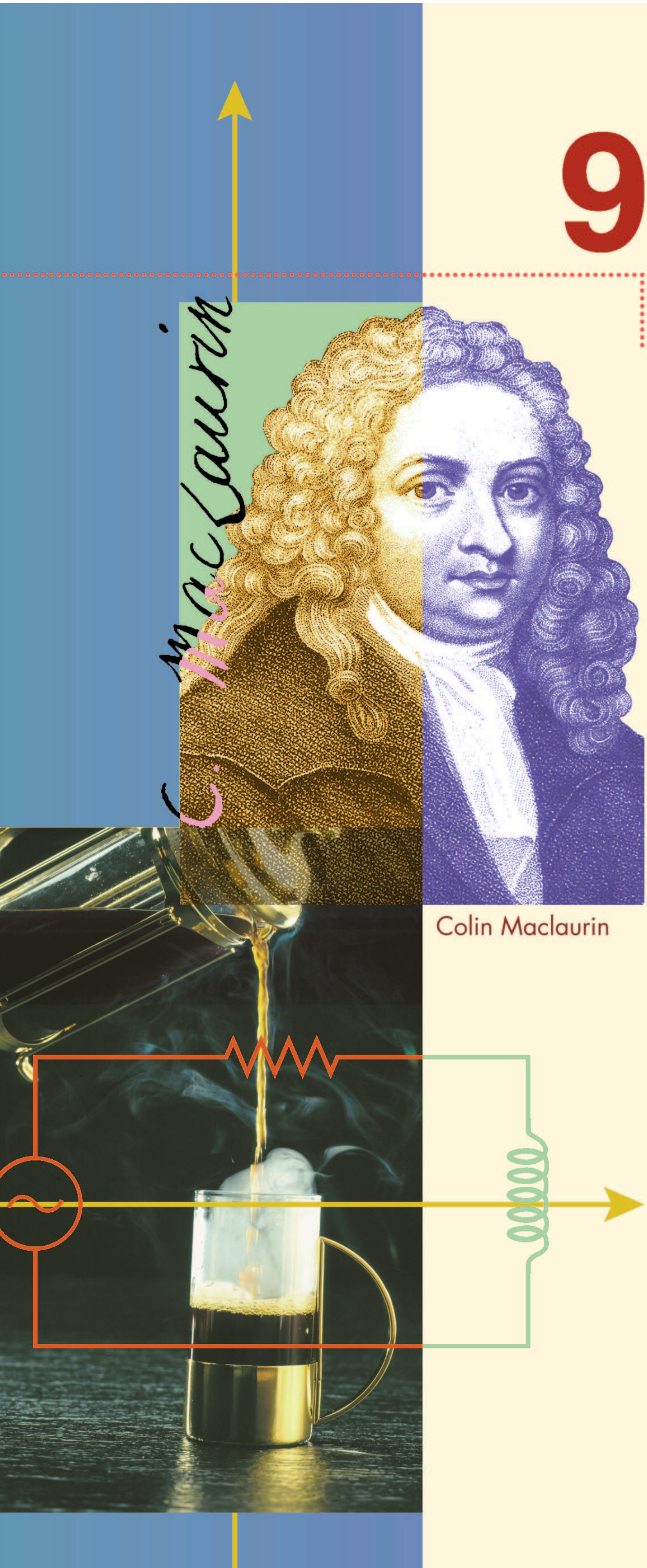


Colin Maclaurin

9

MATHEMATICAL MODELING WITH DIFFERENTIAL EQUATIONS

Many of the principles in science and engineering concern relationships between changing quantities. Since rates of change are represented mathematically by derivatives, it should not be surprising that such principles are often expressed in terms of differential equations. We introduced the concept of a differential equation in Section 5.2, but in this chapter we will go into more detail. We will discuss some important mathematical models that involve differential equations, and we will discuss some methods for solving and approximating solutions of some of the basic types of differential equations. However, we will only be able to touch the surface of this topic, leaving many important topics in differential equations to courses that are devoted completely to the subject.

9.1 FIRST-ORDER DIFFERENTIAL EQUATIONS AND APPLICATIONS

In this section we will introduce some basic terminology and concepts concerning differential equations. We will also discuss methods for solving certain basic types of differential equations, and we will give some applications of our work.

TERMINOLOGY

Recall from Section 5.2 that a ***differential equation*** is an equation involving one or more derivatives of an unknown function. In this section we will denote the unknown function by $y = y(x)$ unless the differential equation arises from an applied problem involving time, in which case we will denote it by $y = y(t)$. The ***order*** of a differential equation is the order of the highest derivative that it contains. Here are some examples:

DIFFERENTIAL EQUATION	ORDER
$\dfrac{dy}{dx} = 3y$	1
$\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 8y = 0$	2
$\dfrac{d^3y}{dx^3} - t\dfrac{dy}{dt} + (t^2 - 1)y = e^t$	3
$y' - y = e^{2x}$	1
$y'' + y' = \cos t$	2

In the last two equations the derivatives of y are expressed in "prime" notation. You will usually be able to tell from the equation itself or the context in which it arises whether to interpret y' as dy/dx or as dy/dt.

SOLUTIONS OF DIFFERENTIAL EQUATIONS

A function $y = y(x)$ is a ***solution*** of a differential equation on an open interval I if the equation is satisfied identically on I when y and its derivatives are substituted into the equation. For example, $y = e^{2x}$ is a solution of the differential equation

$$\frac{dy}{dx} - y = e^{2x} \tag{1}$$

on the interval $I = (-\infty, +\infty)$, since substituting y and its derivative into the left side of this equation yields

$$\frac{dy}{dx} - y = \frac{d}{dx}[e^{2x}] - e^{2x} = 2e^{2x} - e^{2x} = e^{2x}$$

for all real values of x. However, this is not the only solution on I; for example, the function

$$y = Ce^x + e^{2x} \tag{2}$$

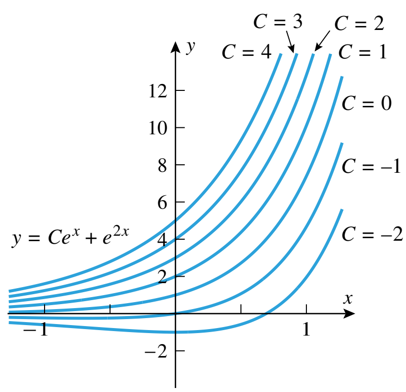
is also a solution for every real value of the constant C, since

$$\frac{dy}{dx} - y = \frac{d}{dx}[Ce^x + e^{2x}] - (Ce^x + e^{2x}) = (Ce^x + 2e^{2x}) - (Ce^x + e^{2x}) = e^{2x}$$

After developing some techniques for solving equations such as (1), we will be able to show that *all* solutions of (1) on $I = (-\infty, +\infty)$ can be obtained by substituting values for the constant C in (2). On a given interval I, a solution of a differential equation from which all solutions on I can be derived by substituting values for arbitrary constants is called a ***general solution*** of the equation on I. Thus (2) is a general solution of (1) on the interval $I = (-\infty, +\infty)$.

REMARK. Usually, the general solution of an nth-order differential equation on an interval will contain n arbitrary constants. Although we will not prove this, it makes sense intuitively because n integrations are needed to recover a function from its nth derivative, and each integration introduces an arbitrary constant. For example, (2) has one arbitrary constant, which is consistent with the fact that it is the general solution of the *first-order* equation (1).

The graph of a solution of a differential equation is called an ***integral curve*** for the equation, so the general solution of a differential equation produces a family of integral curves corresponding to the different possible choices for the arbitrary constants. For example, Figure 9.1.1 shows some integral curves for (1), which were obtained by assigning values to the arbitrary constant in (2).



Integral curves for $\frac{dy}{dx} - y = e^{2x}$

Figure 9.1.1

INITIAL-VALUE PROBLEMS

When an applied problem leads to a differential equation, there are usually conditions in the problem that determine specific values for the arbitrary constants. As a rule of thumb, it requires n conditions to determine values for all n arbitrary constants in the general solution of an nth-order differential equation (one condition for each constant). For a first-order equation, the single arbitrary constant can be determined by specifying the value of the unknown function $y(x)$ at an arbitrary x-value x_0, say $y(x_0) = y_0$. This is called an ***initial condition***, and the problem of solving a first-order equation subject to an initial condition is called a ***first-order initial-value problem***. Geometrically, the initial condition $y(x_0) = y_0$ has the effect of isolating the integral curve that passes through the point (x_0, y_0) from the complete family of integral curves.

Example 1 The solution of the initial-value problem

$$\frac{dy}{dx} - y = e^{2x}, \quad y(0) = 3$$

can be obtained by substituting the initial condition $x = 0$, $y = 3$ in the general solution (2) to find C. We obtain

$$3 = Ce^0 + e^0 = C + 1$$

Thus, $C = 2$, and the solution of the initial-value problem, which is obtained by substituting this value of C in (2), is

$$y = 2e^x + e^{2x}$$

Geometrically, this solution is realized as the integral curve in Figure 9.1.1 that passes through the point (0, 3). ◀

FIRST-ORDER LINEAR EQUATIONS

The simplest first-order equations are those that can be written in the form

$$\frac{dy}{dx} = q(x) \tag{3}$$

Such equations can often be solved by integration. For example, if

$$\frac{dy}{dx} = x^3 \tag{4}$$

then

$$y = \int x^3\,dx = \frac{x^4}{4} + C$$

is the general solution of (4) on the interval $I = (-\infty, +\infty)$. More generally, a first-order differential equation is called ***linear*** if it is expressible in the form

$$\frac{dy}{dx} + p(x)y = q(x) \tag{5}$$

Equation (3) is the special case of (5) that results when the function $p(x)$ is identically 0. Some other examples of first-order linear differential equations are

$$\frac{dy}{dx} + x^2y = e^x, \qquad \frac{dy}{dx} + (\sin x)y + x^3 = 0, \qquad \frac{dy}{dx} + 5y = 2$$

$p(x) = x^2,\ q(x) = e^x$ $\qquad$ $p(x) = \sin x,\ q(x) = -x^3$ $\qquad$ $p(x) = 5,\ q(x) = 2$

Let us assume that the functions $p(x)$ and $q(x)$ are both *continuous* on some common open interval I. We will now describe a procedure for finding a general solution to (5) on I. From the Fundamental Theorem of Calculus (Theorem 5.6.3) it follows that $p(x)$ has an antiderivative $P = P(x)$ on I. That is, there exists a differentiable function $P(x)$ on I such that $dP/dx = p(x)$. Consequently, the function $\mu = \mu(x)$ defined by $\mu = e^{P(x)}$ is differentiable on I with

$$\frac{d\mu}{dx} = \frac{d}{dx}\left(e^{P(x)}\right) = \frac{dP}{dx}e^{P(x)} = \mu p(x)$$

Suppose now that $y = y(x)$ is a solution to (5) on I. Then

$$\frac{d}{dx}(\mu y) = \mu\frac{dy}{dx} + \frac{d\mu}{dx}y = \mu\frac{dy}{dx} + \mu p(x)y = \mu\left(\frac{dy}{dx} + p(x)y\right) = \mu q(x)$$

That is, the function μy is an antiderivative (or integral) of the known function $\mu q(x)$. For this reason, the function $\mu = e^{P(x)}$ is known as an ***integrating factor*** for Equation (5). On the other hand, the function $\mu q(x)$ is continuous on I and therefore possesses an antiderivative $H(x)$. It then follows from Theorem 5.2.2 that $\mu y = H(x) + C$ for some constant C or, equivalently, that

$$y = \frac{1}{\mu}[H(x) + C] \tag{6}$$

Conversely, it is straightforward to check that for any choice of C, Equation (6) defines a solution to (5) on I [Exercise 58(a)]. We conclude that a general solution to (5) on I is given by (6). Since

$$\int \mu q(x)\,dx = H(x) + C$$

this general solution can be expressed as

$$y = \frac{1}{\mu}\int \mu q(x)\,dx \tag{7}$$

We will refer to this process for solving Equation (5) as ***the method of integrating factors***.

Example 2 Solve the differential equation

$$\frac{dy}{dx} - y = e^{2x}$$

Solution. This is a first-order linear differential equation with the functions $p(x) = -1$ and $q(x) = e^{2x}$ that are both continuous on the interval $I = (-\infty, +\infty)$. Thus, we can choose $P(x) = -x$, with $\mu = e^{-x}$, and $\mu q(x) = e^{-x}e^{2x} = e^x$ so that the general solution to this equation on I is given by

$$y = \frac{1}{\mu}\int \mu q(x)\,dx = \frac{1}{e^{-x}}\int e^x\,dx = e^x[e^x + C] = e^{2x} + Ce^x$$

Note that this solution is in agreement with Equation (2) discussed earlier. ◀

It is not necessary to memorize Equation (7) to apply the method of integrating factors; you need only remember the integrating factor $\mu = e^{P(x)}$ and the steps used to derive Equation (7).

Example 3 Solve the initial-value problem

$$x\frac{dy}{dx} - y = x, \quad y(1) = 2$$

Solution. This differential equation can be written in the form of (5) by dividing through by x. This yields

$$\frac{dy}{dx} - \frac{1}{x}y = 1 \tag{8}$$

where $q(x) = 1$ is continuous on $(-\infty, +\infty)$ and $p(x) = -1/x$ is continuous on $(-\infty, 0)$ and $(0, +\infty)$. Since we need $p(x)$ and $q(x)$ to be continuous on a common interval, and since our initial condition presumes a solution for $x = 1$, we will find the general solution of (8) on the interval $(0, +\infty)$. On this interval

$$\int \frac{1}{x}\,dx = \ln x + C$$

so that we can take $P(x) = -\ln x$ with $\mu = e^{P(x)} = e^{-\ln x} = 1/x$ the corresponding integrating factor. Multiplying both sides of Equation (8) by this integrating factor yields

$$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = \frac{1}{x}$$

or

$$\frac{d}{dx}\left[\frac{1}{x}y\right] = \frac{1}{x}$$

Therefore, on the interval $(0, +\infty)$,

$$\frac{1}{x}y = \int \frac{1}{x}\,dx = \ln x + C$$

from which it follows that

$$y = x\ln x + Cx \tag{9}$$

The initial condition $y(1) = 2$ requires that $y = 2$ if $x = 1$. Substituting these values into (9) and solving for C yields $C = 2$ (verify), so the solution of the initial-value problem is

$$y = x\ln x + 2x$$ ◀

The result of Example 3 illustrates an important property of first-order linear initial-value problems: Given any x_0 in I and any value of y_0, there will always *exist* a solution $y = y(x)$ to (5) on I with $y(x_0) = y_0$; furthermore, this solution will be *unique* [Exercise 58(b)]. Such existence and uniqueness results need not hold for nonlinear equations (Exercise 60).

FIRST-ORDER SEPARABLE EQUATIONS

Solving a first-order linear differential equation involves only the integration of functions of x. We will now consider a collection of equations whose solutions require integration of functions of y as well. A first-order ***separable*** differential equation is one that can be written in the form

$$h(y)\frac{dy}{dx} = g(x) \tag{10}$$

For example, the equation

$$(4y - \cos y)\frac{dy}{dx} = 3x^2$$

is a separable equation with

$$h(y) = 4y - \cos y \quad \text{and} \quad g(x) = 3x^2$$

We will assume that the functions $h(y)$ and $g(x)$ both possess antiderivatives in their respective variables y and x. That is, there exists a differentiable function $H(y)$ with $dH/dy = h(y)$ and there exists a differentiable function $G(x)$ with $dG/dx = g(x)$.

Suppose now that $y = y(x)$ is a solution to (10) on an open interval I. Then it follows from the chain rule that

$$\frac{d}{dx}[H(y)] = \frac{dH}{dy}\frac{dy}{dx} = h(y)\frac{dy}{dx} = g(x)$$

In other words, the function $H(y(x))$ is an antiderivative of $g(x)$ on the interval I. By Theorem 5.2.2, there must exist a constant C such that $H(y(x)) = G(x) + C$ on I. Equivalently, the solution $y = y(x)$ to (10) is defined *implicitly* by the equation

$$H(y) = G(x) + C \tag{11}$$

Conversely, suppose that for some choice of C a differentiable function $y = y(x)$ is defined implicitly by (11), Then $y(x)$ will be a solution to (10) (Exercise 59). We conclude that every solution to (10) will be given implicitly by Equation (11) for some appropriate choice of C.

We can express Equation (11) symbolically by writing

$$\int h(y)\,dy = \int g(x)\,dx \tag{12}$$

Informally, we first "multiply" both sides of Equation (10) by dx to "separate" the variables into the equation $h(y)\,dy = g(x)\,dx$. Integrating both sides of this equation then gives Equation (12). This process is called the method of ***separation of variables***. Although separation of variables provides us with a convenient way of recovering Equation (11), it must be interpreted with care. For example, the constant C in Equation (11) is often *not* arbitrary; some choices of C may yield solutions, and others may not. Furthermore, even when solutions do exist, their domains can vary in unexpected ways with the choice of C. It is for reasons such as these that we will not refer to a "general" solution of a separable equation.

In some cases Equation (11) can be solved to yield explicit solutions to (10).

