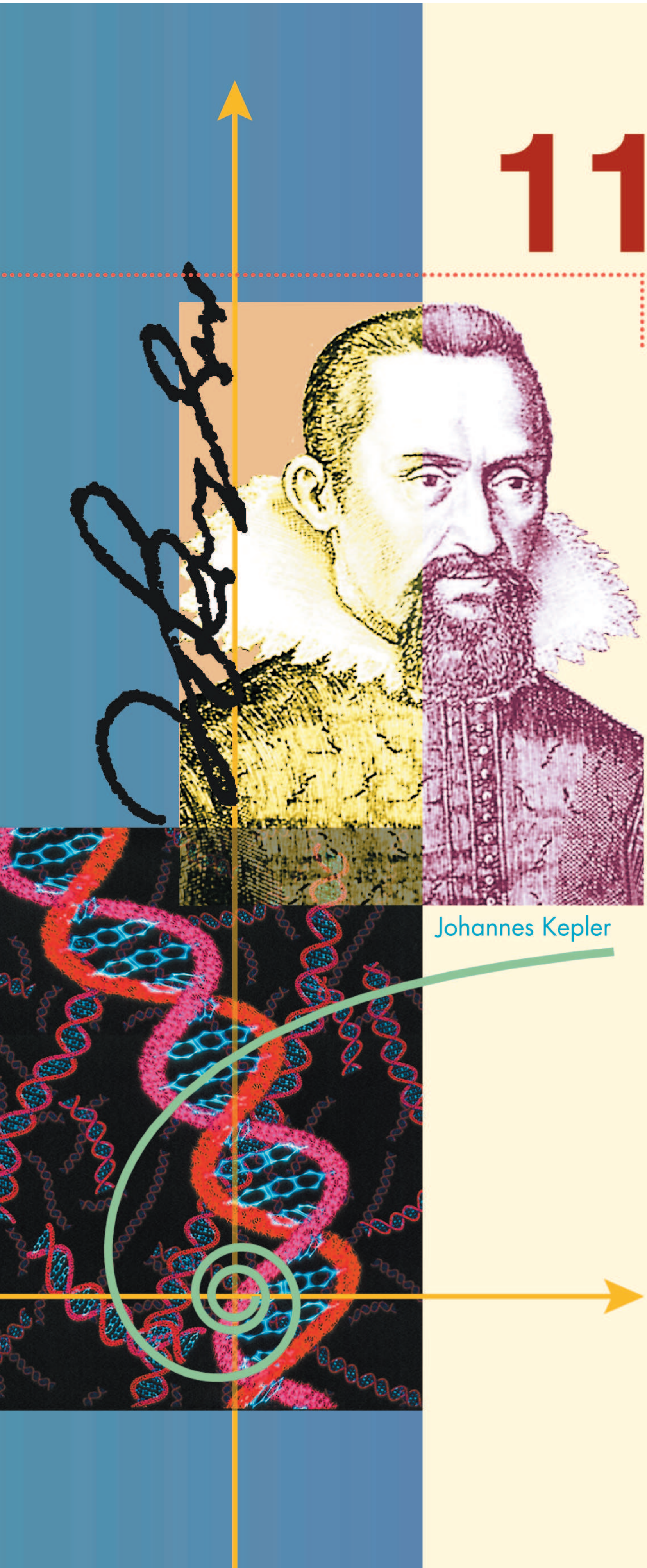

11

ANALYTIC GEOMETRY IN CALCULUS

In this chapter we will study aspects of analytic geometry that are important in applications of calculus. We will begin by introducing *polar coordinate systems*, which are used, for example, in tracking the motion of planets and satellites, in identifying the locations of objects from information on radar screens, and in the design of antennas. We will then discuss relationships between curves in polar coordinates and parametric curves in rectangular coordinates, and we will discuss methods for finding areas in polar coordinates and tangent lines to curves given in polar coordinates or parametrically in rectangular coordinates. We will then review the basic properties of parabolas, ellipses, and hyperbolas and discuss these curves in the context of polar coordinates. Finally, we will give some basic applications of our work in astronomy.

