Knowledge Representation and Reasoning

Exercises on Well-Founded Semantics

1 Well-Founded Semantics

1. Determine the well-founded model of the following normal logic program:

```
a :- not b.     c :- not a.     g :- not h.     d :- not e.    i :- g.
b :- not a.     c :- not c.     h :- not g.     f :- d.        i :- h.
```

Answer:
To determine the well-founded model of a program P, we need to iterate Γ_P^2, starting from the empty interpretation, until we reach a fixed point. Let T be such fixed point. Then, the well-founded model of P is

$$M = T \cup not \ (H_P - \Gamma_P(T))$$

In this example, if we iterate Γ_P, we obtain:

$$\Gamma_P(\{\}) = \{a, b, c, d, f, g, h, i\}$$
$$\Gamma_P(\{a, b, c, d, f, g, h, i\}) = \Gamma_P^2(\{\}) = \{d, f\}$$
$$\Gamma_P(\{d, f\}) = \{a, b, c, d, f, g, h, i\}$$
$$\Gamma_P(\{a, b, c, d, f, g, h, i\}) = \Gamma_P^2\left(\Gamma_P^2(\{\})\right) = \{d, f\}$$

Therefore, the iteration of the Γ_P^2 results in:

$$\Gamma_P^2(\{\}) = \{d, f\}$$
$$\Gamma_P^2(\{d, f\}) = \{d, f\}$$

so, the least fixed point of Γ_P^2 is $\{d, f\}$. Let T denote this least fixed point, which corresponds to the atoms that are true in the well-founded model.
Now, we need to apply Γ_P to T to obtain the atoms that are true or undefined.

$$\Gamma_P(\{d, f\}) = \{a, b, c, d, f, g, h, i\}$$

Since $H_P = \{a, b, c, d, e, f, g, h, i\}$ (the Herbrand base of P), we can now determine the atoms which are false in the well-founded model:

$$H_P - \Gamma_P(\{d, f\}) = \{e\}$$

We finally obtain the well-founded model:

$$M = T \cup not \ (H_P - \Gamma_P(T)) = \{d, not \ e, f\}$$

2. Determine the well-founded model of the following normal logic program.

```
winning(X) :- move(X,Y), loosing(Y).
loosing(X) :- not winning(X).
move(a,b).    move(b,d).    move(a,c).    move(c,c).
```

Answer:
Iterating Γ_P we obtain (with obvious abbreviations):

$$\begin{aligned}
\Gamma_P(\{\}) &= \{m(a,b), m(b,d), m(a,c), m(c,c), l(a), l(b), l(c), l(d), w(a), w(b), w(c)\} \\
\Gamma_P(\Gamma_P(\{\})) = \Gamma_P^2(\{\}) &= \{m(a,b), m(b,d), m(a,c), m(c,c), l(d), w(b)\} \\
\Gamma_P\left(\Gamma_P^2(\{\})\right) &= \{m(a,b), m(b,d), m(a,c), m(c,c), l(a), l(c), l(d), w(a), w(b), w(c)\} \\
\Gamma_P\left(\Gamma_P\left(\Gamma_P^2(\{\})\right)\right) = \Gamma_P^2\left(\Gamma_P^2(\{\})\right) &= \{m(a,b), m(b,d), m(a,c), m(c,c), l(d), w(b)\}
\end{aligned}$$

so the least fixed point of Γ_P^2 is $\{m(a,b), m(b,d), m(a,c), m(c,c), l(d), w(b)\}$. Let T denote this least fixed-point, which corresponds to the atoms that are true in the well-founded model.
Now, we need to apply Γ_P to T to obtain the atoms that are true or undefined.

$$\Gamma_P(T) = \{m(a,b), m(b,d), m(a,c), m(c,c), l(a), l(c), l(d), w(a), w(b), w(c)\}$$

We can now determine the atoms which are false in the well-founded model:

$$H_P - \Gamma_P(T) = \{l(b), w(d)\}$$

And finally obtain the well-founded model:

$$\begin{aligned}
M = T \cup not\ (H_P - \Gamma_P(T)) = \{&m(a,b), m(b,d), m(a,c), m(c,c), l(d), w(b), not\ l(b), not\ w(d), \\
&not\ m(a,a), not\ m(a,d), not\ m(b,a), not\ m(b,b), not\ m(b,c), not\ m(c,a), not\ m(c,b), not\ m(c,d), \\
&not\ m(d,a), not\ m(d,b), not\ m(d,c), not\ m(d,d)\}
\end{aligned}$$

3. Consider the following taxonomic knowledge expressed by the sentences:
   - Normally, big carnivorous are dangerous.
   - Cats are an exception to the above rule.
   - Felines are carnivorous.
   - Both lions and cats are felines.
   - Lions are big.
   - Normally, tamed animals are not dangerous.
   - King is a tamed lion.
   - Tom is a big cat.

   (a) Represent the previous taxonomic knowledge using extended logic programming.

Answer:

$$\begin{aligned}
dangerous(X) &\leftarrow big(X), carnivorous(X), bcd(X), not\ \neg dangerous(X). \\
bcd(X) &\leftarrow not\ \neg bcd(X). \\
\neg bcd(X) &\leftarrow cat(X). \\
carnivorous(X) &\leftarrow feline(X). \\
feline(X) &\leftarrow cat(X). \\
feline(X) &\leftarrow lion(X). \\
big(X) &\leftarrow lion(X). \\
\neg dangerous(X) &\leftarrow tamed(X), tnd(X), not\ dangerous(X). \\
tnd(X) &\leftarrow not\ \neg tnd(X). \\
&lion(king). \\
&tamed(king). \\
&big(tom). \\
&cat(tom).
\end{aligned}$$

(b) Compute the extended well-founded model and explain what you can conclude regarding Tom and King.