Example 4 Solve the differential equation

$$\frac{dy}{dx} = -4xy^2$$

and then solve the initial-value problem

$$\frac{dy}{dx} = -4xy^2, \quad y(0) = 1$$

Solution. For $y \neq 0$ we can write this equation in the form of (10) as

$$\frac{1}{y^2}\frac{dy}{dx} = -4x$$

Separating variables and integrating yields

$$\frac{1}{y^2}\,dy = -4x\,dx$$

$$\int \frac{1}{y^2}\,dy = \int -4x\,dx$$

which is a symbolic expression of the equation

$$-\frac{1}{y} = -2x^2 + C$$

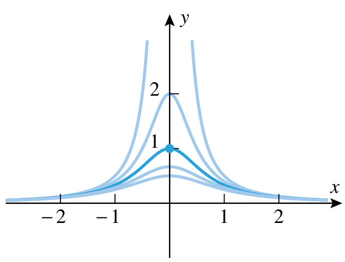
Solving for y as a function of x, we obtain

$$y = \frac{1}{2x^2 - C}$$

The initial condition $y(0) = 1$ requires that $y = 1$ when $x = 0$. Substituting these values into our solution yields $C = -1$ (verify). Thus, a solution to the initial-value problem is

$$y = \frac{1}{2x^2 + 1}$$

Some integral curves and our solution of the initial-value problem are graphed in Figure 9.1.2. ◀



Integral curves for $\frac{dy}{dx} = -4xy^2$

Figure 9.1.2

One aspect of our solution to Example 4 deserves special comment. Had the initial condition been $y(0) = 0$ instead of $y(0) = 1$, the method we used would have failed to yield a solution to the resulting initial-value problem (Exercise 39). This is due to the fact that we assumed $y \neq 0$ in order to rewrite the equation $dy/dx = -4xy^2$ in the form

$$\frac{1}{y^2}\frac{dy}{dx} = -4x$$

It is important to be aware of such assumptions when manipulating a differential equation algebraically.

As a second example, consider the first-order linear equation $dy/dx - 3y = 0$. Using the method of integrating factors, it is easy to see that the general solution of this equation is $y = Ce^{3x}$ (verify). On the other hand, we can also apply the method of separation of variables to this differential equation. For $y \neq 0$ the equation can be written in the form

$$\frac{1}{y}\frac{dy}{dx} = 3$$

Separating the variables and integrating yields

$$\int \frac{dy}{y} = \int 3\,dx$$

$$\ln|y| = 3x + c$$

$$|y| = e^{3x+c} = e^c e^{3x}$$

We have used c as the constant of integration here to reserve C for the constant in the final result.

$$y = \pm e^c e^{3x} = Ce^{3x}$$

Letting $C = \pm e^c$

This appears to be the same solution that we obtained using the method of integrating factors. However, the careful reader may have observed that the constant $C = \pm e^c$ is not truly arbitrary, since $C = 0$ is not an allowable value. Thus, separation of variables missed the solution $y = 0$, which the method of integrating factors did not. The problem occurred because we had to divide by y to separate the variables. (Exercises 7 and 8 ask you to compare the two methods with some other first-order linear equations.)

It is often not possible to solve Equation (11) for y as an explicit function of x. In such cases, it is common to refer to Equation (11) as a "solution" to (10).

Example 5 Solve the initial-value problem

$$(4y - \cos y)\frac{dy}{dx} - 3x^2 = 0, \quad y(0) = 0$$

Solution. We can write this equation in the form of (10) as

$$(4y - \cos y)\frac{dy}{dx} = 3x^2$$

Separating variables and integrating yields

$$(4y - \cos y)\, dy = 3x^2\, dx$$

$$\int (4y - \cos y)\, dy = \int 3x^2\, dx$$

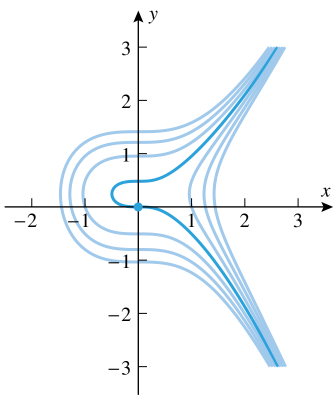
which is a symbolic expression of the equation

$$2y^2 - \sin y = x^3 + C \tag{13}$$

Equation (13) defines solutions of the differential equation implicitly; it cannot be solved explicitly for y as a function of x.

For the initial-value problem, the initial condition $y(0) = 0$ requires that $y = 0$ if $x = 0$. Substituting these values into (13) to determine the constant of integration yields $C = 0$ (verify). Thus, the solution of the initial-value problem is

$$2y^2 - \sin y = x^3$$ ◀



Integral curves for $(4y - \cos y)\frac{dy}{dx} - 3x^2 = 0$

Figure 9.1.3

FOR THE READER. Some computer algebra systems can graph implicit equations. For example, Figure 9.1.3 shows the graphs of (13) for $C = 0, \pm 1, \pm 2$, and ± 3, with emphasis on the solution of the initial-value problem. If you have a CAS that can graph implicit equations, read the documentation on graphing them and try to duplicate this figure. Also, try to determine which values of C produce which curves.

APPLICATIONS IN GEOMETRY

We conclude this section with some applications of first-order differential equations.

Example 6 Find a curve in the xy-plane that passes through (0, 3) and whose tangent line at a point (x, y) has slope $2x/y^2$.

Solution. Since the slope of the tangent line is dy/dx, we have

$$\frac{dy}{dx} = \frac{2x}{y^2} \tag{14}$$

and, since the curve passes through (0, 3), we have the initial condition

$$y(0) = 3 \tag{15}$$

Equation (14) is separable and can be written as

$$y^2\, dy = 2x\, dx$$

so

$$\int y^2\, dy = \int 2x\, dx \quad \text{or} \quad \tfrac{1}{3}y^3 = x^2 + C$$

It follows from the initial condition (15) that $y = 3$ if $x = 0$. Substituting these values into the last equation yields $C = 9$ (verify), so the equation of the desired curve is

$$\tfrac{1}{3}y^3 = x^2 + 9 \quad \text{or} \quad y = (3x^2 + 27)^{1/3}$$ ◀

MIXING PROBLEMS

In a typical mixing problem, a tank is filled to a specified level with a solution that contains a known amount of some soluble substance (say salt). The thoroughly stirred solution is allowed to drain from the tank at a known rate, and at the same time a solution with a known concentration of the soluble substance is added to the tank at a known rate that may or may not differ from the draining rate. As time progresses, the amount of the soluble substance in the tank will generally change, and the usual mixing problem seeks to determine the amount of the substance in the tank at a specified time. This type of problem serves as a model for many kinds of problems: discharge and filtration of pollutants in a river, injection and absorption of medication in the bloodstream, and migrations of species into and out of an ecological system, for example.

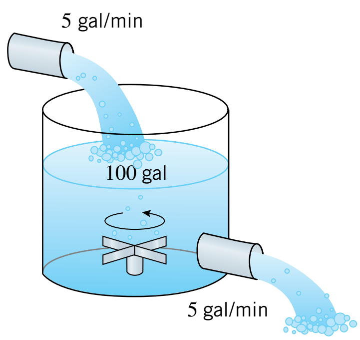


Figure 9.1.4

Example 7 At time $t = 0$, a tank contains 4 lb of salt dissolved in 100 gal of water. Suppose that brine containing 2 lb of salt per gallon of brine is allowed to enter the tank at a rate of 5 gal/min and that the mixed solution is drained from the tank at the same rate (Figure 9.1.4). Find the amount of salt in the tank after 10 minutes.

Solution. Let $y(t)$ be the amount of salt (in pounds) after t minutes. We are given that $y(0) = 4$, and we want to find $y(10)$. We will begin by finding a differential equation that is satisfied by $y(t)$. To do this, observe that dy/dt, which is the rate at which the amount of salt in the tank changes with time, can be expressed as

$$\frac{dy}{dt} = \text{rate in} - \text{rate out} \tag{16}$$

where *rate in* is the rate at which salt enters the tank and *rate out* is the rate at which salt leaves the tank. But the rate at which salt enters the tank is

$$\text{rate in} = (2 \text{ lb/gal}) \cdot (5 \text{ gal/min}) = 10 \text{ lb/min}$$

Since brine enters and drains from the tank at the same rate, the volume of brine in the tank stays constant at 100 gal. Thus, after t minutes have elapsed, the tank contains $y(t)$ lb of salt per 100 gal of brine, and hence the rate at which salt leaves the tank at that instant is

$$\text{rate out} = \left(\frac{y(t)}{100} \text{ lb/gal}\right) \cdot (5 \text{ gal/min}) = \frac{y(t)}{20} \text{ lb/min}$$

Therefore, (16) can be written as

$$\frac{dy}{dt} = 10 - \frac{y}{20} \quad \text{or} \quad \frac{dy}{dt} + \frac{y}{20} = 10$$

which is a first-order linear differential equation satisfied by $y(t)$. Since we are given that $y(0) = 4$, the function $y(t)$ can be obtained by solving the initial-value problem

$$\frac{dy}{dt} + \frac{y}{20} = 10, \quad y(0) = 4$$

The integrating factor for the differential equation is

$$\mu = e^{t/20}$$

If we multiply the differential equation through by μ, then we obtain

$$\frac{d}{dt}(e^{t/20}y) = 10e^{t/20}$$

$$e^{t/20}y = \int 10e^{t/20}dt = 200e^{t/20} + C$$

$$y(t) = 200 + Ce^{-t/20} \tag{17}$$

The initial condition states that $y = 4$ when $t = 0$. Substituting these values into (17) and solving for C yields $C = -196$ (verify), so

$$y(t) = 200 - 196e^{-t/20} \tag{18}$$

Thus, at time $t = 10$ the amount of salt in the tank is

$$y(10) = 200 - 196e^{-0.5} \approx 81.1 \text{ lb}$$ ◀

FOR THE READER. Figure 9.1.5 shows the graph of (18). Observe that $y(t) \to 200$ as $t \to +\infty$, which means that over an extended period of time the amount of salt in the tank tends toward 200 lb. Give an informal physical argument to explain why this result is to be expected.

A MODEL OF FREE-FALL MOTION RETARDED BY AIR RESISTANCE

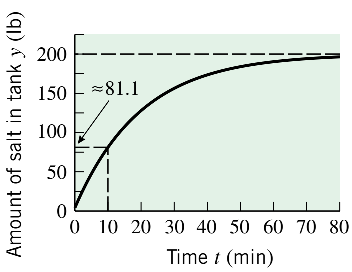


Figure 9.1.5

In Section 4.4 we considered the free-fall model of an object moving along a vertical axis near the surface of the Earth. It was assumed in that model that there is no air resistance and that the only force acting on the object is the Earth's gravity. Our goal here is to find a model that takes air resistance into account. For this purpose we make the following assumptions:

- The object moves along a vertical s-axis whose origin is at the surface of the Earth and whose positive direction is up (Figure 4.4.8).
- At time $t = 0$ the height of the object is s_0 and the velocity is v_0.
- The only forces on the object are the force $F_G = -mg$ of the Earth's gravity acting down and the force F_R of air resistance acting opposite to the direction of motion. The force F_R is called the ***drag force***.

We will also need the following result from physics:

9.1.1 NEWTON'S SECOND LAW OF MOTION. If an object with mass m is subjected to a force F, then the object undergoes an acceleration a that satisfies the equation

$$F = ma \tag{19}$$

In the case of free-fall motion retarded by air resistance, the net force acting on the object is

$$F_G + F_R = -mg + F_R$$

and the acceleration is d^2s/dt^2, so Newton's second law implies that

$$-mg + F_R = m\frac{d^2s}{dt^2} \tag{20}$$

Experimentation has shown that the force F_R of air resistance depends on the shape of the object and its speed—the greater the speed, the greater the drag force. There are many possible models for air resistance, but one of the most basic assumes that the drag force F_R is proportional to the velocity of the object, that is,

$$F_R = -cv$$

where c is a positive constant that depends on the object's shape and properties of the air.* (The minus sign ensures that the drag force is opposite to the direction of motion.) Substituting this in (20) and writing d^2s/dt^2 as dv/dt, we obtain

$$-mg - cv = m\frac{dv}{dt}$$

or on dividing by m and rearranging we obtain

$$\frac{dv}{dt} + \frac{c}{m}v = -g$$

which is a first-order linear differential equation in the unknown function $v = v(t)$ with $p(t) = c/m$ and $q(t) = -g$ [see (5)]. For a specific object, the coefficient c can be determined experimentally, so we can assume that m, g, and c are known constants. Thus,

*Other common models assume that $F_R = -cv^2$ or, more generally, $F_R = -cv^p$ for some value of p.

the velocity function $v = v(t)$ can be obtained by solving the initial-value problem

$$\frac{dv}{dt} + \frac{c}{m}v = -g, \quad v(0) = v_0 \tag{21}$$

Once the velocity function is found, the position function $s = s(t)$ can be obtained by solving the initial-value problem

$$\frac{ds}{dt} = v(t), \quad s(0) = s_0 \tag{22}$$

In Exercise 47 we will ask you to solve (21) and show that

$$v(t) = e^{-ct/m}\left(v_0 + \frac{mg}{c}\right) - \frac{mg}{c} \tag{23}$$

Note that

$$\lim_{t\to+\infty} v(t) = -\frac{mg}{c} \tag{24}$$

(verify). Thus, the speed $|v(t)|$ does not increase indefinitely, as in free fall; rather, because of the air resistance, it approaches a finite limiting speed v_τ given by

$$v_\tau = \left|-\frac{mg}{c}\right| = \frac{mg}{c} \tag{25}$$

This is called the ***terminal speed*** of the object, and (24) is called its ***terminal velocity***.

REMARK. Intuition suggests that near the limiting velocity, the velocity $v(t)$ changes very slowly; that is, $dv/dt \approx 0$. Thus, it should not be surprising that the limiting velocity can be obtained informally from (21) by setting $dv/dt = 0$ in the differential equation and solving for v. This yields

$$v = -\frac{mg}{c}$$

which agrees with (24).

EXERCISE SET 9.1 Graphing Utility C CAS

1. Confirm that $y = 2e^{x^3/3}$ is a solution of the initial-value problem $y' = x^2y$, $y(0) = 2$.
2. Confirm that $y = \frac{1}{4}x^4 + 2\cos x + 1$ is a solution of the initial-value problem $y' = x^3 - 2\sin x$, $y(0) = 3$.

In Exercises 3 and 4, state the order of the differential equation, and confirm that the functions in the given family are solutions.

3. (a) $(1+x)\dfrac{dy}{dx} = y;\ \ y = c(1+x)$
   (b) $y'' + y = 0;\ \ y = c_1\sin t + c_2\cos t$
4. (a) $2\dfrac{dy}{dx} + y = x - 1;\ \ y = ce^{-x/2} + x - 3$
   (b) $y'' - y = 0;\ \ y = c_1e^t + c_2e^{-t}$

In Exercises 5 and 6, use implicit differentiation to confirm that the equation defines implicit solutions of the differential equation.

5. $\ln y = xy + C;\ \ \dfrac{dy}{dx} = \dfrac{y^2}{1-xy}$
6. $x^2 + xy^2 = C;\ \ 2x + y^2 + 2xy\dfrac{dy}{dx} = 0$

The first-order linear equations in Exercises 7 and 8 can be rewritten as first-order separable equations. Solve the equations using both the method of integrating factors and the method of separation of variables, and determine whether the solutions produced are the same.

7. (a) $\dfrac{dy}{dx} + 3y = 0$ (b) $\dfrac{dy}{dt} - 2y = 0$
8. (a) $\dfrac{dy}{dx} - 4xy = 0$ (b) $\dfrac{dy}{dt} + y = 0$

In Exercises 9–14, solve the differential equation by the method of integrating factors.

9. $\dfrac{dy}{dx} + 3y = e^{-2x}$

10. $\dfrac{dy}{dx} + 2xy = x$

11. $y' + y = \cos(e^x)$

12. $2\dfrac{dy}{dx} + 4y = 1$

13. $(x^2+1)\dfrac{dy}{dx} + xy = 0$

14. $\dfrac{dy}{dx} + y - \dfrac{1}{1+e^x} = 0$

In Exercises 15–24, solve the differential equation by separation of variables. Where reasonable, express the family of solutions as explicit functions of x.

15. $\dfrac{dy}{dx} = \dfrac{y}{x}$

16. $\dfrac{dy}{dx} = (1+y^2)x^2$

17. $\dfrac{\sqrt{1+x^2}}{1+y}\dfrac{dy}{dx} = -x$

18. $(1+x^4)\dfrac{dy}{dx} = \dfrac{x^3}{y}$

19. $(1+y^2)y' = e^x y$

20. $y' = -xy$

21. $e^{-y}\sin x - y'\cos^2 x = 0$

22. $y' - (1+x)(1+y^2) = 0$

23. $\dfrac{dy}{dx} - \dfrac{y^2 - y}{\sin x} = 0$

24. $3\tan y - \dfrac{dy}{dx}\sec x = 0$

25. In each part, find the solution of the differential equation

$$x\frac{dy}{dx} + y = x$$

that satisfies the initial condition.

(a) $y(1) = 2$ (b) $y(-1) = 2$

26. In each part, find the solution of the differential equation

$$\frac{dy}{dx} = xy$$

that satisfies the initial condition.

(a) $y(0) = 1$ (b) $y(0) = \frac{1}{2}$

In Exercises 27–32, solve the initial-value problem by any method.

27. $\dfrac{dy}{dx} - xy = x, \quad y(0) = 3$

28. $\dfrac{dy}{dt} + y = 2, \quad y(0) = 1$

29. $y' = \dfrac{4x^2}{y + \cos y}, \quad y(1) = \pi$

30. $y' - xe^y = 2e^y, \quad y(0) = 0$

31. $\dfrac{dy}{dt} = \dfrac{2t+1}{2y-2}, \quad y(0) = -1$

32. $y'\cosh x + y\sinh x = \cosh^2 x, \quad y(0) = \frac{1}{4}$

33. (a) Sketch some typical integral curves of the differential equation $y' = y/2x$.
(b) Find an equation for the integral curve that passes through the point $(2, 1)$.

34. (a) Sketch some typical integral curves of the differential equation $y' = -x/y$.
(b) Find an equation for the integral curve that passes through the point $(3, 4)$.

In Exercises 35 and 36, solve the differential equation, and then use a graphing utility to generate five integral curves for the equation.

35. $(x^2+4)\dfrac{dy}{dx} + xy = 0$

36. $y' + 2y - 3e^t = 0$

If you have a CAS that can graph implicit equations, solve the differential equations in Exercises 37 and 38, and then use the CAS to generate five integral curves for the equation.