11.1 POLAR COORDINATES

Up to now we have specified the location of a point in the plane by means of coordinates relative to two perpendicular coordinate axes. However, sometimes a moving point has a special affinity for some fixed point, such as a planet moving in an orbit under the central attraction of the Sun. In such cases, the path of the particle is best described by its angular direction and its distance from the fixed point. In this section we will discuss a new kind of coordinate system that is based on this idea.

POLAR COORDINATE SYSTEMS

A ***polar coordinate system*** in a plane consists of a fixed point O, called the ***pole*** (or ***origin***), and a ray emanating from the pole, called the ***polar axis***. In such a coordinate system we can associate with each point P in the plane a pair of ***polar coordinates*** (r, θ), where r is the distance from P to the pole and θ is an angle from the polar axis to the ray OP (Figure 11.1.1). The number r is called the ***radial coordinate*** of P and the number θ the ***angular coordinate*** (or ***polar angle***) of P. In Figure 11.1.2, the points $(6, 45^\circ)$, $(5, 120^\circ)$, $(3, 225^\circ)$, and $(4, 330^\circ)$ are plotted in polar coordinate systems. If P is the pole, then $r = 0$, but there is no clearly defined polar angle. We will agree that an arbitrary angle can be used in this case; that is, $(0, \theta)$ are polar coordinates of the pole for all choices of θ.

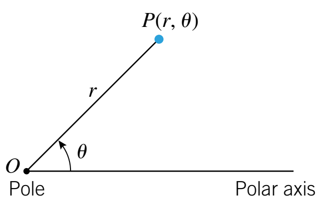

Figure 11.1.1

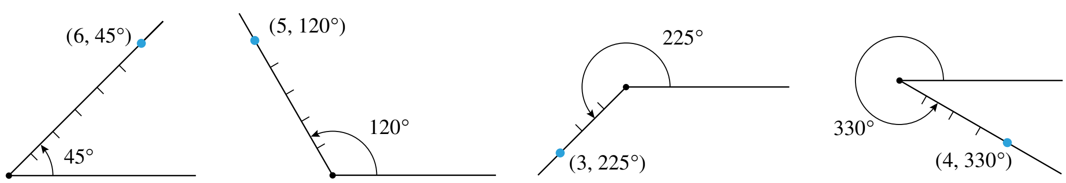

Figure 11.1.2

The polar coordinates of a point are not unique. For example, the polar coordinates

$$(1, 315^\circ), \quad (1, -45^\circ), \quad \text{and} \quad (1, 675^\circ)$$

all represent the same point (Figure 11.1.3). In general, if a point P has polar coordinates (r, θ), then

$$(r, \theta + n \cdot 360^\circ) \quad \text{and} \quad (r, \theta - n \cdot 360^\circ)$$

are also polar coordinates of P for any nonnegative integer n. Thus, every point has infinitely many pairs of polar coordinates.

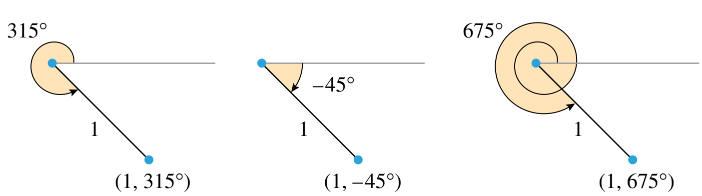

Figure 11.1.3

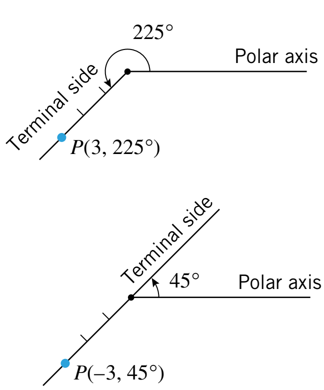

Figure 11.1.4

As defined above, the radial coordinate r of a point P is nonnegative, since it represents the distance from P to the pole. However, it will be convenient to allow for negative values of r as well. To motivate an appropriate definition, consider the point P with polar coordinates $(3, 225^\circ)$. As shown in Figure 11.1.4, we can reach this point by rotating the polar axis through an angle of 225° and then moving 3 units from the pole along the terminal side of the angle, or we can reach the point P by rotating the polar axis through an angle of 45° and then moving 3 units from the pole along the extension of the terminal side. This suggests that the point $(3, 225^\circ)$ might also be denoted by $(-3, 45^\circ)$, with the minus sign serving to indicate that the point is on the *extension* of the angle's terminal side rather than on the terminal side itself.

