Knowledge Representation and Reasoning

Exercises on Description Logics

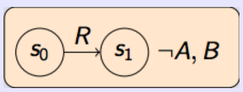
1 (Un)Satisfiability and Validity of $\mathcal{ALC}$

For each of the following concepts, indicate if it is valid, satisfiable or unsatisfiable. If it is valid, or unsatisfiable, provide a proof. If it is satisfiable (and not valid), then exhibit a model that interprets the concept in a non-empty set:

1. $\neg(\forall r.A \sqcup \exists r.(\neg A \sqcap \neg B))$.
2. $\exists r.(\forall s.C) \sqcap \forall r.(\exists s.\neg C)$.
3. $(\exists s.C \sqcap \exists s.D) \sqcap \forall s.(\neg C \sqcup \neg D)$.
4. $\exists s.(C \sqcap D) \sqcap (\forall s.\neg C \sqcup \forall s.\neg D)$.
5. $C \sqcap \exists r.A \sqcap \exists r.B \sqcap \neg\exists r.(A \sqcap B)$.

Answer:

1. $\neg(\forall r.A \sqcup \exists r.(\neg A \sqcap \neg B))$. Satisfiable.



   - $s_0 \in (\neg(\forall r.A \sqcup \exists r.(\neg A \sqcap \neg B))^{\mathcal{I}}$
   - $s_1 \notin (\neg(\forall r.A \sqcup \exists r.(\neg A \sqcap \neg B))^{\mathcal{I}}$

2. $\exists r.(\forall s.C) \sqcap \forall r.(\exists s.\neg C)$. Unsatisfiable.
   Since $\exists r.(\forall s.C) \equiv \neg\forall r.(\neg\forall s.C) \equiv \neg\forall r.(\exists s.(\neg C))$, this implies that $\exists r.(\forall s.C) \sqcap \forall r.(\exists s.\neg C)$ is equivalent to $\neg\forall r.(\exists s.(\neg C)) \sqcap \forall r.(\exists s.(\neg C))$, which is a concept of the form $\neg B \sqcap B$ which is always unsatisfiable.

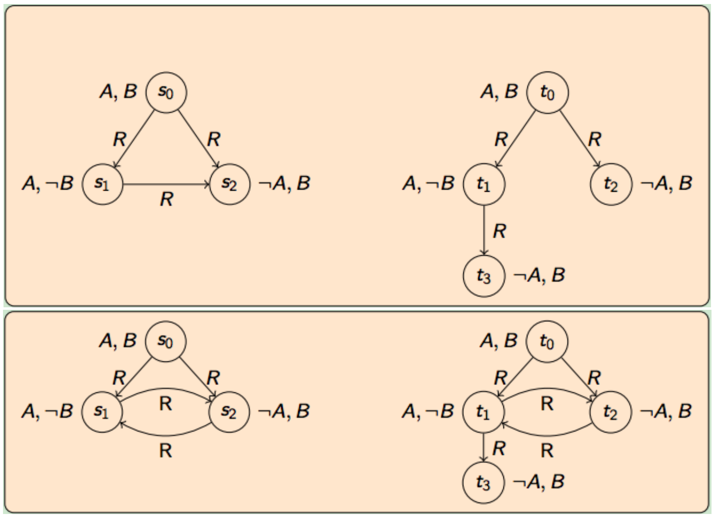
3. $(\exists s.C \sqcap \exists s.D) \sqcap \forall s.(\neg C \sqcup \neg D)$. Satisfiable.

4. $\exists s.(C \sqcap D) \sqcap (\forall s.\neg C \sqcup \forall s.\neg D)$. Unatisfiable.

5. $C \sqcap \exists r.A \sqcap \exists r.B \sqcap \neg\exists r.(A \sqcap B)$. Satisfiable.

2 Bissimulation

For each of the following pairs of models, check if they are bisimular. If yes, find the bisimulation relation, if not, find a formula that is true in the first model and false in the second.