C **37.** $y' = \dfrac{x^2}{1-y^2}$

C **38.** $y' = \dfrac{y}{1+y^2}$

39. Suppose that the initial condition in Example 4 had been $y(0) = 0$. Show that none of the solutions generated in Example 4 satisfy this initial condition, and then solve the initial-value problem

$$\frac{dy}{dx} = -4xy^2, \quad y(0) = 0$$

Why does the method of Example 4 fail to produce this particular solution?

40. Find all ordered pairs (x_0, y_0) such that if the initial condition in Example 4 is replaced by $y(x_0) = y_0$, the solution of the resulting initial-value problem is defined for all real numbers.

41. Find an equation of a curve with x-intercept 2 whose tangent line at any point (x, y) has slope xe^y.

42. Use a graphing utility to generate a curve that passes through the point $(1, 1)$ and whose tangent line at (x, y) is perpendicular to the line through (x, y) with slope $-2y/(3x^2)$.

43. At time $t = 0$, a tank contains 25 ounces of salt dissolved in 50 gal of water. Then brine containing 4 ounces of salt per gallon of brine is allowed to enter the tank at a rate of 2 gal/min and the mixed solution is drained from the tank at the same rate.
(a) How much salt is in the tank at an arbitrary time t?
(b) How much salt is in the tank after 25 min?

44. A tank initially contains 200 gal of pure water. Then at time $t = 0$ brine containing 5 lb of salt per gallon of brine is allowed to enter the tank at a rate of 10 gal/min and the mixed solution is drained from the tank at the same rate.
(a) How much salt is in the tank at an arbitrary time t?
(b) How much salt is in the tank after 30 min?

45. A tank with a 1000-gal capacity initially contains 500 gal of water that is polluted with 50 lb of particulate matter. At time $t = 0$, pure water is added at a rate of 20 gal/min and the mixed solution is drained off at a rate of 10 gal/min. How much particulate matter is in the tank when it reaches the point of overflowing?

46. The water in a polluted lake initially contains 1 lb of mercury salts per 100,000 gal of water. The lake is circular with diameter 30 m and uniform depth 3 m. Polluted water is pumped from the lake at a rate of 1000 gal/h and is replaced with fresh water at the same rate. Construct a table that shows the amount of mercury in the lake (in lb) at the end of each hour over a 12-hour period. Discuss any assumptions you made. [Use 264 gal/m^3.]

47. (a) Use the method of integrating factors to derive solution (23) to the initial-value problem (21). [*Note:* Keep in mind that c, m, and g are constants.]
(b) Show that (23) can be expressed in terms of the terminal speed (25) as

$$v(t) = e^{-gt/v_\tau}(v_0 + v_\tau) - v_\tau$$

(c) Show that if $s(0) = s_0$, then the position function of the object can be expressed as

$$s(t) = s_0 - v_\tau t + \frac{v_\tau}{g}(v_0 + v_\tau)(1 - e^{-gt/v_\tau})$$

48. Suppose a fully equipped sky diver weighing 240 lb has a terminal speed of 120 ft/s with a closed parachute and 24 ft/s with an open parachute. Suppose further that this sky diver is dropped from an airplane at an altitude of 10,000 ft, falls for 25 s with a closed parachute, and then falls the rest of the way with an open parachute.
(a) Assuming that the sky diver's initial vertical velocity is zero, use Exercise 47 to find the sky diver's vertical velocity and height at the time the parachute opens. [Take $g = 32$ ft/s^2.]
(b) Use a calculating utility to find a numerical solution for the total time that the sky diver is in the air.

49. The accompanying figure is a schematic diagram of a basic RL series electrical circuit that contains a power source with a time-dependent voltage of $V(t)$ volts (V), a resistor with a constant resistance of R ohms (Ω), and an inductor with a constant inductance of L henrys (H). If you don't know anything about electrical circuits, don't worry; all you need to know is that electrical theory states that a current of $I(t)$ amperes (A) flows through the circuit where $I(t)$ satisfies the differential equation

$$L\frac{dI}{dt} + RI = V(t)$$

(a) Find $I(t)$ if $R = 10\,\Omega$, $L = 4$ H, V is a constant 12 V, and $I(0) = 0$ A.
(b) What happens to the current over a long period of time?

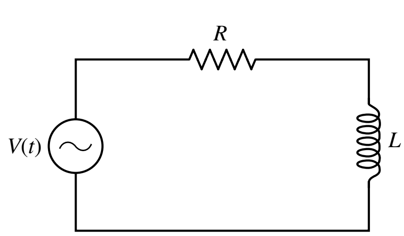


Figure Ex-49

50. Find $I(t)$ for the electrical circuit in Exercise 49 if $R = 6\,\Omega$, $L = 3$ H, $V(t) = 3\sin t$ V, and $I(0) = 15$ A.

51. A rocket, fired upward from rest at time $t = 0$, has an initial mass of m_0 (including its fuel). Assuming that the fuel is consumed at a constant rate k, the mass m of the rocket, while fuel is being burned, will be given by $m = m_0 - kt$. It can be shown that if air resistance is neglected and the fuel gases are expelled at a constant speed c relative to the rocket, then the velocity v of the rocket will satisfy the equation

$$m\frac{dv}{dt} = ck - mg$$

where g is the acceleration due to gravity.
(a) Find $v(t)$ keeping in mind that the mass m is a function of t.
(b) Suppose that the fuel accounts for 80% of the initial mass of the rocket and that all of the fuel is consumed in 100 s. Find the velocity of the rocket in meters per second at the instant the fuel is exhausted. [Take $g = 9.8$ m/s^2 and $c = 2500$ m/s.]

52. A bullet of mass m, fired straight up with an initial velocity of v_0, is slowed by the force of gravity and a drag force of air resistance kv^2, where g is the constant acceleration due to gravity and k is a positive constant. As the bullet moves upward, its velocity v satisfies the equation

$$m\frac{dv}{dt} = -(kv^2 + mg)$$

(a) Show that if $x = x(t)$ is the height of the bullet above the barrel opening at time t, then

$$mv\frac{dv}{dx} = -(kv^2 + mg)$$

(b) Express x in terms of v given that $x = 0$ when $v = v_0$.
(c) Assuming that

$$v_0 = 988 \text{ m/s}, \quad g = 9.8 \text{ m/s}^2$$
$$m = 3.56 \times 10^{-3} \text{ kg}, \quad k = 7.3 \times 10^{-6} \text{ kg/m}$$

use the result in part (b) to find out how high the bullet rises. [*Hint:* Find the velocity of the bullet at its highest point.]

The following discussion is needed for Exercises 53 and 54. Suppose that a tank containing a liquid is vented to the air at the top and has an outlet at the bottom through which the liquid can drain. It follows from ***Torricelli's law*** in physics that if the outlet is opened at time $t = 0$, then at each instant the depth of the liquid $h(t)$ and the area $A(h)$ of the liquid's surface are related by

$$A(h)\frac{dh}{dt} = -k\sqrt{h}$$

where k is a positive constant that depends on such factors as the viscosity of the liquid and the cross-sectional area of the outlet. Use this result in Exercises 53 and 54, assuming that h is in feet, $A(h)$ is in square feet, and t is in seconds.

53. Suppose that the cylindrical tank in the accompanying figure is filled to a depth of 4 feet at time $t = 0$ and that the constant in Torricelli's law is $k = 0.025$.
(a) Find $h(t)$.
(b) How many minutes will it take for the tank to drain completely?

54. Follow the directions of Exercise 53 for the cylindrical tank in the accompanying figure, assuming that the tank is filled to a depth of 4 feet at time $t = 0$ and that the constant in Torricelli's law is $k = 0.025$.

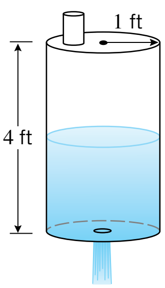


Figure Ex-53

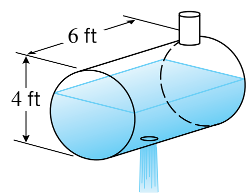


Figure Ex-54

55. Suppose that a particle moving along the x-axis encounters a resisting force that results in an acceleration of $a = dv/dt = -0.04v^2$. Given that $x = 0$ cm and $v = 50$ cm/s at time $t = 0$, find the velocity v and position x as a function of t for $t \geq 0$.

56. Suppose that a particle moving along the x-axis encounters a resisting force that results in an acceleration of $a = dv/dt = -0.02\sqrt{v}$. Given that $x = 0$ cm and $v = 9$ cm/s at time $t = 0$, find the velocity v and position x as a function of t for $t \geq 0$.

57. Find an initial-value problem whose solution is

$$y = \cos x + \int_0^x e^{-t^2}\,dt$$

58. (a) Prove that if C is an arbitrary constant, then any function $y = y(x)$ defined by Equation (6) will be a solution to (5) on the interval I.
(b) Consider the initial-value problem

$$\frac{dy}{dx} + p(x)y = q(x), \quad y(x_0) = y_0$$

where the functions $p(x)$ and $q(x)$ are both continuous on some open interval I. Using the general solution for a first-order linear equation, prove that this initial-value problem has a unique solution on I.

59. Use implicit differentiation to prove that any differentiable function defined implicitly by Equation (11) will be a solution to (10).

60. (a) Prove that solutions need not be unique for nonlinear initial-value problems by finding two solutions to

$$y\frac{dy}{dx} = x, \quad y(0) = 0$$

(b) Prove that solutions need not exist for nonlinear initial-value problems by showing that there is no solution for

$$y\frac{dy}{dx} = -x, \quad y(0) = 0$$

61. In our derivation of Equation (6) we did not consider the possibility of a solution $y = y(x)$ to (5) that was defined on an open subset $I_1 \subseteq I$, $I_1 \neq I$. Prove that there was no loss of generality in our analysis by showing that any such solution must extend to a solution to (5) on the entire interval I.

9.2 DIRECTION FIELDS; EULER'S METHOD

In this section we will reexamine the concept of a direction field, and we will discuss a method for approximating solutions of first-order equations numerically. Numerical approximations are important in cases where the differential equation cannot be solved exactly.

FUNCTIONS OF TWO VARIABLES

We will be concerned here with first-order equations that are expressed with the derivative by itself on one side of the equation. For example,

$$y' = x^3 \quad \text{and} \quad y' = \sin(xy)$$

The first of these equations involves only x on the right side, so it has the form $y' = f(x)$. However, the second equation involves both x and y on the right side, so it has the form $y' = f(x, y)$, where the symbol $f(x, y)$ stands for a function of the two variables x and y. Later in the text we will study functions of two variables in more depth, but for now it will suffice to think of $f(x, y)$ as a formula that produces a unique output when values of x and

y are given as inputs. For example, if

$$f(x, y) = x^2 + 3y$$

and if the inputs are $x = 2$ and $y = -4$, then the output is

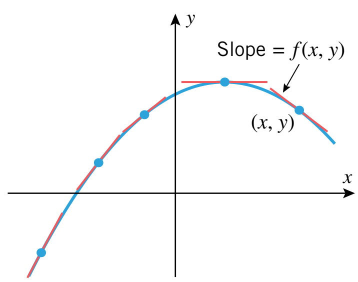
$$f(2, -4) = 2^2 + 3(-4) = 4 - 12 = -8$$

REMARK. In applied problems involving time, it is usual to use t as the independent variable, in which case we would be concerned with equations of the form $y' = f(t, y)$, where $y' = dy/dt$.

DIRECTION FIELDS

In Section 5.2 we introduced the concept of a direction field in the context of differential equations of the form $y' = f(x)$; the same principles apply to differential equations of the form

$$y' = f(x, y)$$



At each point (x, y) on an integral curve of $y' = f(x, y)$, the tangent line has slope $f(x, y)$.

Figure 9.2.1

To see why this is so, let us review the basic idea. If we interpret y' as the slope of a tangent line, then the differential equation states that at each point (x, y) on an integral curve, the slope of the tangent line is equal to the value of f at that point (Figure 9.2.1). For example, suppose that $f(x, y) = y - x$, in which case we have the differential equation

$$y' = y - x \tag{1}$$

A geometric description of the set of integral curves can be obtained by choosing a rectangular grid of points in the xy-plane, calculating the slopes of the tangent lines to the integral curves at the gridpoints, and drawing small segments of the tangent lines at those points. The resulting picture is called a ***direction field*** or a ***slope field*** for the differential equation because it shows the "direction" or "slope" of the integral curves at the gridpoints. The more gridpoints that are used, the better the description of the integral curves. For example, Figure 9.2.2 shows two direction fields for (1)—the first was obtained by hand calculation using the 49 gridpoints shown in the accompanying table, and the second, which gives a clearer picture of the integral curves, was obtained using 625 gridpoints and a CAS.

VALUES OF $f(x, y) = y - x$

	$y = -3$	$y = -2$	$y = -1$	$y = 0$	$y = 1$	$y = 2$	$y = 3$
$x = -3$	0	1	2	3	4	5	6
$x = -2$	–1	0	1	2	3	4	5
$x = -1$	–2	–1	0	1	2	3	4
$x = 0$	–3	–2	–1	0	1	2	3
$x = 1$	–4	–3	–2	–1	0	1	2
$x = 2$	–5	–4	–3	–2	–1	0	1
$x = 3$	–6	–5	–4	–3	–2	–1	0

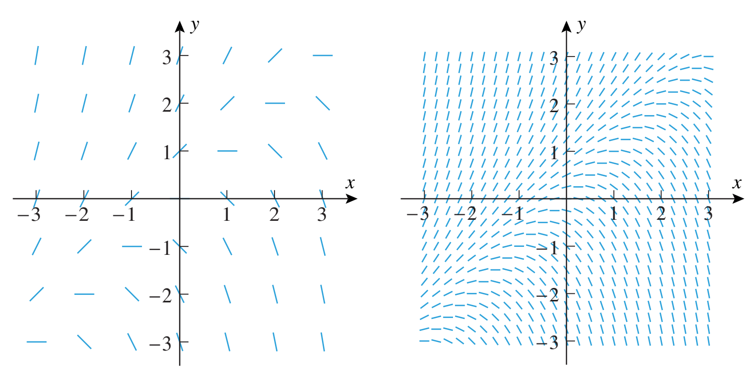


Figure 9.2.2

It so happens that Equation (1) can be solved exactly, since it can be written as

$$y' - y = -x$$

which, by comparison with Equation (5) in Section 9.1, is a first-order linear equation with $p(x) = -1$ and $q(x) = -x$. We leave it for you to use the method of integrating factors to show that the general solution of this equation is

$$y = x + 1 + Ce^x \tag{2}$$

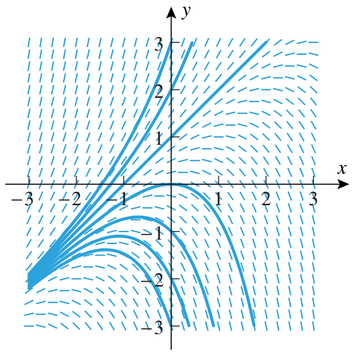


Figure 9.2.3

Figure 9.2.3 shows some of the integral curves superimposed on the direction field. Observe, however, that it was not necessary to have the general solution to construct the direction field. Indeed, direction fields are important precisely because they can be constructed in cases where the differential equation cannot be solved exactly.

FOR THE READER. Confirm that the first direction field in Figure 9.2.2 is consistent with the values in the accompanying table.

Example 1 In Example 7 of Section 9.1 we considered a mixing problem in which the amount of salt $y(t)$ in a tank at time t was shown to satisfy the differential equation

$$\frac{dy}{dt} + \frac{y}{20} = 10$$

which can be rewritten as

$$y' = 10 - \frac{y}{20} \tag{3}$$

We subsequently found the general solution of this equation to be

$$y(t) = 200 + Ce^{-t/20} \tag{4}$$

and then we found the value of the arbitrary constant C from the initial condition in the problem [the known amount of salt $y(0)$ at time $t = 0$]. However, it follows from (4) that

$$\lim_{t \to +\infty} y(t) = 200$$

for all values of C, so regardless of the amount of salt that is present in the tank initially, the amount of salt in the tank will eventually begin to stabilize at 200 lb. This can also be seen geometrically from the direction field for (3) shown in Figure 9.2.4. This direction field suggests that if the amount of salt present in the tank is greater than 200 lb initially, then the amount of salt will decrease steadily over time toward a limiting value of 200 lb; and if it is less than 200 lb initially, then it will increase steadily toward a limiting value of 200 lb. The direction field also suggests that if the amount present initially is exactly 200 lb, then the amount of salt in the tank will stay constant at 200 lb. This can also be seen from (4), since $C = 0$ in this case (verify). ◀

Observe that for the direction field shown in Figure 9.2.4 the tangent segments along any horizontal line are parallel. This occurs because the differential equation has the form $y' = f(y)$ with t absent from the right side [see (3)]. Thus, for a fixed y the slope y' does not change as time varies. Because of this time independence of slope, differential equations of the form $y' = f(y)$ are said to be ***autonomous*** (from the Greek word *autonomous*, meaning "independent").

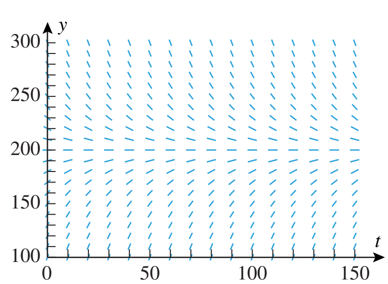


Figure 9.2.4

EULER'S METHOD

Our next objective is to develop a method for approximating the solution of an initial-value problem of the form

$$y' = f(x, y), \quad y(x_0) = y_0$$

We will not attempt to approximate $y(x)$ for all values of x; rather, we will choose some small increment Δx and focus on approximating the values of $y(x)$ at a succession of

x-values spaced Δx units apart, starting from x_0. We will denote these x-values by

$$x_1 = x_0 + \Delta x, \quad x_2 = x_1 + \Delta x, \quad x_3 = x_2 + \Delta x, \quad x_4 = x_3 + \Delta x, \ldots$$

and we will denote the approximations of $y(x)$ at these points by

$$y_1 \approx y(x_1), \quad y_2 \approx y(x_2), \quad y_3 \approx y(x_3), \quad y_4 \approx y(x_4), \ldots$$

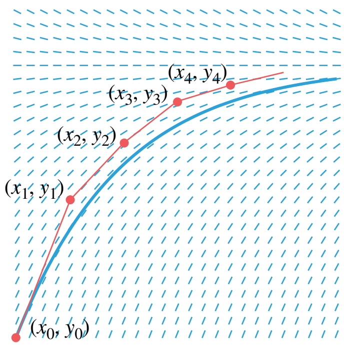


Figure 9.2.5

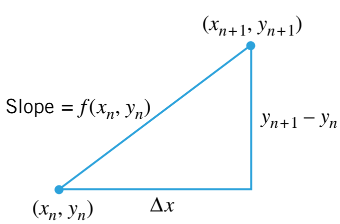


Figure 9.2.6

The technique that we will describe for obtaining these approximations is called ***Euler's Method***. Although there are better approximation methods available, many of them use Euler's Method as a starting point, so the underlying concepts are important to understand.