In general, the terminal side of the angle $\theta + 180^\circ$ is the extension of the terminal side of θ, so we define negative radial coordinates by agreeing that

$$(-r, \theta) \quad \text{and} \quad (r, \theta + 180^\circ)$$

are polar coordinates of the same point.

FOR THE READER. For many purposes it does not matter whether polar angles are measured in degrees or radians. However, in problems that involve derivatives or integrals they must be measured in radians, since the derivatives of the trigonometric functions were derived under this assumption. Henceforth, we will use radian measure for polar angles, except in certain applications where it is not required and degree measure is more convenient.

RELATIONSHIP BETWEEN POLAR AND RECTANGULAR COORDINATES

Frequently, it will be useful to superimpose a rectangular xy-coordinate system on top of a polar coordinate system, making the positive x-axis coincide with the polar axis. If this is done, then every point P will have both rectangular coordinates (x, y) and polar coordinates (r, θ). As suggested by Figure 11.1.5, these coordinates are related by the equations

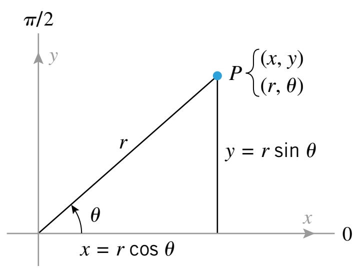

Figure 11.1.5

$$x = r\cos\theta, \quad y = r\sin\theta \tag{1}$$

These equations are well suited for finding x and y when r and θ are known. However, to find r and θ when x and y are known, it is preferable to use the identities $\sin^2\theta + \cos^2\theta = 1$ and $\tan\theta = \sin\theta/\cos\theta$ to rewrite (1) as

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x} \tag{2}$$

Example 1 Find the rectangular coordinates of the point P whose polar coordinates are $(6, 2\pi/3)$.

Solution. Substituting the polar coordinates $r = 6$ and $\theta = 2\pi/3$ in (1) yields

$$x = 6\cos\frac{2\pi}{3} = 6\left(-\frac{1}{2}\right) = -3$$

$$y = 6\sin\frac{2\pi}{3} = 6\left(\frac{\sqrt{3}}{2}\right) = 3\sqrt{3}$$

Thus, the rectangular coordinates of P are $(-3, 3\sqrt{3})$ (Figure 11.1.6). ◀

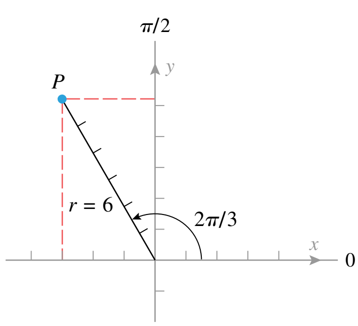

Figure 11.1.6

Example 2 Find polar coordinates of the point P whose rectangular coordinates are $(-2, 2\sqrt{3})$.