Answer:
To compute the well-founded model, we need to iterate $\Gamma_P \Gamma_{P_S}$, starting from the empty interpretation, until we reach a fixed point. Let T be such fixed point. Then, the well-founded model of P is

$$M = T \cup not \ \left(H_P - \Gamma_{P_S}(T)\right)$$

Recal that Γ_{P_S} operates over the semi-normal version of P, obtained by adding $not\ \neg L$ to every rule of P with head L (where L is a literal, i.e. A or $\neg A$, and $\neg\neg L = L$). You can use a simplified version of the semi-normal program and only modify those rules for literal L such that there exists some rule for $\neg L$). In this example, the simplified semi-normal version is:

$$\begin{aligned}
dangerous(X) &\leftarrow big(X), carnivorous(X), bcd(X), not\ \neg dangerous(X).\\
bcd(X) &\leftarrow not\ \neg bcd(X).\\
\neg bcd(X) &\leftarrow cat(X), not\ bcd(X).\\
carnivorous(X) &\leftarrow feline(X).\\
feline(X) &\leftarrow cat(X).\\
feline(X) &\leftarrow lion(X).\\
big(X) &\leftarrow lion(X).\\
\neg dangerous(X) &\leftarrow tamed(X), tnd(X), not\ dangerous(X).\\
tnd(X) &\leftarrow not\ \neg tnd(X).\\
& lion(king).\\
& tamed(king).\\
& big(tom).\\
& cat(tom).
\end{aligned}$$

The iteration is then (omitting the atoms $lion(king)$, $tamed(king)$, $big(king)$, $big(tom)$, $cat(tom)$, $feline(king)$, $feline(tom)$, $carnivorous(king)$, $carnivorous(tom)$, $tnd(king)$, and $tnd(tom)$ which belong to every iteration):

$$\begin{aligned}
\Gamma_{P_S}(\{\}) &= \left\{ \begin{array}{c} bcd(king), bcd(tom), \neg bcd(tom), \\ dangerous(king), dangerous(tom), \neg dangerous(king) \end{array} \right\}\\
\Gamma_P(\Gamma_{P_S}(\{\})) &= \left\{ \ bcd(king), \neg bcd(tom) \ \right\}\\
\Gamma_{P_S}(\Gamma_P(\Gamma_{P_S}(\{\}))) &= \left\{ \begin{array}{c} bcd(king), \neg bcd(tom), \\ dangerous(king), \neg dangerous(king) \end{array} \right\}\\
\Gamma_P(\Gamma_{P_S}(\Gamma_P(\Gamma_{P_S}(\{\})))) &= \left\{ \ bcd(king), \neg bcd(tom) \ \right\}
\end{aligned}$$

Therefore, the iteration of the $\Gamma_P \Gamma_{P_S}$ results in:

$$\begin{aligned}
\Gamma_P \Gamma_{P_S}(\{\}) &= \left\{ \ bcd(king), \neg bcd(tom) \ \right\}\\
\Gamma_P \Gamma_{P_S}(\Gamma_P \Gamma_{P_S}(\{\})) &= \left\{ \ bcd(king), \neg bcd(tom) \ \right\}
\end{aligned}$$

so, the least fixed point of $\Gamma_P \Gamma_{P_S}$ is

$$\{bcd(king), \neg bcd(tom)\}$$

Let T denote this least fixed-point. Also, we have that

$$\Gamma_{P_S}(T) = \left\{ \begin{array}{c} bcd(king), \neg bcd(tom), \\ dangerous(king), \neg dangerous(king) \end{array} \right\}$$

Since $T \subseteq \Gamma_{P_S}(T)$, this is the well-founded model according to WFSX, i.e.,

$$M = T \cup not\ \left(H_P - \Gamma_{P_S}(T)\right)$$

which, in this case, is (omitting explicitly negated literals not present in the program, such as $not\ \neg lion\,(king)$, $not\ \neg tamed\,(king)$, as well as negations of atoms not present in the program such as $\neg lion(tom)$ or $\neg cat(king)$ etc...):

$$M = \left\{ \begin{array}{c} lion\,(king)\,, tamed\,(king)\,, big\,(king)\,, big\,(tom)\,, cat\,(tom)\,, feline\,(king)\,, \\ feline\,(tom)\,, carnivorous\,(king)\,, carnivorous\,(tom)\,, bcd\,(king)\,, \neg bcd\,(tom)\,, \\ tnd\,(king)\,, tnd\,(tom)\,, not\ \neg tnd\,(king)\,, not\ \neg tnd\,(tom)\,, not\ \neg bcd\,(king)\,, \\ not\ bcd\,(tom)\,, not\ tamed\,(tom)\,, not\ dangerous\,(tom)\,, not\ \neg dangerous\,(tom) \end{array} \right\}$$

According to the model, Tom is not known to be dangerous (because $not\ dangerous\,(tom)$ belongs to the model and $dangerous\,(tom)$ does not). Furthermore, Tom is not known to be (explicitly) not dangerous since $not\ \neg dangerous\,(tom)$ belongs to the model and $\neg dangerous\,(tom)$ does not. In other words, there is no evidence that Tom is dangerous, but there is no certainty that Tom is (explicitly) not dangerous. Our knowledge about King is undefined, since neither $dangerous\,(king)$ and $\neg dangerous\,(king)$ nor their default negations belong to the model.