(A \sqcap B) \equiv \forall r.A \sqcap \forall r.B$.
2. $\forall r.(A \sqcup B) \equiv \forall r.A \sqcup \forall r.B$.
3. $\exists r.(A \sqcap B) \equiv \exists r.A \sqcap \exists r.B$.

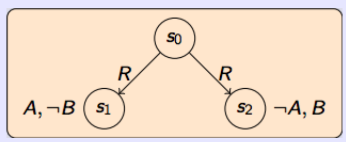
4. $\exists r.(A \sqcup B) \equiv \exists r.A \sqcup \exists r.B$.

Answer:

1. $\forall r.(A \sqcap B) \equiv \forall r.A \sqcap \forall r.B$ is valid. We can prove that $(\forall r.(A \sqcap B))^{\mathcal{I}} = (\forall r.A \sqcap \forall r.B)^{\mathcal{I}}$ for all interpretations $\mathcal{I}$.

$$\begin{aligned}(\forall r.(A \sqcap B))^{\mathcal{I}} &= \{x \in \Delta^{\mathcal{I}} \mid \forall y : (x,y) \in r^{\mathcal{I}} \rightarrow y \in (A \sqcap B)^{\mathcal{I}}\} \\ &= \{x \in \Delta^{\mathcal{I}} \mid \forall y : (x,y) \in r^{\mathcal{I}} \rightarrow y \in (A^{\mathcal{I}} \cap B^{\mathcal{I}})\} \\ &= \{x \in \Delta^{\mathcal{I}} \mid \forall y : (x,y) \in r^{\mathcal{I}} \rightarrow y \in A^{\mathcal{I}}\} \cap \{x \in \Delta^{\mathcal{I}} \mid \forall y : (x,y) \in r^{\mathcal{I}} \rightarrow y \in B^{\mathcal{I}}\} \\ &= (\forall r.A)^{\mathcal{I}} \cap (\forall r.B)^{\mathcal{I}} \\ &= (\forall r.A \sqcap \forall r.B)^{\mathcal{I}}\end{aligned}$$

2. $\forall r.(A \sqcup B) \equiv \forall r.A \sqcup \forall r.B$ is not valid. The following model is such that $(\forall r.(A \sqcup B))^{\mathcal{I}} \neq (\forall r.A \sqcup \forall r.B)^{\mathcal{I}}$.



   - $s_0 \in (\forall r.(A \sqcup B))^{\mathcal{I}}$ but
   - $s_0 \notin (\forall r.A)^{\mathcal{I}}$ and
   - $s_0 \notin (\forall r.B)^{\mathcal{I}}$.

   However, notice that $\forall r.A \sqcup \forall r.B \sqsubseteq \forall r.(A \sqcup B)$ is valid.

3. $\exists r.(A \sqcap B) \equiv \exists r.A \sqcap \exists r.B$ is not valid. The previous model is such that $(\exists r.(A \sqcap B))^{\mathcal{I}} \neq (\exists r.A \sqcap \exists r.B)^{\mathcal{I}}$.

   - $s_0 \in (\exists r.A)^{\mathcal{I}}$ and
   - $s_0 \in (\exists r.B)^{\mathcal{I}}$ but
   - $s_0 \notin (\exists r.(A \sqcap B))^{\mathcal{I}}$.

   However, notice that $\exists r.(A \sqcap B) \sqsubseteq \exists r.A \sqcap \exists r.B$ is valid.

4. $\exists r.(A \sqcup B) \equiv \exists r.A \sqcup \exists r.B$ is valid. We could provide a similar proof to the case $\forall r.(A \sqcap B) \equiv \forall r.A \sqcap \forall r.B$, but we show here an alternative proof which is based on other equivalences.

$$\begin{aligned}\exists r.(A \sqcup B) &\equiv \neg\forall r.(\neg(A \sqcup B)) \\ &\equiv \neg\forall r.(\neg A \sqcap \neg B)) \\ &\equiv \neg(\forall r.(\neg A) \sqcap \forall r.(\neg B)) \\ &\equiv \neg\forall r.(\neg A) \sqcup \neg\forall r.(\neg B)) \\ &\equiv \exists r.A \sqcup \exists r.B\end{aligned}$$