Solution. We will find the polar coordinates (r, θ) of P that satisfy the conditions $r > 0$ and $0 \le \theta < 2\pi$. From the first equation in (2),

$$r^2 = x^2 + y^2 = (-2)^2 + (2\sqrt{3})^2 = 4 + 12 = 16$$

so $r = 4$. From the second equation in (2),

$$\tan\theta = \frac{y}{x} = \frac{2\sqrt{3}}{-2} = -\sqrt{3}$$

From this and the fact that $(-2, 2\sqrt{3})$ lies in the second quadrant, it follows that the angle satisfying the requirement $0 \le \theta < 2\pi$ is $\theta = 2\pi/3$. Thus, $(4, 2\pi/3)$ are polar coordinates of P. All other polar coordinates of P are expressible in the form

$$\left(4, \frac{2\pi}{3} + 2n\pi\right) \quad \text{or} \quad \left(-4, \frac{5\pi}{3} + 2n\pi\right)$$

where n is an integer. ◀

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GRAPHS IN POLAR COORDINATES

We will now consider the problem of graphing equations of the form $r = f(\theta)$ in polar coordinates, where θ is assumed to be measured in radians. Some examples of such equations are

$$r = 2\cos\theta, \quad r = \frac{4}{1 - 3\sin\theta}, \quad r = \theta$$

In a rectangular coordinate system the graph of an equation $y = f(x)$ consists of all points whose coordinates (x, y) satisfy the equation. However, in a polar coordinate system, points have infinitely many different pairs of polar coordinates, so that a given point may have some polar coordinates that satisfy the equation $r = f(\theta)$ and others that do not. Taking this into account, we define the ***graph of r = f(θ) in polar coordinates*** to consist of all points with *at least one* pair of coordinates (r, θ) that satisfy the equation.

The most elementary way to graph an equation $r = f(\theta)$ in polar coordinates is to plot points. The idea is to choose some typical values of θ, calculate the corresponding values of r, and then plot the resulting pairs (r, θ) in a polar coordinate system. Here are some examples.

Example 3 Sketch the graph of the equation $r = \sin\theta$ in polar coordinates by plotting points.

Solution. Table 11.1.1 shows the coordinates of points on the graph at increments of $\pi/6\ (= 30^\circ)$.

Table 11.1.1

θ (RADIANS)	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$r = \sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0
(r, θ)	$(0, 0)$	$\left(\frac{1}{2}, \frac{\pi}{6}\right)$	$\left(\frac{\sqrt{3}}{2}, \frac{\pi}{3}\right)$	$\left(1, \frac{\pi}{2}\right)$	$\left(\frac{\sqrt{3}}{2}, \frac{2\pi}{3}\right)$	$\left(\frac{1}{2}, \frac{5\pi}{6}\right)$	$(0, \pi)$	$\left(-\frac{1}{2}, \frac{7\pi}{6}\right)$	$\left(-\frac{\sqrt{3}}{2}, \frac{4\pi}{3}\right)$	$\left(-1, \frac{3\pi}{2}\right)$	$\left(-\frac{\sqrt{3}}{2}, \frac{5\pi}{3}\right)$	$\left(-\frac{1}{2}, \frac{11\pi}{6}\right)$	$(0, 2\pi)$

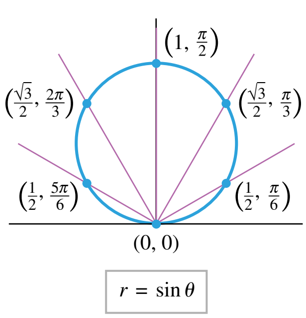

Figure 11.1.7

These points are plotted in Figure 11.1.7. Note, however, that there are 13 points listed in the table but only 6 distinct plotted points. This is because the pairs from $\theta = \pi$ on yield duplicates of the preceding points. For example, $(-1/2, 7\pi/6)$ and $(1/2, \pi/6)$ represent the same point. ◀

Observe that the points in Figure 11.1.7 appear to lie on a circle. We can confirm that this is so by expressing the polar equation $r = \sin\theta$ in terms of x and y. To do this, we multiply the equation through by r to obtain

$$r^2 = r\sin\theta$$

which now allows us to apply Formulas (1) and (2) to rewrite the equation as

$$x^2 + y^2 = y$$

Rewriting this equation as $x^2 + y^2 - y = 0$ and then completing the square yields

$$x^2 + \left(y - \tfrac{1}{2}\right)^2 = \tfrac{1}{4}$$

which is a circle of radius $\frac{1}{2}$ centered at the point $\left(0, \frac{1}{2}\right)$ in the xy-plane.