The basic idea behind Euler's Method is to start at the known initial point (x_0, y_0) and draw a line segment in the direction determined by the direction field until we reach the point (x_1, y_1) with x-coordinate $x_1 = x_0 + \Delta x$ (Figure 9.2.5). If Δx is small, then it is reasonable to expect that this line segment will not deviate much from the integral curve $y = y(x)$, and thus y_1 should closely approximate $y(x_1)$. To obtain the subsequent approximations, we repeat the process using the direction field as a guide at each step. Starting at the endpoint (x_1, y_1), we draw a line segment determined by the direction field until we reach the point (x_2, y_2) with x-coordinate $x_2 = x_1 + \Delta x$, and from that point we draw a line segment determined by the direction field to the point (x_3, y_3) with x-coordinate $x_3 = x_2 + \Delta x$, and so forth. As indicated in Figure 9.2.5, this procedure produces a polygonal path that tends to follow the integral curve closely, so it is reasonable to expect that the y-values $y_2, y_3, y_4, \ldots$ will closely approximate $y(x_2), y(x_3), y(x_4), \ldots$.

To explain how the approximations $y_1, y_2, y_3, \ldots$ can be computed, let us focus on a typical line segment. As indicated in Figure 9.2.6, assume that we have found the point (x_n, y_n), and we are trying to determine the next point (x_{n+1}, y_{n+1}), where $x_{n+1} = x_n + \Delta x$. Since the slope of the line segment joining the points is determined by the direction field at the starting point, the slope is $f(x_n, y_n)$, and hence

$$\frac{y_{n+1} - y_n}{x_{n+1} - x_n} = \frac{y_{n+1} - y_n}{\Delta x} = f(x_n, y_n)$$

which we can rewrite as

$$y_{n+1} = y_n + f(x_n, y_n)\Delta x$$

This formula, which is the heart of Euler's Method, tells us how to use each approximation to compute the next approximation.

Euler's Method

To approximate the solution of the initial-value problem

$$y' = f(x, y), \quad y(x_0) = y_0$$

proceed as follows:

Step 1. Choose a nonzero number Δx to serve as an ***increment*** or ***step size*** along the x-axis, and let

$$x_1 = x_0 + \Delta x, \quad x_2 = x_1 + \Delta x, \quad x_3 = x_2 + \Delta x, \ldots$$

Step 2. Compute successively

$$\begin{aligned} y_1 &= y_0 + f(x_0, y_0)\Delta x \\ y_2 &= y_1 + f(x_1, y_1)\Delta x \\ y_3 &= y_2 + f(x_2, y_2)\Delta x \\ &\vdots \\ y_{n+1} &= y_n + f(x_n, y_n)\Delta x \end{aligned}$$

The numbers $y_1, y_2, y_3, \ldots$ in these equations are the approximations of $y(x_1), y(x_2), y(x_3), \ldots$.

Example 2 Use Euler's Method with a step size of 0.1 to make a table of approximate values of the solution of the initial-value problem

$$y' = y - x, \quad y(0) = 2 \tag{5}$$

over the interval $0 \leq x \leq 1$.

Solution. In this problem we have $f(x, y) = y - x$, $x_0 = 0$, and $y_0 = 2$. Moreover, since the step size is 0.1, the x-values at which the approximate values will be obtained are

$$x_1 = 0.1, \quad x_2 = 0.2, \quad x_3 = 0.3, \ldots, \quad x_9 = 0.9, \quad x_{10} = 1$$

The first three approximations are

$$y_1 = y_0 + f(x_0, y_0)\Delta x = 2 + (2 - 0)(0.1) = 2.2$$

$$y_2 = y_1 + f(x_1, y_1)\Delta x = 2.2 + (2.2 - 0.1)(0.1) = 2.41$$

$$y_3 = y_2 + f(x_2, y_2)\Delta x = 2.41 + (2.41 - 0.2)(0.1) = 2.631$$

Here is a way of organizing all 10 approximations rounded to five decimal places:

EULER'S METHOD FOR $y' = y - x$, $y(0) = 2$ WITH $\Delta x = 0.1$

n	x_n	y_n	$f(x_n, y_n)\Delta x$	$y_{n+1} = y_n + f(x_n, y_n)\Delta x$
0	0	2.00000	0.20000	2.20000
1	0.1	2.20000	0.21000	2.41000
2	0.2	2.41000	0.22100	2.63100
3	0.3	2.63100	0.23310	2.86410
4	0.4	2.86410	0.24641	3.11051
5	0.5	3.11051	0.26105	3.37156
6	0.6	3.37156	0.27716	3.64872
7	0.7	3.64872	0.29487	3.94359
8	0.8	3.94359	0.31436	4.25795
9	0.9	4.25795	0.33579	4.59374
10	1.0	4.59374	—	—

Observe that each entry in the last column becomes the next entry in the third column. ◀

ACCURACY OF EULER'S METHOD

It follows from (5) and the initial condition $y(0) = 2$ that the exact solution of the initial-value problem in Example 2 is

$$y = x + 1 + e^x$$

Thus, in this case we can compare the approximate values of $y(x)$ produced by Euler's Method with decimal approximations of the exact values (Table 9.2.1). In Table 9.2.1 the ***absolute error*** is calculated as

$$|\text{exact value} - \text{approximation}|$$

and the ***percentage error*** as

$$\frac{|\text{exact value} - \text{approximation}|}{|\text{exact value}|} \times 100\%$$

REMARK. As a rough rule of thumb, the absolute error in an approximation produced by Euler's Method is proportional to the step size; thus, reducing the step size by half reduces the absolute error (and hence the percentage error) by roughly half. However, reducing the step size also increases the amount of computation, thereby increasing the potential for roundoff error. We will leave a detailed study of error issues for courses in differential equations or numerical analysis.

Table 9.2.1

x	EXACT SOLUTION	EULER APPROXIMATION	ABSOLUTE ERROR	PERCENTAGE ERROR
0	2.00000	2.00000	0.00000	0.00
0.1	2.20517	2.20000	0.00517	0.23
0.2	2.42140	2.41000	0.01140	0.47
0.3	2.64986	2.63100	0.01886	0.71
0.4	2.89182	2.86410	0.02772	0.96
0.5	3.14872	3.11051	0.03821	1.21
0.6	3.42212	3.37156	0.05056	1.48
0.7	3.71375	3.64872	0.06503	1.75
0.8	4.02554	3.94359	0.08195	2.04
0.9	4.35960	4.25795	0.10165	2.33
1.0	4.71828	4.59374	0.12454	2.64

EXERCISE SET 9.2 Graphing Utility C CAS

1. Sketch the direction field for $y' = xy/8$ at the gridpoints (x, y), where $x = 0, 1, \ldots, 4$ and $y = 0, 1, \ldots, 4$.

2. Sketch the direction field for $y' + y = 2$ at the gridpoints (x, y), where $x = 0, 1, \ldots, 4$ and $y = 0, 1, \ldots, 4$.

3. A direction field for the differential equation $y' = 1 - y$ is shown in the accompanying figure. In each part, sketch the graph of the solution that satisfies the initial condition.
   (a) $y(0) = -1$ (b) $y(0) = 1$ (c) $y(0) = 2$

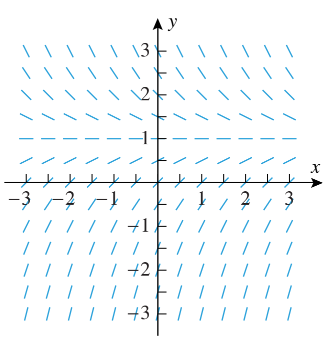


Figure Ex-3

4. Solve the initial-value problems in Exercise 3, and use a graphing utility to confirm that the integral curves for these solutions are consistent with the sketches you obtained from the direction field.

5. A direction field for the differential equation $y' = 2y - x$ is shown in the accompanying figure. In each part, sketch the graph of the solution that satisfies the initial condition.
   (a) $y(1) = 1$ (b) $y(0) = -1$ (c) $y(-1) = 0$

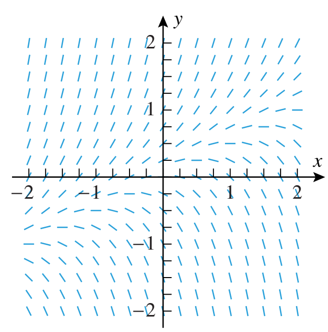


Figure Ex-5

6. Solve the initial-value problems in Exercise 5, and use a graphing utility to confirm that the integral curves for these solutions are consistent with the sketches you obtained from the direction field.

7. Use the direction field in Exercise 3 to make a conjecture about the behavior of the solutions of $y' = 1 - y$ as $x \to +\infty$, and confirm your conjecture by examining the general solution of the equation.

8. Use the direction field in Exercise 5 to make a conjecture about the effect of y_0 on the behavior of the solution of the initial-value problem $y' = 2y - x$, $y(0) = y_0$ as $x \to +\infty$, and check your conjecture by examining the solution of the initial-value problem.

9. In each part, match the differential equation with the direction field (see next page), and explain your reasoning.
   (a) $y' = 1/x$ (b) $y' = 1/y$ (c) $y' = e^{-x^2}$
   (d) $y' = y^2 - 1$ (e) $y' = \dfrac{x+y}{x-y}$
   (f) $y' = (\sin x)(\sin y)$

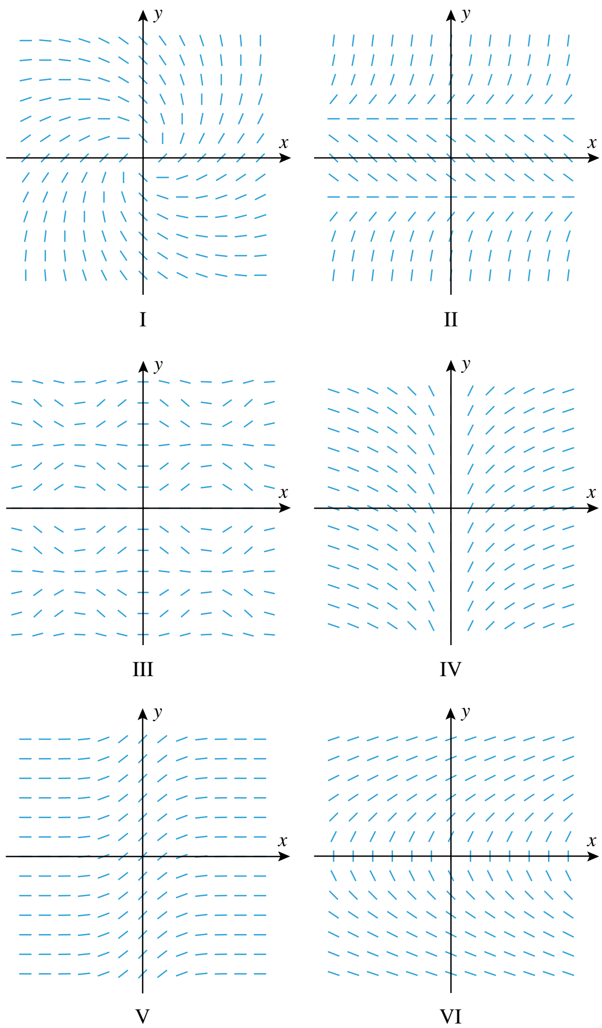


Figure Ex-9

C **10.** If you have a CAS or a graphing utility that can generate direction fields, read the documentation on how to do it and check your answers in Exercise 9 by generating the direction fields for the differential equations.

11. (a) Use Euler's Method with a step size of $\Delta x = 0.2$ to approximate the solution of the initial-value problem

$$y' = x + y, \quad y(0) = 1$$

over the interval $0 \le x \le 1$.

(b) Solve the initial-value problem exactly, and calculate the error and the percentage error in each of the approximations in part (a).

(c) Sketch the exact solution and the approximate solution together.

12. It was stated at the end of this section that reducing the step size in Euler's Method by half reduces the error in each approximation by about half. Confirm that the error in $y(1)$ is reduced by about half if a step size of $\Delta x = 0.1$ is used in Exercise 11.

In Exercises 13–16, use Euler's Method with the given step size Δx to approximate the solution of the initial-value problem over the stated interval. Present your answer as a table and as a graph.

13. $dy/dx = \sqrt{y},\ y(0) = 1,\ 0 \le x \le 4,\ \Delta x = 0.5$

14. $dy/dx = x - y^2,\ y(0) = 1,\ 0 \le x \le 2,\ \Delta x = 0.25$

15. $dy/dt = \sin y,\ y(0) = 1,\ 0 \le t \le 2,\ \Delta x = 0.5$

16. $dy/dt = e^{-y},\ y(0) = 0,\ 0 \le t \le 1,\ \Delta x = 0.1$

17. Consider the initial-value problem

$$y' = \cos 2\pi t, \quad y(0) = 1$$

Use Euler's Method with five steps to approximate $y(1)$.

18. (a) Show that the solution of the initial-value problem $y' = e^{-x^2}$, $y(0) = 0$ is

$$y(x) = \int_0^x e^{-t^2}\,dt$$

(b) Use Euler's Method with $\Delta x = 0.05$ to approximate the value of

$$y(1) = \int_0^1 e^{-t^2}\,dt$$

and compare the answer to that produced by a calculating utility with a numerical integration capability.

19. The accompanying figure shows a direction field for the differential equation $y' = -x/y$.

(a) Use the direction field to estimate $y\left(\frac{1}{2}\right)$ for the solution that satisfies the given initial condition $y(0) = 1$.

(b) Compare your estimate to the exact value of $y\left(\frac{1}{2}\right)$.

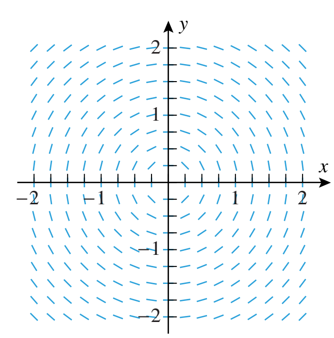


Figure Ex-19

20. Consider the initial-value problem

$$\frac{dy}{dx} = \frac{\sqrt{y}}{2}, \quad y(0) = 1$$

(a) Use Euler's Method with step sizes of $\Delta x = 0.2, 0.1$, and 0.05 to obtain three approximations of $y(1)$.

(b) Plot the three approximations versus Δx, and make a conjecture about the exact value of $y(1)$. Explain your reasoning.

(c) Check your conjecture by finding $y(1)$ exactly.

9.3 MODELING WITH FIRST-ORDER DIFFERENTIAL EQUATIONS

Since many of the fundamental laws of the physical and social sciences involve rates of change, it should not be surprising that such laws are modeled by differential equations. In this section we will discuss the general idea of modeling with differential equations, and we will investigate some important models that can be applied to population growth, carbon dating, medicine, and ecology.

POPULATION GROWTH

One of the simplest models of population growth is based on the observation that when populations (people, plants, bacteria, and fruit flies, for example) are not constrained by environmental limitations, they tend to grow at a rate that is proportional to the size of the population—the larger the population, the more rapidly it grows.

To translate this principle into a mathematical model, suppose that $y = y(t)$ denotes the population at time t. At each point in time, the rate of increase of the population with respect to time is dy/dt, so the assumption that the rate of growth is proportional to the population is described by the differential equation

$$\frac{dy}{dt} = ky \tag{1}$$

where k is a positive constant of proportionality that can usually be determined experimentally. Thus, if the population is known at some point in time, say $y = y_0$ at time $t = 0$, then a general formula for the population $y(t)$ can be obtained by solving the initial-value problem

$$\frac{dy}{dt} = ky, \quad y(0) = y_0$$

PHARMACOLOGY

When a drug (say, penicillin or aspirin) is administered to an individual, it enters the bloodstream and then is absorbed by the body over time. Medical research has shown that the amount of a drug that is present in the bloodstream tends to decrease at a rate that is proportional to the amount of the drug present—the more of the drug that is present in the bloodstream, the more rapidly it is absorbed by the body.

To translate this principle into a mathematical model, suppose that $y = y(t)$ is the amount of the drug present in the bloodstream at time t. At each point in time, the rate of change in y with respect to t is dy/dt, so the assumption that the rate of decrease is proportional to the amount y in the bloodstream translates into the differential equation

$$\frac{dy}{dt} = -ky \tag{2}$$

where k is a positive constant of proportionality that depends on the drug and can be determined experimentally. The negative sign is required because y decreases with time. Thus, if the initial dosage of the drug is known, say $y = y_0$ at time $t = 0$, then a general formula for $y(t)$ can be obtained by solving the initial-value problem

$$\frac{dy}{dt} = -ky, \quad y(0) = y_0$$

SPREAD OF DISEASE

Suppose that a disease begins to spread in a population of L individuals. Logic suggests that at each point in time the rate at which the disease spreads will depend on how many individuals are already affected and how many are not—as more individuals are affected, the opportunity to spread the disease tends to increase, but at the same time there are fewer individuals who are not affected, so the opportunity to spread the disease tends to decrease. Thus, there are two conflicting influences on the rate at which the disease spreads.