Answer:

- The first pair of models is bisimilar and the bisimulation is $\{(s_0, t_0), (s_1, t_1), (s_2, t_2), (s_2, t_3)\}$.
- The second pair of models is not bisimular on s_0 and t_0. Note that (s_0, t_0) would have to belong to the bisimulation. However, we have that $s_0 \in (\forall r.(\forall r.(\exists r.\top)))^{\mathcal{I}_1}$ and $t_0 \notin (\forall r.(\forall r.(\exists r.\top)))^{\mathcal{I}_2}$, where $\mathcal{I}_1$ and $\mathcal{I}_2$ and the interpretations shown above.

3 $\mathcal{ALC}$ Tableaux

Check by means of tableaux whether the following subsumption is valid:

1. $\neg\forall r.A \sqcap \forall r.\left((\forall r.B) \sqcup A\right) \sqsubseteq \forall r.\neg\left(\exists r.A\right) \sqcup \exists r.\left(\exists r.B\right)$

Answer: To check whether the given subsumption is valid we can use tableaux to verify whether the following concept is insatisfiable:

$$\neg\forall r.A \sqcap \forall r.\left(\forall r.B \sqcup A\right) \sqcap \neg\left(\forall r.\neg\left(\exists r.A\right) \sqcup \exists r.\left(\exists r.B\right)\right)$$

We have to transform into negation normal form first:

$$C_0 = \exists r.\neg A \sqcap \forall r.\left(\forall r.B \sqcup A\right) \sqcap \exists r.\exists r.A \sqcap \forall r.\forall r.\neg B$$

We apply the tableaux algorithm starting with C_0:

	$\mathcal{A}_0 = \{C_0(x_0)\}$		
$\rightarrow_\sqcap$	$\mathcal{A}_1 = \mathcal{A}_0 \cup \{\exists r.\neg A \sqcap \forall r.\left(\forall r.B \sqcup A\right)(x_0), \exists r.\exists r.A \sqcap \forall r.\forall r.\neg B(x_0)\}$		
$\rightarrow_\sqcap$	$\mathcal{A}_2 = \mathcal{A}_1 \cup \{\exists r.\neg A(x_0), \forall r.\left(\forall r.B \sqcup A\right)(x_0)\}$		
$\rightarrow_\sqcap$	$\mathcal{A}_3 = \mathcal{A}_2 \cup \{\exists r.\exists r.A(x_0), \forall r.\forall r.\neg B(x_0)\}$		
$\rightarrow_\exists$	$\mathcal{A}_4 = \mathcal{A}_3 \cup \{r(x_0, x_1), \exists r.A(x_1)\}$		
$\rightarrow_\exists$	$\mathcal{A}_5 = \mathcal{A}_4 \cup \{r(x_1, x_2), A(x_2)\}$		
$\rightarrow_\forall$	$\mathcal{A}_6 = \mathcal{A}_5 \cup \{\forall r.\neg B(x_1)\}$		
$\rightarrow_\forall$	$\mathcal{A}_7 = \mathcal{A}_6 \cup \{\neg B(x_2)\}$		
$\rightarrow_\forall$	$\mathcal{A}_8 = \mathcal{A}_7 \cup \{\forall r.B \sqcup A(x_1)\}$		
$\rightarrow_\sqcup$	$\mathcal{A}_9 = \mathcal{A}_8 \cup \{A(x_1)\}$		$\mathcal{A}_{9'} = \mathcal{A}_8 \cup \{\forall r.B(x_1)\}$
$\rightarrow_\exists$	$\mathcal{A}_{10} = \mathcal{A}_9 \cup \{r(x_0, x_3), \neg A(x_3)\}$	$\rightarrow_\forall$	$\mathcal{A}_{10'} = \mathcal{A}_{9'} \cup \{B(x_2)\}\times$
$\rightarrow_\forall$	$\mathcal{A}_{11} = \mathcal{A}_{10} \cup \{\forall r.B \sqcup A(x_3)\}$		
$\rightarrow_\sqcup$	$\mathcal{A}_{12} = \mathcal{A}_{11} \cup \{\forall r.B(x_3)\}$		$\mathcal{A}_{12''} = \mathcal{A}_{11} \cup \{A(x_3)\}\times$
$\rightarrow_\forall$	$\mathcal{A}_{13} = \mathcal{A}_{12} \cup \{\forall r.\neg B(x_3)\}\checkmark$		

Since $\mathcal{A}_{13}$ is complete and clash-free, C_0 is satisfiable, and the initial subsumption inclusion is not valid. We can provide the canonical interpretation $\mathcal{I}_{\mathcal{A}_{13}}$ as follows:

- $\Delta^{\mathcal{I}_{\mathcal{A}_{13}}} = \{x_0, x_1, x_2, x_3\}$
- $A^{\mathcal{I}_{\mathcal{A}_{13}}} = \{x_1, x_2\}$
- $B^{\mathcal{I}_{\mathcal{A}_{13}}} = \emptyset$
- $r^{\mathcal{I}_{\mathcal{A}_{13}}} = \{(x_0, x_1), (x_1, x_2), (x_0, x_3)\}$

4 $\mathcal{ALC}$ Tableaux

Which of the following statements are true? Explain your answer.

Answer: Left as an exercise...

5 $\mathcal{ALC}$ Tableaux with cyclic TBoxes

Check by means of tableaux whether the following subsumption is valid w.r.t. the TBox $\{A \sqsubseteq \exists r.(A \sqcup B)\}$:

1. $A \sqsubseteq \exists r.B$

Answer: To check whether the given subsumption is valid we can use tableaux extended by the $\mathcal{T}$ rule (using $\neg A \sqcup \exists r.(A \sqcup B)$) to verify whether the following concept is insatisfiable:

$$A \sqcap \neg\exists r.B$$

We have to transform into negation normal form first:

$$C_0 = A \sqcap \forall r.\neg B$$

We apply the tableaux algorithm starting with C_0:

	$\mathcal{A}_0 = \{C_0(x_0)\}$	
$\rightarrow_\sqcap$	$\mathcal{A}_1 = \mathcal{A}_0 \cup \{A(x_0), \forall r.\neg B(x_0)\}$	
$\rightarrow_\mathcal{T}$	$\mathcal{A}_2 = \mathcal{A}_1 \cup \{(\neg A \sqcup \exists r.(A \sqcup B))(x_0)\}$	
$\rightarrow_\sqcup$	$\mathcal{A}_3 = \mathcal{A}_2 \cup \{(\exists r.(A \sqcup B))(x_0)\}$	$\mathcal{A}_{3'} = \mathcal{A}_2 \cup \{\neg A(x_0)\}\times$
$\rightarrow_\exists$	$\mathcal{A}_4 = \mathcal{A}_3 \cup \{r(x_0, x_1), (A \sqcup B)(x_1)\}$	
$\rightarrow_\forall$	$\mathcal{A}_5 = \mathcal{A}_4 \cup \{\neg B(x_1)\}$	
$\rightarrow_\sqcup$	$\mathcal{A}_6 = \mathcal{A}_5 \cup \{(A(x_1)\}$	$\mathcal{A}_{6'} = \mathcal{A}_5 \cup \{B(x_1)\}\times$
$\rightarrow_\mathcal{T}$	$\mathcal{A}_7 = \mathcal{A}_6 \cup \{(\neg A \sqcup \exists r.(A \sqcup B))(x_1)\}$	
$\rightarrow_\sqcup$	$\mathcal{A}_8 = \mathcal{A}_7 \cup \{\exists r.(A \sqcup B)(x_1)\}$	$\mathcal{A}_{8'} = \mathcal{A}_7 \cup \{\neg A(x_1)\}\times$
$\rightarrow_\exists$	$\mathcal{A}_9 = \mathcal{A}_8 \cup \{r(x_1, x_2), (A \sqcup B)(x_2)\}-blocked$	

Since $\mathcal{A}_9$ is complete and clash-free, C_0 is satisfiable, and the initial subsumption inclusion is not valid. We can provide the canonical interpretation $\mathcal{I}_{\mathcal{A}_9}$ as follows:

- $\Delta^{\mathcal{I}_{\mathcal{A}_9}} = \{x_0, x_1\}$
- $A^{\mathcal{I}_{\mathcal{A}_9}} = \{x_0, x_1\}$
- $B^{\mathcal{I}_{\mathcal{A}_9}} = \emptyset$
- $r^{\mathcal{I}_{\mathcal{A}_9}} = \{(x_0, x_1), (x_1, x_1)\}$