To translate this into a mathematical model, suppose that $y = y(t)$ is the number of individuals who have the disease at time t, so of necessity the number of individuals who do not have the disease at time t is $L - y$. As the value of y increases, the value of $L - y$

decreases, so the conflicting influences of the two factors on the rate of spread dy/dt are taken into account by the differential equation

$$\frac{dy}{dt} = ky(L - y)$$

where k is a positive constant of proportionality that depends on the nature of the disease and the behavior patterns of the individuals and can be determined experimentally. Thus, if the number of affected individuals is known at some point in time, say $y = y_0$ at time $t = 0$, then a general formula for $y(t)$ can be obtained by solving the initial-value problem

$$\frac{dy}{dt} = ky(L - y), \quad y(0) = y_0 \tag{3}$$

INHIBITED POPULATION GROWTH

The population growth model that we discussed at the beginning of this section was predicated on the assumption that the population $y = y(t)$ is not constrained by the environment. For this reason, it is sometimes called the ***uninhibited growth model***. However, in the real world this assumption is usually not valid—populations generally grow within ecological systems that can only support a certain number of individuals; the number L of such individuals is called the ***carrying capacity*** of the system. Thus, when $y > L$, the population exceeds the capacity of the ecological system and tends to decrease toward L; when $y < L$, the population is below the capacity of the ecological system and tends to increase toward L; and when $y = L$, the population is in balance with the capacity of the ecological system and tends to remain stable.

To translate this into a mathematical model, we must look for a differential equation in which

$$\frac{dy}{dt} < 0 \quad \text{if} \quad \frac{y}{L} > 1$$

$$\frac{dy}{dt} > 0 \quad \text{if} \quad \frac{y}{L} < 1$$

$$\frac{dy}{dt} = 0 \quad \text{if} \quad \frac{y}{L} = 1$$

Moreover, logic suggests that when the population is far below the carrying capacity (i.e., $y/L \approx 0$), then the environmental constraints should have little effect, and the growth rate should behave very much like the uninhibited model. Thus, we want

$$\frac{dy}{dt} \approx ky \quad \text{if} \quad \frac{y}{L} \approx 0$$

A simple differential equation that meets all of these requirements is

$$\frac{dy}{dt} = k\left(1 - \frac{y}{L}\right)y$$

where k is a positive constant of proportionality. Thus, if k and L can be determined experimentally, and if the population is known at some point in time, say $y(0) = y_0$, then a general formula for the population $y(t)$ can be determined by solving the initial-value problem

$$\frac{dy}{dt} = k\left(1 - \frac{y}{L}\right)y, \quad y(0) = y_0 \tag{4}$$

This theory of population growth is due to the Belgian mathematician, P. F. Verhulst (1804–1849), who introduced it in 1838 and described it as "logistic growth."* Thus, the differential equation in (4) is called the ***logistic differential equation***, and the growth model described by (4) is called the ***logistic model*** or the ***inhibited growth model***.

* Verhulst's model fell into obscurity for nearly a hundred years because he did not have sufficient census data to test its validity. However, interest in the model was revived in the 1930s when biologists used it successfully to describe the growth of fruit fly and flour beetle populations. Verhulst himself used the model to predict that an upper limit on Belgium's population would be approximately 9,400,000. In 1998 the population was about 10,175,000.

REMARK. Observe that the differential equation in (3) can be expressed as

$$\frac{dy}{dt} = kL\left(1 - \frac{y}{L}\right)y$$

which is a logistic equation with kL rather than k as the constant of proportionality. Thus, this model for the spread of disease is also a logistic or inhibited growth model.

EXPONENTIAL GROWTH AND DECAY MODELS

Equations (1) and (2) are examples of a general class of models called *exponential models*. In general, exponential models arise in situations where a quantity increases or decreases at a rate that is proportional to the amount of the quantity present. More precisely, we make the following definition:

9.3.1 DEFINITION. A quantity $y = y(t)$ is said to have an ***exponential growth model*** if it increases at a rate that is proportional to the amount of the quantity present, and it is said to have an ***exponential decay model*** if it decreases at a rate that is proportional to the amount of the quantity present. Thus, for an exponential growth model, the quantity $y(t)$ satisfies an equation of the form

$$\frac{dy}{dt} = ky \quad (k > 0) \tag{5}$$

and for an exponential decay model, the quantity $y(t)$ satisfies an equation of the form

$$\frac{dy}{dt} = -ky \quad (k > 0) \tag{6}$$

The constant k is called the ***growth constant*** or the ***decay constant***, as appropriate.

Equations (5) and (6) are first-order linear equations, since they can be rewritten as

$$\frac{dy}{dt} - ky = 0 \quad \text{and} \quad \frac{dy}{dt} + ky = 0$$

both of which have the form of Equation (5) in Section 9.1 (but with t rather than x as the independent variable); in the first equation we have $p(t) = -k$ and $q(t) = 0$, and in the second we have $p(t) = k$ and $q(t) = 0$.

To illustrate how these equations can be solved, suppose that a quantity $y = y(t)$ has an exponential growth model and we know the amount of the quantity at some point in time, say $y = y_0$ when $t = 0$. Thus, a general formula for $y(t)$ can be obtained by solving the initial-value problem

$$\frac{dy}{dt} - ky = 0, \quad y(0) = y_0$$

Multiplying the differential equation through by the integrating factor

$$\mu = e^{-kt}$$

yields

$$\frac{d}{dt}(e^{-kt}y) = 0$$

and then integrating with respect to t yields

$$e^{-kt}y = C \quad \text{or} \quad y = Ce^{kt}$$

The initial condition implies that $y = y_0$ when $t = 0$, from which it follows that $C = y_0$ (verify). Thus, the solution of the initial-value problem is

$$y = y_0e^{kt} \tag{7}$$

We leave it for you to show that if $y = y(t)$ has an exponential decay model, and if $y(0) = y_0$, then

$$y = y_0e^{-kt} \tag{8}$$

INTERPRETING THE GROWTH AND DECAY CONSTANTS

The significance of the constant k in Formulas (7) and (8) can be understood by reexamining the differential equations that gave rise to these formulas. For example, in the case of the exponential growth model, Equation (5) can be rewritten as

$$k = \frac{dy/dt}{y}$$

which states that the growth rate as a fraction of the entire population remains constant over time, and this constant is k. For this reason, k is called the ***relative growth rate*** of the population. It is usual to express the relative growth rate as a percentage. Thus, a relative growth rate of 3% per unit of time in an exponential growth model means that $k = 0.03$. Similarly, the constant k in an exponential decay model is called the ***relative decay rate***.

REMARK. It is standard practice in applications to call the relative growth rate the *growth rate*, even though it is not really correct (the growth rate is dy/dt). However, the practice is so common that we will follow it here.

Example 1 According to United Nations data, the world population in 1998 was approximately 5.9 billion and growing at a rate of about 1.33% per year. Assuming an exponential growth model, estimate the world population at the beginning of the year 2023.

Solution. We assume that the population at the beginning of 1998 was 5.9 billion and let

t = time elapsed from the beginning of 1998 (in years)

y = world population (in billions)

Since the beginning of 1998 corresponds to $t = 0$, it follows from the given data that

$$y_0 = y(0) = 5.9 \text{ (billion)}$$

Since the growth rate is 1.33% ($k = 0.0133$), it follows from (7) that the world population at time t will be

$$y(t) = y_0e^{kt} = 5.9e^{0.0133t} \tag{9}$$

Since the beginning of the year 2023 corresponds to an elapsed time of $t = 25$ years $(2023 - 1998 = 25)$, it follows from (9) that the world population by the year 2023 will be

$$y(25) = 5.9e^{0.0133(25)} \approx 8.2$$

which is a population of approximately 8.2 billion. ◀

REMARK. In this example, the growth rate was given, so there was no need to calculate it. If the growth rate or decay rate in an exponential model is unknown, then it can be calculated using the initial condition and the value of y at one other point in time (Exercise 34).

DOUBLING TIME AND HALF-LIFE

If a quantity y has an exponential growth model, then the time required for the original size to double is called the ***doubling time***, and if y has an exponential decay model, then the time required for the original size to reduce by half is called the ***half-life***. As it turns out, doubling time and half-life depend only on the growth or decay rate and not on the amount present initially. To see why this is so, suppose that $y = y(t)$ has an exponential growth model

$$y = y_0e^{kt} \tag{10}$$

and let T denote the amount of time required for y to double in size. Thus, at time $t = T$ the value of y will be $2y_0$, and hence from (10)

$$2y_0 = y_0e^{kT} \quad \text{or} \quad e^{kT} = 2$$

Taking the natural logarithm of both sides yields $kT = \ln 2$, which implies that the doubling time is

$$T = \frac{1}{k}\ln 2 \tag{11}$$

We leave it as an exercise to show that Formula (11) also gives the half-life of an exponential decay model. Observe that this formula does not involve the initial amount y_0, so that in an exponential growth or decay model, the quantity y doubles (or reduces by half) every T units (Figure 9.3.1).

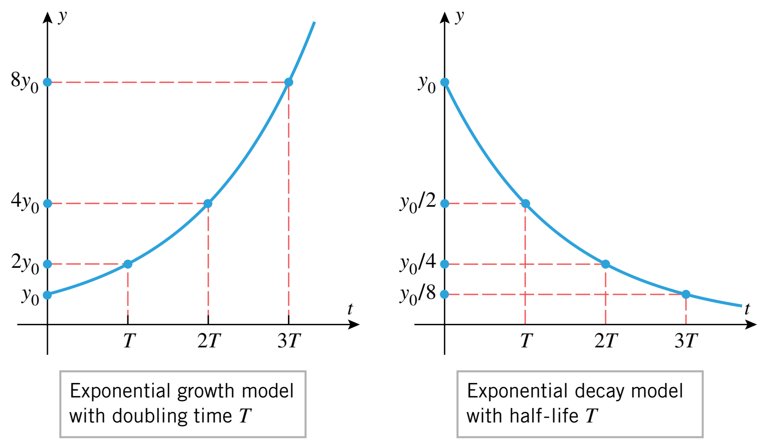


Figure 9.3.1

Example 2 It follows from (11) that with a continued growth rate of 1.33% per year, the doubling time for the world population will be

$$T = \frac{1}{0.0133}\ln 2 \approx 52.116$$

or approximately 52 years. Thus, with a continued 1.33% annual growth rate the population of 5.9 billion in 1998 will double to 11.8 billion by the year 2050 and will double again to 23.6 billion by 2102. ◀

RADIOACTIVE DECAY

It is a fact of physics that radioactive elements disintegrate spontaneously in a process called ***radioactive decay***. Experimentation has shown that the rate of disintegration is proportional to the amount of the element present, which implies that the amount $y = y(t)$ of a radioactive element present as a function of time has an exponential decay model.

Every radioactive element has a specific half-life; for example, the half-life of radioactive carbon-14 is about 5730 years. Thus, from (11), the decay constant for this element is

$$k = \frac{1}{T}\ln 2 = \frac{\ln 2}{5730} \approx 0.000121$$

and this implies that if there are y_0 units of carbon-14 present at time $t = 0$, then the number of units present after t years will be approximately

$$y(t) = y_0 e^{-0.000121t} \tag{12}$$

Example 3 If 100 grams of radioactive carbon-14 are stored in a cave for 1000 years, how many grams will be left at that time?

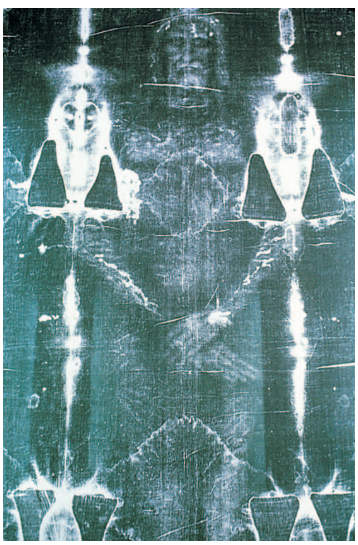
Solution. From (12) with $y_0 = 100$ and $t = 1000$, we obtain

$$y(1000) = 100e^{-0.000121(1000)} = 100e^{-0.121} \approx 88.6$$

Thus, about 88.6 grams will be left. ◀

CARBON DATING

When the nitrogen in the Earth's upper atmosphere is bombarded by cosmic radiation, the radioactive element carbon-14 is produced. This carbon-14 combines with oxygen to form carbon dioxide, which is ingested by plants, which in turn are eaten by animals. In this way all living plants and animals absorb quantities of radioactive carbon-14. In 1947 the American nuclear scientist W. F. Libby* proposed the theory that the percentage of carbon-14 in the atmosphere and in living tissues of plants is the same. When a plant or animal dies, the carbon-14 in the tissue begins to decay. Thus, the age of an artifact that contains plant or animal material can be estimated by determining what percentage of its original carbon-14 content remains. Various procedures, called ***carbon dating*** or ***carbon-14 dating***, have been developed for measuring this percentage.



The Shroud of Turin

Example 4 In 1988 the Vatican authorized the British Museum to date a cloth relic known as the Shroud of Turin, possibly the burial shroud of Jesus of Nazareth. This cloth, which first surfaced in 1356, contains the negative image of a human body that was widely believed to be that of Jesus. The report of the British Museum showed that the fibers in the cloth contained between 92% and 93% of their original carbon-14. Use this information to estimate the age of the shroud.

Solution. From (12), the fraction of the original carbon-14 that remains after t years is

$$\frac{y(t)}{y_0} = e^{-0.000121t}$$

Taking the natural logarithm of both sides and solving for t, we obtain

$$t = -\frac{1}{0.000121}\ln\left(\frac{y(t)}{y_0}\right)$$

Thus, taking $y(t)/y_0$ to be 0.93 and 0.92, we obtain

$$t = -\frac{1}{0.000121}\ln(0.93) \approx 600$$

$$t = -\frac{1}{0.000121}\ln(0.92) \approx 689$$

This means that when the test was done in 1988, the shroud was between 600 and 689 years old, thereby placing its origin between 1299 A.D. and 1388 A.D. Thus, if one accepts the validity of carbon-14 dating, the Shroud of Turin cannot be the burial shroud of Jesus of Nazareth. ◀

LOGISTIC MODELS

Recall that the logistic model of population growth in an ecological system with carrying capacity L is determined by initial-value problem (4). To illustrate how this initial-value problem can be solved for $y(t)$, let us focus on the differential equation

$$\frac{dy}{dt} = k\left(1 - \frac{y}{L}\right)y \tag{13}$$

Note that the constant functions $y = 0$ and $y = L$ are particular solutions of (13). To find nonconstant solutions, it will be convenient to rewrite Equation (13) as

$$\frac{dy}{dt} = \frac{k}{L}(L - y)y = \frac{k}{L}y(L - y)$$

This equation is separable, since it can be rewritten in differential form as

$$\frac{L}{y(L-y)}\,dy = k\,dt$$

Integrating both sides yields the equation

$$\int \frac{L}{y(L-y)}\,dy = \int k\,dt$$

* W. F. Libby, "Radiocarbon Dating," *American Scientist*, Vol. 44, 1956, pp. 98–112.

Using partial fractions on the left side, we can rewrite this equation as (verify)

$$\int \left(\frac{1}{y} + \frac{1}{L-y}\right) dy = \int k\,dt$$

Integrating and rearranging the form of the result, we obtain

$$\ln|y| - \ln|L-y| = kt + C$$

$$\ln\left|\frac{y}{L-y}\right| = kt + C$$

$$\left|\frac{y}{L-y}\right| = e^{kt+C}$$

$$\left|\frac{L-y}{y}\right| = e^{-kt-C} = e^{-C}e^{-kt}$$

$$\frac{L-y}{y} = \pm e^{-C}e^{-kt}$$

$$\frac{L}{y} - 1 = Ae^{-kt} \quad (\text{where } A = \pm e^{-C})$$

Solving this equation for y yields (verify)

$$y = \frac{L}{1 + Ae^{-kt}} \tag{14}$$

As the final step, we want to use the initial condition in (4) to determine the constant A. But the initial condition implies that $y = y_0$ if $t = 0$, so from (14)

$$y_0 = \frac{L}{1 + A}$$

from which we obtain

$$A = \frac{L - y_0}{y_0}$$

Thus, the solution of the initial-value problem (4) is

$$y = \frac{L}{1 + \left(\dfrac{L - y_0}{y_0}\right) e^{-kt}}$$

which can be rewritten more simply as

$$y = \frac{y_0 L}{y_0 + (L - y_0)e^{-kt}} \tag{15}$$

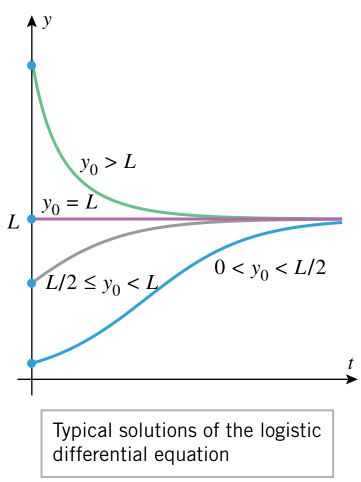


Figure 9.3.2

Note that the constant solutions of (13) are also given in (15); they correspond to the initial conditions $Y_0 = 0$ and $y_0 = L$.

The graph of (15) has one of four general shapes, depending on the relationship between the initial population y_0 and the carrying capacity L (Figure 9.3.2).

Example 5 Figure 9.3.3 shows the graph of a population $y = y(t)$ with a logistic growth model. Estimate the values of y_0, L, and k, and use the estimates to deduce a formula for y as a function of t.

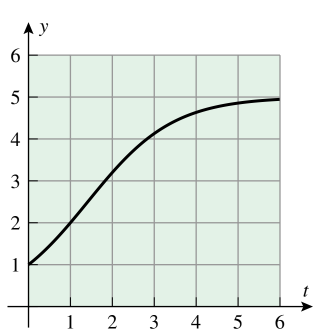


Figure 9.3.3

Solution. The graph suggests that the carrying capacity is $L = 5$, and the population at time $t = 0$ is $y_0 = 1$. Thus, from (15), the equation has the form

$$y = \frac{5}{1 + 4e^{-kt}} \tag{16}$$

where k must still be determined. However, the graph passes through the point $(1, 2)$, which tells us that $y = 2$ if $t = 1$. Substituting these values in (16) yields

$$2 = \frac{5}{1 + 4e^{-k}}$$

Solving for k we obtain (verify)

$$k = \ln \tfrac{8}{3} \approx 0.98$$

and substituting this in (16) yields

$$y = \frac{5}{1 + 4e^{-0.98t}}$$

◀

EXERCISE SET 9.3 Graphing Utility

1. (a) Suppose that a quantity $y = y(t)$ increases at a rate that is proportional to the square of the amount present, and suppose that at time $t = 0$, the amount present is y_0. Find an initial-value problem whose solution is $y(t)$.
   (b) Suppose that a quantity $y = y(t)$ decreases at a rate that is proportional to the square of the amount present, and suppose that at a time $t = 0$, the amount present is y_0. Find an initial-value problem whose solution is $y(t)$.
2. (a) Suppose that a quantity $y = y(t)$ changes in such a way that $dy/dt = k\sqrt{y}$, where $k > 0$. Describe how y changes in words.
   (b) Suppose that a quantity $y = y(t)$ changes in such a way that $dy/dt = -ky^3$, where $k > 0$. Describe how y changes in words.
3. (a) Suppose that a particle moves along an s-axis in such a way that its velocity $v(t)$ is always half of $s(t)$. Find a differential equation whose solution is $s(t)$.
   (b) Suppose that an object moves along an s-axis in such a way that its acceleration $a(t)$ is always twice the velocity. Find a differential equation whose solution is $s(t)$.
4. Suppose that a body moves along an s-axis through a resistive medium in such a way that the velocity $v = v(t)$ decreases at a rate that is twice the square of the velocity.
   (a) Find a differential equation whose solution is the velocity $v(t)$.
   (b) Find a differential equation whose solution is the position $s(t)$.
5. Suppose that an initial population of 10,000 bacteria grows exponentially at a rate of 1% per hour and that $y = y(t)$ is the number of bacteria present t hours later.
   (a) Find an initial-value problem whose solution is $y(t)$.
   (b) Find a formula for $y(t)$.
   (c) How long does it take for the initial population of bacteria to double?
   (d) How long does it take for the population of bacteria to reach 45,000?
6. A cell of the bacterium *E. coli* divides into two cells every 20 minutes when placed in a nutrient culture. Let $y = y(t)$ be the number of cells that are present t minutes after a single cell is placed in the culture. Assume that the growth of the bacteria is approximated by a continuous exponential growth model.
   (a) Find an initial-value problem whose solution is $y(t)$.
   (b) Find a formula for $y(t)$.
   (c) How many cells are present after 2 hours?
   (d) How long does it take for the number of cells to reach 1,000,000?
7. Radon-222 is a radioactive gas with a half-life of 3.83 days. This gas is a health hazard because it tends to get trapped in the basements of houses, and many health officials suggest that homeowners seal their basements to prevent entry of the gas. Assume that 5.0×10^7 radon atoms are trapped in a basement at the time it is sealed and that $y(t)$ is the number of atoms present t days later.
   (a) Find an initial-value problem whose solution is $y(t)$.
   (b) Find a formula for $y(t)$.
   (c) How many atoms will be present after 30 days?
   (d) How long will it take for 90% of the original quantity of gas to decay?
8. Polonium-210 is a radioactive element with a half-life of 140 days. Assume that 10 milligrams of the element are placed in a lead container and that $y(t)$ is the number of milligrams present t days later.
   (a) Find an initial-value problem whose solution is $y(t)$.
   (b) Find a formula for $y(t)$.
   (c) How many milligrams will be present after 10 weeks?
   (d) How long will it take for 70% of the original sample to decay?
9. Suppose that 100 fruit flies are placed in a breeding container that can support at most 5000 flies. Assuming that the population grows exponentially at a rate of 2% per day, how long will it take for the container to reach capacity?
10. Suppose that the town of Grayrock had a population of 10,000 in 1987 and a population of 12,000 in 1997. Assuming an exponential growth model, in what year will the population reach 20,000?
11. A scientist wants to determine the half-life of a certain radioactive substance. She determines that in exactly 5 days a 10.0-milligram sample of the substance decays to 3.5 milligrams. Based on these data, what is the half-life?
12. Suppose that 40% of a certain radioactive substance decays in 5 years.
    (a) What is the half-life of the substance in years?
    (b) Suppose that a certain quantity of this substance is stored in a cave. What percentage of it will remain after t years?

13. In each part, find an exponential growth model $y = y_0e^{kt}$ that satisfies the stated conditions.
(a) $y_0 = 2$; doubling time $T = 5$
(b) $y(0) = 5$; growth rate 1.5%
(c) $y(1) = 1$; $y(10) = 100$
(d) $y(1) = 1$; doubling time $T = 5$

14. In each part, find an exponential decay model $y = y_0e^{-kt}$ that satisfies the stated conditions.
(a) $y_0 = 10$; half-life $T = 5$
(b) $y(0) = 10$; decay rate 1.5%
(c) $y(1) = 100$; $y(10) = 1$
(d) $y(1) = 10$; half-life $T = 5$

15. (a) Make a conjecture about the effect on the graphs of $y = y_0e^{kt}$ and $y = y_0e^{-kt}$ of varying k and keeping y_0 fixed. Confirm your conjecture with a graphing utility.
(b) Make a conjecture about the effect on the graphs of $y = y_0e^{kt}$ and $y = y_0e^{-kt}$ of varying y_0 and keeping k fixed. Confirm your conjecture with a graphing utility.

16. (a) What effect does increasing y_0 and keeping k fixed have on the doubling time or half-life of an exponential model? Justify your answer.
(b) What effect does increasing k and keeping y_0 fixed have on the doubling time and half-life of an exponential model? Justify your answer.

17. (a) There is a trick, called the ***Rule of 70***, that can be used to get a quick estimate of the doubling time or half-life of an exponential model. According to this rule, the doubling time or half-life is roughly 70 divided by the percentage growth or decay rate. For example, we showed in Example 2 that with a continued growth rate of 1.33% per year the world population would double every 52 years. This result agrees with the Rule of 70, since $70/1.33 \approx 52.6$. Explain why this rule works.
(b) Use the Rule of 70 to estimate the doubling time of a population that grows exponentially at a rate of 1% per year.
(c) Use the Rule of 70 to estimate the half-life of a population that decreases exponentially at a rate of 3.5% per hour.
(d) Use the Rule of 70 to estimate the growth rate that would be required for a population growing exponentially to double every 10 years.

18. Find a formula for the tripling time of an exponential growth model.

19. In 1950, a research team digging near Folsom, New Mexico, found charred bison bones along with some leaf-shaped projectile points (called the "Folsom points") that had been made by a Paleo-Indian hunting culture. It was clear from the evidence that the bison had been cooked and eaten by the makers of the points, so that carbon-14 dating of the bones made it possible for the researchers to determine when the hunters roamed North America. Tests showed that the bones contained between 27% and 30% of their original carbon-14. Use this information to show that the hunters lived roughly between 9000 B.C. and 8000 B.C.

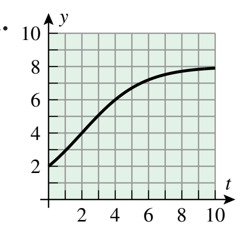
20. (a) Use a graphing utility to make a graph of p_{rem} versus t, where p_{rem} is the percentage of carbon-14 that remains in an artifact after t years.
(b) Use the graph to estimate the percentage of carbon-14 that would have to have been present in the 1988 test of the Shroud of Turin for it to have been the burial shroud of Jesus. [See Example 4.]

In Exercises 21 and 22, the graph of a logistic model

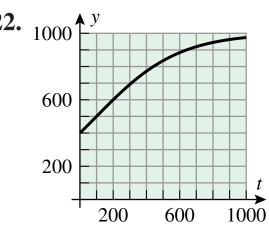
$$y = \frac{y_0L}{y_0 + (L - y_0)e^{-kt}}$$

is shown. Estimate y_0, L, and k.

21.



22.



23. Suppose that the growth of a population $y = y(t)$ is given by the logistic equation

$$y = \frac{60}{5 + 7e^{-t}}$$

(a) What is the population at time $t = 0$?
(b) What is the carrying capacity L?
(c) What is the constant k?
(d) When does the population reach half of the carrying capacity?
(e) Find an initial-value problem whose solution is $y(t)$.

24. Suppose that the growth of a population $y = y(t)$ is given by the logistic equation

$$y = \frac{1000}{1 + 999e^{-0.9t}}$$

(a) What is the population at time $t = 0$?
(b) What is the carrying capacity L?
(c) What is the constant k?
(d) When does the population reach 75% of the carrying capacity?
(e) Find an initial-value problem whose solution is $y(t)$.

25. Suppose that a population $y(t)$ grows in accordance with the logistic model

$$\frac{dy}{dt} = 10(1 - 0.1y)y$$

(a) What is the carrying capacity?
(b) What is the value of k?
(c) For what value of y is the population growing most rapidly?

26. Suppose that a population $y(t)$ grows in accordance with the logistic model

$$\frac{dy}{dt} = 50y - 0.001y^2$$

(a) What is the carrying capacity?
(b) What is the value of k?
(c) For what value of y is the population growing most rapidly?

27. Suppose that a college residence hall houses 1000 students. Following the semester break, 20 students in the hall return with the flu, and 5 days later 35 students have the flu.
(a) Use model (4) to set up an initial-value problem whose solution is the number of students who will have had the flu t days after the return from the break. [*Note:* The differential equation in this case will involve a constant of proportionality.]
(b) Solve the initial-value problem, and use the given data to find the constant of proportionality.
(c) Make a table that illustrates how the flu spreads day to day over a 2-week period.
(d) Use a graphing utility to generate a graph that illustrates how the flu spreads over a 2-week period.

28. It has been observed experimentally that at a constant temperature the rate of change of the atmospheric pressure p with respect to the altitude h above sea level is proportional to the pressure.
(a) Assuming that the pressure at sea level is p_0, find an initial-value problem whose solution is $p(h)$. [*Note:* The differential equation in this case will involve a constant of proportionality.]
(b) Find a formula for $p(h)$ in atmospheres (atm) if the pressure at sea level is 1 atm and the pressure at 5000 ft above sea level is 0.83 atm.

Newton's Law of Cooling states that the rate at which the temperature of a cooling object decreases and the rate at which a warming object increases are proportional to the difference between the temperature of the object and the temperature of the surrounding medium. Use this result in Exercises 29–32.

29. A cup of water with a temperature of 95°C is placed in a room with a constant temperature 21°C.
(a) Assuming that Newton's Law of Cooling applies, set up and solve an initial-value problem whose solution is the temperature of the water t minutes after it is placed in the room. [*Note:* The differential equation will involve a constant of proportionality.]
(b) How many minutes will it take for the water to reach a temperature of 51°C if it cools to 85°C in 1 minute?

30. A glass of lemonade with a temperature of 40°F is placed in a room with a constant temperature of 70°F, and 1 hour later its temperature is 52°F. We stated in Example 4 of Section 7.4 that t hours after the lemonade is placed in the room its temperature is approximated by $T = 70 - 30e^{-0.5t}$. Confirm this using Newton's Law of Cooling and the method used in Exercise 29.

31. The great detective Sherlock Holmes and his assistant Dr. Watson are discussing the murder of actor Cornelius McHam. McHam was shot in the head, and his understudy, Barry Moore, was found standing over the body with the murder weapon in hand. Let's listen in.

Watson: Open-and-shut case Holmes—Moore is the murderer.
Holmes: Not so fast Watson—you are forgetting Newton's Law of Cooling!
Watson: Huh?
Holmes: Elementary my dear Watson—Moore was found standing over McHam at 10:06 P.M., at which time the coroner recorded a body temperature of 77.9°F and noted that the room thermostat was set to 72°F. At 11:06 P.M. the coroner took another reading and recorded a body temperature of 75.6°F. Since McHam's normal temperature is 98.6°F, and since Moore was on stage between 6:00 P.M. and 8:00 P.M., Moore is obviously innocent.
Watson: Huh?
Holmes: Sometimes you are so dull Watson. Ask any calculus student to figure it out for you.
Watson: Hrrumph. . . .

32. Suppose that at time $t = 0$ an object with temperature T_0 is placed in a room with constant temperature T_a. If $T_0 < T_a$, then the temperature of the object will increase, and if $T_0 > T_a$, then the temperature will decrease. Assuming that Newton's Law of Cooling applies, show that in both cases the temperature $T(t)$ at time t is given by

$$T(t) = T_a + (T_0 - T_a)e^{-kt}$$

where k is a positive constant.

33. (a) Show that if $b > 1$, then the equation $y = y_0 b^t$ can be expressed as $y = y_0 e^{kt}$ for some positive constant k. [*Note:* This shows that if $b > 1$, and if y grows in accordance with the equation $y = y_0 b^t$, then y has an exponential growth model.]
(b) Show that if $0 < b < 1$, then the equation $y = y_0 b^t$ can be expressed as $y = y_0 e^{-kt}$ for some positive constant k. [*Note:* This shows that if $0 < b < 1$, and if y decays in accordance with the equation $y = y_0 b^t$, then y has an exponential decay model.]
(c) Express $y = 4(2^t)$ in the form $y = y_0 e^{kt}$.
(d) Express $y = 4(0.5^t)$ in the form $y = y_0 e^{-kt}$.

34. Suppose that a quantity y has an exponential growth model $y = y_0 e^{kt}$ or an exponential decay model $y = y_0 e^{-kt}$, and it is known that $y = y_1$ if $t = t_1$. In each case find a formula for k in terms of y_0, y_1, and t_1, assuming that $t_1 \neq 0$.

9.4 SECOND-ORDER LINEAR HOMOGENEOUS DIFFERENTIAL EQUATIONS; THE VIBRATING SPRING

In this section we will show how to solve an important collection of second-order differential equations. As an application, we will study the motion of a vibrating spring.

SECOND-ORDER LINEAR HOMOGENEOUS DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

A ***second-order linear differential equation*** is one of the form

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x) \tag{1}$$

or in alternative notation,

$$y'' + p(x)y' + q(x)y = r(x)$$

If $r(x)$ is identically 0, then (1) reduces to

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0$$

which is called the second-order linear ***homogeneous*** differential equation.

In order to discuss the solutions to a second-order linear homogeneous differential equation, it will be useful to introduce some terminology. Two functions f and g are said to be ***linearly dependent*** if one is a *constant* multiple of the other. If neither is a constant multiple of the other, then they are called ***linearly independent***. Thus,

$$f(x) = \sin x \quad \text{and} \quad g(x) = 3\sin x$$

are linearly dependent, but

$$f(x) = x \quad \text{and} \quad g(x) = x^2$$

are linearly independent. The following theorem is central to the study of second-order linear homogeneous differential equations.

9.4.1 THEOREM. *Consider the homogeneous equation*

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0 \tag{2}$$

where the functions $p(x)$ and $q(x)$ are continuous on some common open interval I. Then there exist linearly independent solutions $y_1(x)$ and $y_2(x)$ to (2) *on I. Furthermore, given any such pair of linearly independent solutions $y_1(x)$ and $y_2(x)$, a general solution of* (2) *on I is given by*

$$y(x) = c_1y_1(x) + c_2y_2(x) \tag{3}$$

That is, every solution of (2) *on I can be obtained from* (3) *by choosing appropriate values of the constants c_1 and c_2; conversely,* (3) *is a solution of* (2) *for all choices of c_1 and c_2.*

A complete proof of this theorem is best left for a course in differential equations. (Readers interested in portions of the argument are referred to Chapter 3 of *Elementary Differential Equations*, 6th ed., John Wiley & Sons, New York, 1997, by William E. Boyce and Richard C. DiPrima.)

We will restrict our attention to second-order linear homogeneous equations of the form

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0 \tag{4}$$

where p and q are *constants*. Since the constant functions $p(x) = p$ and $q(x) = q$ are continuous on $I = (-\infty, +\infty)$, it follows from Theorem 9.4.1 that to determine a general solution to (4) we need only find two linearly independent solutions $y_1(x)$ and $y_2(x)$ on I. The general solution will then be given by $y(x) = c_1y_1(x) + c_2y_2(x)$, where c_1 and c_2 are arbitrary constants.

We will start by looking for solutions to (4) of the form $y = e^{mx}$. This is motivated by the fact that the first and second derivatives of this function are multiples of y, suggesting that a solution of (4) might result by choosing m appropriately. To find such an m, we substitute

$$y = e^{mx}, \quad \frac{dy}{dx} = me^{mx}, \quad \frac{d^2y}{dx^2} = m^2e^{mx} \tag{5}$$

into (4) to obtain

$$(m^2 + pm + q)e^{mx} = 0 \tag{6}$$

which is satisfied if and only if

$$m^2 + pm + q = 0 \tag{7}$$

since $e^{mx} \neq 0$ for every x.

Equation (7), which is called the ***auxiliary equation*** for (4), can be obtained from (4) by replacing d^2y/dx^2 by m^2, dy/dx by $m\ (= m^1)$, and y by $1\ (= m^0)$. The solutions, m_1 and m_2, of the auxiliary equation can be obtained by factoring or by the quadratic formula. These solutions are

$$m_1 = \frac{-p + \sqrt{p^2 - 4q}}{2}, \quad m_2 = \frac{-p - \sqrt{p^2 - 4q}}{2} \tag{8}$$

Depending on whether $p^2 - 4q$ is positive, zero, or negative, these roots will be distinct and real, equal and real, or complex conjugates.* We will consider each of these cases separately.

DISTINCT REAL ROOTS

If m_1 and m_2 are distinct real roots, then (4) has the two solutions

$$y_1 = e^{m_1x}, \quad y_2 = e^{m_2x}$$

Neither of the functions e^{m_1x} and e^{m_2x} is a constant multiple of the other (Exercise 29), so the general solution of (4) in this case is

$$y(x) = c_1e^{m_1x} + c_2e^{m_2x} \tag{9}$$

Example 1 Find the general solution of $y'' - y' - 6y = 0$.

Solution. The auxiliary equation is

$$m^2 - m - 6 = 0 \quad \text{or equivalently,} \quad (m+2)(m-3) = 0$$

so its roots are $m = -2$, $m = 3$. Thus, from (9) the general solution of the differential equation is

$$y = c_1e^{-2x} + c_2e^{3x}$$

where c_1 and c_2 are arbitrary constants. ◀

EQUAL REAL ROOTS

If m_1 and m_2 are equal real roots, say $m_1 = m_2\ (= m)$, then the auxiliary equation yields only one solution of (4):

$$y_1(x) = e^{mx}$$

We will now show that

$$y_2(x) = xe^{mx} \tag{10}$$

is a second linearly independent solution. To see that this is so, note that $p^2 - 4q = 0$ in

*Recall that the complex solutions of a polynomial equation, and in particular of a quadratic equation, occur as conjugate pairs $a + bi$ and $a - bi$.

(8) since the roots are equal. Thus,

$$m = m_1 = m_2 = -p/2$$

and (10) becomes

$$y_2(x) = xe^{(-p/2)x}$$

Differentiating yields

$$y_2'(x) = \left(1 - \frac{p}{2}x\right)e^{(-p/2)x} \quad \text{and} \quad y_2''(x) = \left(\frac{p^2}{4}x - p\right)e^{-(p/2)x}$$

so

$$\begin{aligned} y_2''(x) + py_2'(x) + qy_2(x) &= \left[\left(\frac{p^2}{4}x - p\right) + p\left(1 - \frac{p}{2}x\right) + qx\right]e^{(-p/2)x} \\ &= \left[-\frac{p^2}{4} + q\right]xe^{(-p/2)x} \end{aligned} \tag{11}$$

But $p^2 - 4q = 0$ implies that $(-p^2/4) + q = 0$, so (11) becomes

$$y_2''(x) + py_2'(x) + qy_2(x) = 0$$

which tells us that $y_2(x)$ is a solution of (4). It can be shown that

$$y_1(x) = e^{mx} \quad \text{and} \quad y_2(x) = xe^{mx}$$

are linearly independent (Exercise 29), so the general solution of (4) in this case is

$$y = c_1e^{mx} + c_2xe^{mx} \tag{12}$$

Example 2 Find the general solution of $y'' - 8y' + 16y = 0$.

Solution. The auxiliary equation is

$$m^2 - 8m + 16 = 0 \quad \text{or equivalently,} \quad (m - 4)^2 = 0$$

so $m = 4$ is the only root. Thus, from (12) the general solution of the differential equation is

$$y = c_1e^{4x} + c_2xe^{4x}$$ ◀

COMPLEX ROOTS

If the auxiliary equation has complex roots $m_1 = a + bi$ and $m_2 = a - bi$, then both $y_1(x) = e^{ax}\cos bx$ and $y_2(x) = e^{ax}\sin bx$ are linearly independent solutions of (4) and

$$y = e^{ax}(c_1\cos bx + c_2\sin bx) \tag{13}$$

is the general solution. The proof is discussed in the exercises (Exercise 30).

Example 3 Find the general solution of $y'' + y' + y = 0$.

Solution. The auxiliary equation $m^2 + m + 1 = 0$ has roots

$$m_1 = \frac{-1 + \sqrt{1-4}}{2} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$m_2 = \frac{-1 - \sqrt{1-4}}{2} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

Thus, from (13) with $a = -1/2$ and $b = \sqrt{3}/2$, the general solution of the differential equation is

$$y = e^{-x/2}\left(c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right)$$ ◀

INITIAL-VALUE PROBLEMS

When a physical problem leads to a second-order differential equation, there are usually two conditions in the problem that determine specific values for the two arbitrary constants in the general solution of the equation. Conditions that specify the value of the solution $y(x)$ and its derivative $y'(x)$ at $x = x_0$ are called ***initial conditions***. A second-order differential equation with initial conditions is called a ***second-order initial-value problem***.

Example 4 Solve the initial-value problem

$$y'' - y = 0, \quad y(0) = 1, \quad y'(0) = 0$$

Solution. We must first solve the differential equation. The auxiliary equation

$$m^2 - 1 = 0$$

has distinct real roots $m_1 = 1$, $m_2 = -1$, so from (9) the general solution is

$$y(x) = c_1e^x + c_2e^{-x} \tag{14}$$

and the derivative of this solution is

$$y'(x) = c_1e^x - c_2e^{-x} \tag{15}$$

Substituting $x = 0$ in (14) and (15) and using the initial conditions $y(0) = 1$ and $y'(0) = 0$ yields the system of equations

$$c_1 + c_2 = 1$$
$$c_1 - c_2 = 0$$

Solving this system yields $c_1 = \frac{1}{2}$, $c_2 = \frac{1}{2}$, so from (14) the solution of the initial-value problem is

$$y(x) = \tfrac{1}{2}e^x + \tfrac{1}{2}e^{-x} = \cosh x$$ ◀

The following summary is included as a ready reference for the solution of second-order homogeneous linear differential equations with constant coefficients.

Summary

EQUATION: $y'' + py' + qy = 0$

AUXILIARY EQUATION: $m^2 + pm + q = 0$

CASE	GENERAL SOLUTION
Distinct real roots m_1, m_2 of the auxiliary equation	$y = c_1e^{m_1x} + c_2e^{m_2x}$
Equal real roots $m_1 = m_2\ (= m)$ of the auxiliary equation	$y = c_1e^{mx} + c_2xe^{mx}$
Complex roots $m_1 = a + bi$, $m_2 = a - bi$ of the auxiliary equation	$y = e^{ax}(c_1 \cos bx + c_2 \sin bx)$

VIBRATIONS OF SPRINGS

We conclude this section with an engineering model that leads to a second-order differential equation of type (4).

As shown in Figure 9.4.1, consider a block of mass M that is suspended from a vertical spring and allowed to settle into an ***equilibrium position***. Assume that the block is then set into vertical vibratory motion by pulling or pushing on it and releasing it at time $t = 0$. We will be interested in finding a mathematical model that describes the vibratory motion of the block over time.

To translate this problem into mathematical form, we introduce a vertical y-axis whose positive direction is up and whose origin is at the connection of the spring to the block when the block is in equilibrium (Figure 9.4.2). Our goal is to find the coordinate $y = y(t)$ of the top of the block as a function of time. For this purpose we will need Newton's Second Law

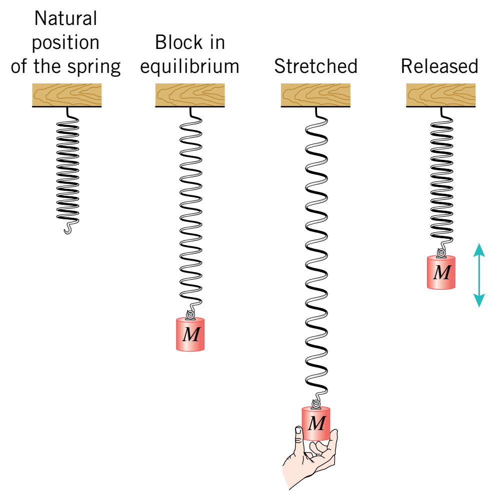


Figure 9.4.1

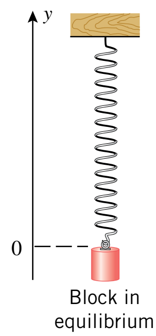


Figure 9.4.2

of Motion, which we will write as

$$F = Ma$$

rather than $F = ma$, as in Formula (19) of Section 9.1. This is to avoid a conflict with the letter "m" in the auxiliary equation. We will also need the following two results from physics:

> **9.4.2** HOOKE'S LAW. If a spring is stretched (or compressed) ℓ units beyond its natural position, then it pulls (or pushes) with a force of magnitude
>
> $$F = k\ell$$
>
> where k is a positive constant, called the ***spring constant***. This constant, which is measured in units of force per unit length, depends on such factors as the thickness of the spring and its composition. The force exerted by the spring is called the ***restoring force***.

> **9.4.3** WEIGHT. The gravitational force exerted by the Earth on an object is called the object's ***weight*** (or more precisely, its ***Earth weight***). It follows from Newton's Second Law of Motion that an object with mass M has a weight w of magnitude Mg, where g is the acceleration due to gravity. However, if the positive direction is up, as we are assuming here, then the force of the Earth's gravity is in the negative direction, so
>
> $$w = -Mg$$
>
> The weight of an object is measured in units of force.

The motion of the block in Figure 9.4.1 will depend on how far it is stretched or compressed initially and the forces that act on it while it moves. In our model we will assume that there are only two such forces: its weight w and the restoring force F_s of the spring. In particular, we will ignore such forces as air resistance, internal frictional forces in the spring, forces due to movement of the spring support, and so forth. With these assumptions, the model is called the ***simple harmonic model*** and the motion of the block is called ***simple harmonic motion***.

Our goal is to produce a differential equation whose solution gives the position function $y(t)$ of the block as a function of time. We will do this by determining the net force $F(t)$ acting on the block at a general time t and then applying Newton's Second Law of Motion.

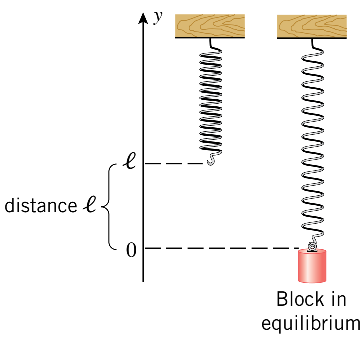


Figure 9.4.3

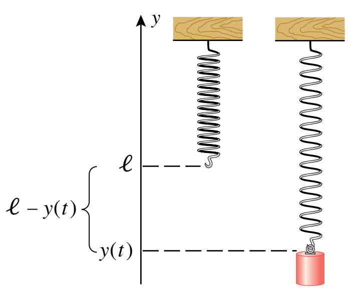


Figure 9.4.4

Since the only forces acting on the block are its weight $w = -Mg$ and the restoring force F_s of the spring, and since the acceleration of the block at time t is $y''(t)$, it follows from Newton's Second Law that

$$F_s(t) - Mg = My''(t) \tag{16}$$

To express $F_s(t)$ in terms of $y(t)$, we will begin by examining the forces on the block when it is in its equilibrium position. In this position the downward force of the weight is perfectly balanced by the upward restoring force of the spring, so that the sum of these two forces must be zero. Thus, if we assume that the spring constant is k and that the spring is stretched a distance of ℓ units beyond its natural length when the block is in equilibrium (Figure 9.4.3), then

$$k\ell - Mg = 0 \tag{17}$$

Now let us examine the restoring force acting on the block when the connection point has coordinate $y(t)$. At this point the end of the spring is displaced $\ell - y(t)$ units from its natural position (Figure 9.4.4), so Hooke's law implies that the restoring force is

$$F_s(t) = k(\ell - y(t)) = k\ell - ky(t)$$

which from (17) can be rewritten as

$$F_s(t) = Mg - ky(t)$$

Substituting this in (16) and canceling the Mg terms yields

$$-ky(t) = My''(t)$$

which we can rewrite as the homogeneous equation

$$y''(t) + \left(\frac{k}{M}\right) y(t) = 0 \tag{18}$$

The auxiliary equation for (18) is

$$m^2 + \frac{k}{M} = 0$$

which has imaginary roots $m_1 = \sqrt{k/M}\,i$, $m_2 = -\sqrt{k/M}\,i$ (since k and M are positive). It follows that the general solution of (18) is

$$y(t) = c_1 \cos\left(\sqrt{\frac{k}{M}}\,t\right) + c_2 \sin\left(\sqrt{\frac{k}{M}}\,t\right) \tag{19}$$

FOR THE READER. Confirm that the functions in family (19) are solutions of (18).

To determine the constants c_1 and c_2 in (19) we will take as our initial conditions the position and velocity at time $t = 0$. Specifically, we will ask you to show in Exercise 40 that if the position of the block at time $t = 0$ is y_0, and if the initial velocity of the block is zero (i.e., it is *released* from rest), then

$$y(t) = y_0 \cos\left(\sqrt{\frac{k}{M}}\,t\right) \tag{20}$$

This formula describes a periodic vibration with an amplitude of $|y_0|$, a period T given by

$$T = \frac{2\pi}{\sqrt{k/M}} = 2\pi\sqrt{M/k} \tag{21}$$

and a frequency f given by

$$f = \frac{1}{T} = \frac{\sqrt{k/M}}{2\pi} \tag{22}$$

(Figure 9.4.5).

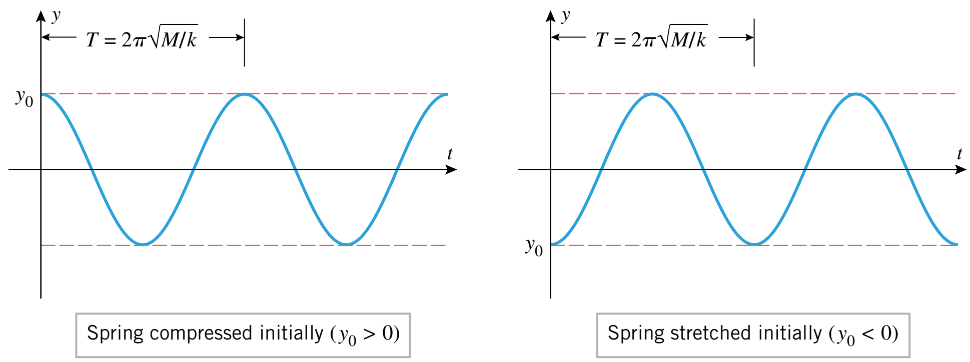


Figure 9.4.5

Example 5 Suppose that the block in Figure 9.4.2 stretches the spring 0.2 m in equilibrium. Suppose also that the block is pulled 0.5 m below its equilibrium position and released at time $t = 0$.

(a) Find the position function $y(t)$ of the block.

(b) Find the amplitude, period, and frequency of the vibration.

Solution (a). The appropriate formula is (20). Although we are not given the mass M of the block or the spring constant k, it does not matter because we can use the equilibrium condition (17) to find the ratio k/M without having values for k and M. Specifically, we are given that in equilibrium the block stretches the spring $\ell = 0.2$ m, and we know that $g = 9.8\ \text{m/s}^2$. Thus, (17) implies that

$$\frac{k}{M} = \frac{g}{\ell} = \frac{9.8}{0.2} = 49\ \text{s}^{-2} \tag{23}$$

Substituting this in (20) yields

$$y(t) = y_0 \cos 7t$$

where y_0 is the coordinate of the block at time $t = 0$. However, we are given that the block is initially 0.5 m *below* the equilibrium position, so $y_0 = -0.5$ and hence the position function of the block is $y(t) = -0.5 \cos 7t$.

Solution (b). The amplitude of the vibration is

$$\text{amplitude} = |y_0| = |-0.5| = 0.5\ \text{m}$$

and from (21), (22), and (23) the period and frequency are

$$\text{period} = T = 2\pi\sqrt{\frac{M}{k}} = 2\pi\sqrt{\frac{1}{49}} = \frac{2\pi}{7}\ \text{s}, \quad \text{frequency} = f = \frac{1}{T} = \frac{7}{2\pi}\ \text{Hz}$$ ◀

EXERCISE SET 9.4 Graphing Utility C CAS

1. Verify that the following are solutions of the differential equation $y'' - y' - 2y = 0$ by substituting these functions into the equation.
(a) e^{2x} and e^{-x}
(b) $c_1e^{2x} + c_2e^{-x}$ (c_1, c_2 constants)

2. Verify that the following are solutions of the differential equation $y'' + 4y' + 4y = 0$ by substituting these functions into the equation.
(a) e^{-2x} and xe^{-2x}
(b) $c_1e^{-2x} + c_2xe^{-2x}$ (c_1, c_2 constants)

In Exercises 3–16, find the general solution of the differential equation.

3. $y'' + 3y' - 4y = 0$

4. $y'' + 6y' + 5y = 0$

5. $y'' - 2y' + y = 0$

6. $y'' + 6y' + 9y = 0$

7. $y'' + 5y = 0$

8. $y'' + y = 0$

9. $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} = 0$

10. $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} = 0$

11. $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + 4y = 0$

12. $\dfrac{d^2y}{dt^2} - 10\dfrac{dy}{dt} + 25y = 0$

13. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 13y = 0$

14. $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 25y = 0$

15. $8y'' - 2y' - y = 0$

16. $9y'' - 6y' + y = 0$

In Exercises 17–22, solve the initial-value problem.

17. $y'' + 2y' - 3y = 0;\quad y(0) = 1,\ y'(0) = 5$

18. $y'' - 6y' - 7y = 0;\quad y(0) = 5,\ y'(0) = 3$

19. $y'' - 6y' + 9y = 0;\quad y(0) = 2,\ y'(0) = 1$

20. $y'' + 4y' + y = 0;\quad y(0) = 5,\ y'(0) = 4$

21. $y'' + 4y' + 5y = 0;\quad y(0) = -3,\ y'(0) = 0$

22. $y'' - 6y' + 13y = 0;\quad y(0) = -1,\ y'(0) = 1$

23. In each part find a second-order linear homogeneous differential equation with constant coefficients that has the given functions as solutions.
(a) $y_1 = e^{5x},\ y_2 = e^{-2x}$ (b) $y_1 = e^{4x},\ y_2 = xe^{4x}$
(c) $y_1 = e^{-x}\cos 4x,\ y_2 = e^{-x}\sin 4x$

24. Show that if e^x and e^{-x} are solutions of a second-order linear homogeneous differential equation, then so are $\cosh x$ and $\sinh x$.

25. Find all values of k for which the differential equation $y'' + ky' + ky = 0$ has a general solution of the given form.
(a) $y = c_1e^{ax} + c_2e^{bx}$ (b) $y = c_1e^{ax} + c_2xe^{ax}$
(c) $y = c_1e^{ax}\cos bx + c_2e^{ax}\sin bx$

26. The equation

$$x^2\frac{d^2y}{dx^2} + px\frac{dy}{dx} + qy = 0 \quad (x > 0)$$

where p and q are constants, is called *Euler's equidimensional equation*. Show that the substitution $x = e^z$ transforms this equation into the equation

$$\frac{d^2y}{dz^2} + (p-1)\frac{dy}{dz} + qy = 0$$

27. Use the result in Exercise 26 to find the general solution of
(a) $x^2\dfrac{d^2y}{dx^2} + 3x\dfrac{dy}{dx} + 2y = 0 \quad (x > 0)$
(b) $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - 2y = 0 \quad (x > 0)$.

28. Let $y(x)$ be a solution of $y'' + py' + qy = 0$. Prove: If p and q are positive constants, then $\lim_{x\to+\infty} y(x) = 0$.

29. Prove that the following functions are linearly independent.
(a) $y_1 = e^{m_1x},\ y_2 = e^{m_2x} \quad (m_1 \neq m_2)$
(b) $y_1 = e^{mx},\ y_2 = xe^{mx}$

30. Prove: If the auxiliary equation of

$$y'' + py' + qy = 0$$

has complex roots $a + bi$ and $a - bi$, then the general solution of this differential equation is

$$y(x) = e^{ax}(c_1\cos bx + c_2\sin bx)$$

[*Hint:* Using substitution, verify that $y_1 = e^{ax}\cos bx$ and $y_2 = e^{ax}\sin bx$ are solutions of the differential equation. Then prove that y_1 and y_2 are linearly independent.]

31. Suppose that the auxiliary equation of the equation $y'' + py' + qy = 0$ has distinct real roots μ and m.
(a) Show that the function

$$g_\mu(x) = \frac{e^{\mu x} - e^{mx}}{\mu - m}$$

is a solution of the differential equation.
(b) Use L'Hôpital's rule to show that

$$\lim_{\mu\to m} g_\mu(x) = xe^{mx}$$

[*Note:* Can you see how the result in part (b) makes it plausible that the function $y(x) = xe^{mx}$ is a solution of $y'' + py' + qy = 0$ when m is a repeated root of the auxiliary equation?]

32. Consider the problem of solving the differential equation

$$y'' + \lambda y = 0$$

subject to the conditions $y(0) = 0$, $y(\pi) = 0$.
(a) Show that if $\lambda \le 0$, then $y = 0$ is the only solution.
(b) Show that if $\lambda > 0$, then the solution is

$$y = c\sin\sqrt{\lambda}x$$

where c is an arbitrary constant, if

$$\lambda = 1, 2^2, 3^2, 4^2, \ldots$$

and the only solution is $y = 0$ otherwise.

Exercises 33–38 involve vibrations of the block pictured in Figure 9.4.1. Assume that the y-axis is as shown in Figure 9.4.2 and that the simple harmonic model applies.

33. Suppose that the block has a mass of 1 kg, the spring constant is $k = 0.25$ N/m, and the block is pushed 0.3 m above its equilibrium position and released at time $t = 0$.
(a) Find the position function $y(t)$ of the block.
(b) Find the period and frequency of the vibration.
(c) Sketch the graph of $y(t)$.
(d) At what time does the block first pass through the equilibrium position?
(e) At what time does the block first reach its maximum distance below the equilibrium position?

34. Suppose that the block has a weight of 64 lb, the spring constant is $k = 0.25$ lb/ft, and the block is pushed 1 ft above its equilibrium position and released at time $t = 0$.
(a) Find the position function $y(t)$ of the block.
(b) Find the period and frequency of the vibration.
(c) Sketch the graph of $y(t)$.
(d) At what time does the block first pass through the equilibrium position?
(e) At what time does the block first reach its maximum distance below the equilibrium position?

35. Suppose that the block stretches the spring 0.05 m in equilibrium, and the block is pulled 0.12 m below the equilibrium position and released at time $t = 0$.
(a) Find the position function $y(t)$ of the block.
(b) Find the period and frequency of the vibration.
(c) Sketch the graph of $y(t)$.
(d) At what time does the block first pass through the equilibrium position?
(e) At what time does the block first reach its maximum distance above the equilibrium position?

36. Suppose that the block stretches the spring 0.5 ft in equilibrium, and is pulled 1.5 ft below the equilibrium position and released at time $t = 0$.
(a) Find the position function $y(t)$ of the block.
(b) Find the period and frequency of the vibration.
(c) Sketch the graph of $y(t)$.
(d) At what time does the block first pass through the equilibrium position?
(e) At what time does the block first reach its maximum distance above the equilibrium position?

37. (a) For what values of y would you expect the block in Exercise 36 to have its maximum speed? Confirm your answer to this question mathematically.
(b) For what values of y would you expect the block to have its minimum speed? Confirm your answer to this question mathematically.

38. Suppose that the block weighs w pounds and vibrates with a period of 3 s when it is pulled below the equilibrium position and released. Suppose also that if the process is repeated with an additional 4 lb of weight, then the period is 5 s.
(a) Find the spring constant. (b) Find w.

39. As shown in the accompanying figure, suppose that a toy cart of mass M is attached to a wall by a spring with spring constant k, and let a horizontal x-axis be introduced with its origin at the connection point of the spring and cart when the cart is in equilibrium. Suppose that the cart is pulled or pushed horizontally to a point x_0 and then released at time $t = 0$. Find an initial-value problem whose solution is the position function of the cart, and state any assumptions you have made.

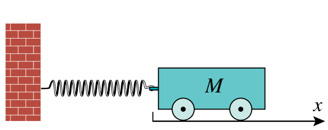


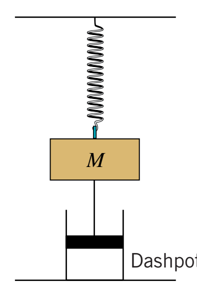
Figure Ex-39

40. Use the initial position $y(0) = y_0$ and the initial velocity $v(0) = 0$ to find the constants c_1 and c_2 in (19).

The accompanying figure shows a mass–spring system in which an object of mass M is suspended by a spring and linked to a piston that moves in a *dashpot* containing a viscous fluid. If there are no external forces acting on the system, then the object is said to have ***free motion*** and the motion of the object is completely determined by the displacement and velocity of the object at time $t = 0$, the stiffness of the spring as measured by the spring constant k, and the viscosity of the fluid in the dashpot as measured by a *damping constant* c. Mathematically, the displacement $y = y(t)$ of the object from its equilibrium position is the solution of an initial-value problem of the form

$$y'' + Ay' + By = 0, \quad y(0) = y_0, \quad y'(0) = v_0$$

where the coefficient A is determined by M and c and the coefficient B is determined by M and k. In our derivation of Equation (21) we considered only motion in which the coefficient A is zero and in which the object is released from rest, that is, $v_0 = 0$. In Exercises 41–45, you are asked to consider initial-value problems for which both the coefficient A and the initial velocity v_0 are nonzero.



41. (a) Solve the initial-value problem $y''+2.4y'+1.44y = 0$, $y(0) = 1$, $y'(0) = 2$ and graph $y = y(t)$ on the interval $[0, 5]$.
(b) Find the maximum distance above the equilibrium position attained by the object.
(c) The graph of $y(t)$ suggests that the object does not pass through the equilibrium position. Show that this is so.

42. (a) Solve the initial-value problem $y'' + 5y' + 2y = 0$, $y(0) = 1/2$, $y'(0) = -4$ and graph $y = y(t)$ on the interval $[0, 5]$.
(b) Find the maximum distance below the equilibrium position attained by the object.
(c) The graph of $y(t)$ suggests that the object passes through the equilibrium position exactly once. With what speed does the object pass through the equilibrium position?

43. (a) Solve the initial-value problem $y'' + y' + 5y = 0$, $y(0) = 1$, $y'(0) = -3.5$ and graph $y = y(t)$ on the interval $[0, 8]$.

(b) Find the maximum distance below the equilibrium position attained by the object.
(c) Find the velocity of the object when it passes through the equilibrium position the first time.
(d) Find, by inspection, the acceleration of the object when it passes through the equilibrium position the first time. [*Hint:* Examine the differential equation and use the result in part (c).]

C **44.** (a) Solve the initial-value problem $y'' + y' + 3y = 0$, $y(0) = -2$, $y'(0) = v_0$.
(b) Find the largest positive value of v_0 for which the object will rise no higher than 1 unit above the equilibrium position. [*Hint:* Use a trial-and-error strategy. Estimate v_0 to the nearest hundredth.]
(c) Graph the solution of the initial-value problem on the interval [0, 8] using the value of v_0 obtained in part (b).

45. (a) Solve the initial-value problem $y'' + 3.5y' + 3y = 0$, $y(0) = 1$, $y'(0) = v_0$.
(b) Use the result in part (a) to find the solutions for $v_0 = 2$, $v_0 = -1$, and $v_0 = -4$ and graph all three solutions on the interval [0, 4] in the same coordinate system.
(c) Discuss the effect of the initial velocity on the motion of the object.

46. Consider the first-order linear homogeneous equation

$$\frac{dy}{dx} + p(x)y = 0$$

where $p(x)$ is a continuous function on some open interval I. By analogy to the results of Theorem 9.4.1, we might expect the general solution of this equation to be of the form

$$y = cy_1(x)$$

where $y_1(x)$ is a solution of the equation on the interval I and c is an arbitrary constant. Prove this to be the case.

SUPPLEMENTARY EXERCISES

C CAS

1. We have seen that the general solution of a first-order linear equation involves a single arbitrary constant and that the general solution of a second-order linear differential equation involves two arbitrary constants. Give an informal explanation of why one might expect the number of arbitrary constants to equal the order of the equation.

2. Write a paragraph that describes Euler's Method.

3. (a) List the steps in the method of integrating factors for solving first-order linear differential equations.
(b) What would you do if you had to solve an important initial-value problem involving a first-order linear differential equation whose integrating factor could not be obtained because of the complexity of the integration?

4. Which of the following differential equations are separable?
(a) $\dfrac{dy}{dx} = f(x)g(y)$ (b) $\dfrac{dy}{dx} = \dfrac{f(x)}{g(y)}$
(c) $\dfrac{dy}{dx} = f(x) + g(y)$ (d) $\dfrac{dy}{dx} = \sqrt{f(x)g(y)}$

5. Classify the following first-order differential equations as separable, linear, both, or neither.
(a) $\dfrac{dy}{dx} - 3y = \sin x$ (b) $\dfrac{dy}{dx} + xy = x$
(c) $y\dfrac{dy}{dx} - x = 1$ (d) $\dfrac{dy}{dx} + xy^2 = \sin(xy)$

6. Determine whether the methods of integrating factors and separation of variables produce the same solution of the differential equation

$$\frac{dy}{dx} - 4xy = x$$

7. Consider the model $dy/dt = ky(L - y)$ for the spread of a disease, where $k > 0$ and $0 < y \le L$. For what value of y is the disease spreading most rapidly, and at what rate is it spreading?

8. (a) Show that if a quantity $y = y(t)$ has an exponential model, and if $y(t_1) = y_1$ and $y(t_2) = y_2$, then the doubling time or the half-life T is

$$T = \left| \frac{(t_2 - t_1)\ln 2}{\ln(y_2/y_1)} \right|$$

(b) In a certain 1-hour period the number of bacteria in a colony increases by 25%. Assuming an exponential growth model, what is the doubling time for the colony?

9. Assume that a spherical meteoroid burns up at a rate that is proportional to its surface area. Given that the radius is originally 4 m and 1 min later its radius is 3 m, find a formula for the radius as a function of time.

10. A tank contains 1000 gal of fresh water. At time $t = 0$ min, brine containing 5 ounces of salt per gallon of brine is poured into the tank at a rate of 10 gal/min, and the mixed solution is drained from the tank at the same rate. After 15 min that process is stopped and fresh water is poured into the tank at the rate of 5 gal/min, and the mixed solution is drained from the tank at the same rate. Find the amount of salt in the tank at time $t = 30$ min.

11. Suppose that a room containing 1200 ft^3 of air is free of carbon monoxide. At time $t = 0$ cigarette smoke containing 4% carbon monoxide is introduced at the rate of 0.1 ft^3/min, and the well-circulated mixture is vented from the room at the same rate.

(a) Find a formula for the percentage of carbon monoxide in the room at time t.
(b) Extended exposure to air containing 0.012% carbon monoxide is considered dangerous. How long will it take to reach this level? [This is based on a problem from William E. Boyce and Richard C. DiPrima, *Elementary Differential Equations*, 6th ed., John Wiley & Sons, New York, 1997.]

In Exercises 12–16, solve the initial-value problem.

12. $y' = 1 + y^2, \quad y(0) = 1$

13. $y' = \dfrac{y^5}{x(1+y^4)}, \quad y(1) = 1$

14. $xy' + 2y = 4x^2, \quad y(1) = 2$

15. $y' = 4y^2 \sec^2 2x, \quad y(\pi/8) = 1$

16. $y' = 6 - 5y + y^2, \quad y(0) = \ln 2$

C **17.** (a) Solve the initial-value problem

$$y' - y = x \sin 3x, \quad y(0) = 1$$

by the method of integrating factors, using a CAS to perform any difficult integrations.
(b) Use the CAS to solve the initial-value problem directly, and confirm that the answer is consistent with that obtained in part (a).
(c) Graph the solution.

C **18.** Use a CAS to derive Formula (23) of Section 9.1 by solving initial-value problem (21).

19. (a) It is currently accepted that the half-life of carbon-14 might vary ±40 years from its nominal value of 5730 years. Does this variation make it possible that the Shroud of Turin dates to the time of Jesus of Nazareth? [See Example 4 of Section 9.3.]
(b) Review the subsection of Section 3.8 entitled Error Propagation in Applications, and then estimate the percentage error that results in the computed age of an artifact from an r% error in the half-life of carbon-14.

20. (a) Use Euler's Method with a step-size of $\Delta x = 0.1$ to approximate the solution of the initial-value problem

$$y' = 1 + 5t - y, \quad y(1) = 5$$

over the interval [1, 2].
(b) Find the percentage error in the values computed.

21. Find the general solution of each differential equation.
(a) $y'' - 3y' + 2y = 0$ (b) $4y'' - 4y' + y = 0$
(c) $y'' + y' + 2y = 0$

22. (a) Sketch the integral curve of $2yy' = 1$ that passes through the point (0, 1) and the integral curve that passes through the point (0, −1).
(b) Sketch the integral curve of $y' = -2xy^2$ that passes through the point (0, 1).

23. Suppose that a herd of 19 deer is moved to a small island whose estimated carrying capacity is 95 deer, and assume that the population has a logistic growth model.
(a) Given that 1 year later the population is 25, how long will it take for the deer population to reach 80% of the island's carrying capacity?
(b) Find an initial-value problem whose solution gives the deer population as a function of time.

C **24.** If the block in Figure 9.4.1 is displaced y_0 units from its equilibrium position and given an initial velocity of v_0, rather than being released with an initial velocity of 0, then its position function $y(t)$ given in Equation (19) of Section 9.4 must satisfy the initial conditions $y(0) = y_0$ and $y'(0) = v_0$.
(a) Show that

$$y(t) = y_0 \cos\left(\sqrt{\frac{k}{M}}t\right) + v_0\sqrt{\frac{M}{k}} \sin\left(\sqrt{\frac{k}{M}}t\right)$$

(b) Suppose that a block with a mass of 1 kg stretches the spring 0.5 m in equilibrium. Use a graphing utility to graph the position function of the block if it is set in motion by pulling it down 1 m and imparting it an initial upward velocity of 0.25 m/s.
(c) What is the maximum displacement of the block from the equilibrium position?

25. A block attached to a vertical spring is displaced from its equilibrium position and released, thereby causing it to vibrate with amplitude $|y_0|$ and period T.
(a) Show that the velocity of the block has maximum magnitude $2\pi|y_0|/T$ and that the maximum occurs when the block is at its equilibrium position.
(b) Show that the acceleration of the block has maximum magnitude $4\pi^2|y_0|/T^2$ and that the maximum occurs when the block is at a top or bottom point of its motion.

26. Suppose that P dollars is invested at an annual interest rate of $r \times 100\%$. If the accumulated interest is credited to the account at the end of the year, then the interest is said to be *compounded annually*; if it is credited at the end of each 6-month period, then it is said to be *compounded semiannually*; and if it is credited at the end of each 3-month period, then it is said to be *compounded quarterly*. The more frequently the interest is compounded, the better it is for the investor since more of the interest is itself earning interest.
(a) Show that if interest is compounded n times a year at equally spaced intervals, then the value A of the investment after t years is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

(b) One can imagine interest to be compounded each day, each hour, each minute, and so forth. Carried to the limit one can conceive of interest compounded at each instant of time; this is called ***continuous compounding***. Thus, from part (a), the value A of P dollars after t years when invested at an annual rate of $r \times 100\%$, compounded continuously, is

$$A = \lim_{n\to+\infty} P\left(1+\frac{r}{n}\right)^{nt}$$

Use the fact that $\lim_{x\to 0}(1+x)^{1/x} = e$ to prove that $A = Pe^{rt}$.

(c) Use the result in part (b) to show that money invested at continuous compound interest increases at a rate proportional to the amount present.

27. (a) If $1000 is invested at 8% per year compounded continuously (Exercise 26), what will the investment be worth after 5 years?

(b) If it is desired that an investment at 8% per year compounded continuously should have a value of $10,000 after 10 years, how much should be invested now?

(c) How long does it take for an investment at 8% per year compounded continuously to double in value?

28. Prove Theorem 9.4.1 in the special case where $q(x)$ is identically zero.

29. Assume that the motion of a block of mass M is governed by the simple harmonic model (18) in Section 9.4. Define the *potential energy* of the block at time t to be $\frac{1}{2}k[y(t)]^2$, and define the *kinetic energy* of the block at time t to be $\frac{1}{2}M[y'(t)]^2$. Prove that the sum of the potential energy of the block and the kinetic energy of the block is constant.