Just because an equation $r = f(\theta)$ involves the variables r and θ does not mean that it has to be graphed in a polar coordinate system. When useful, this equation can also be graphed in a rectangular coordinate system. For example, Figure 11.1.8 shows the graph of $r = \sin\theta$ in a rectangular θr-coordinate system. This graph can actually help to visualize how the polar graph in Figure 11.1.7 is generated:

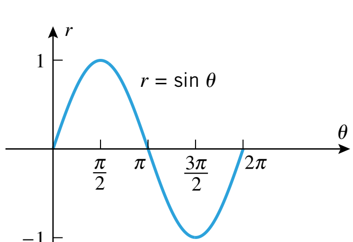

Figure 11.1.8

- At $\theta = 0$ we have $r = 0$, which corresponds to the pole $(0, 0)$ on the polar graph.
- As θ varies from 0 to $\pi/2$, the value of r increases from 0 to 1, so the point (r, θ) moves along the circle from the pole to the high point at $(1, \pi/2)$.
- As θ varies from $\pi/2$ to π, the value of r decreases from 1 back to 0, so the point (r, θ) moves along the circle from the high point back to the pole.
- As θ varies from π to $3\pi/2$, the values of r are negative, varying from 0 to -1. Thus, the point (r, θ) moves along the circle from the pole to the high point at $(1, \pi/2)$, which is the same as the point $(-1, 3\pi/2)$. This duplicates the motion that occurred for $0 \le \theta \le \pi/2$.
- As θ varies from $3\pi/2$ to 2π, the value of r varies from -1 to 0. Thus, the point (r, θ) moves along the circle from the high point back to the pole, duplicating the motion that occurred for $\pi/2 \le \theta \le \pi$.

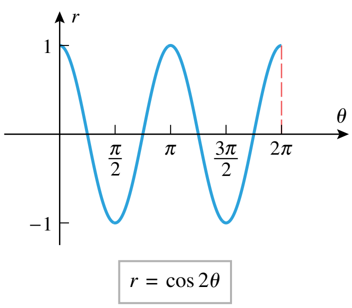

Figure 11.1.9

Example 4 Sketch the graph of $r = \cos 2\theta$ in polar coordinates.

Solution. Instead of plotting points, we will use the graph of $r = \cos 2\theta$ in rectangular coordinates (Figure 11.1.9) to visualize how the polar graph of this equation is generated. The analysis and the resulting polar graph are shown in Figure 11.1.10. This curve is called a *four-petal rose*. ◀

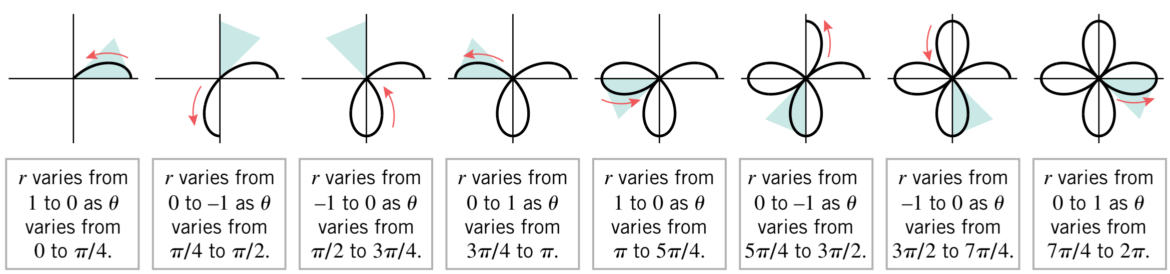

Figure 11.1.10

SYMMETRY TESTS

Observe that the polar graph of $r = \cos 2\theta$ in Figure 11.1.10 is symmetric about the x-axis and the y-axis. This symmetry could have been predicted from the following theorem, which is suggested by Figure 11.1.11 (we omit the proof).

> **11.1.1 THEOREM** (***Symmetry Tests***).
>
> (a) *A curve in polar coordinates is symmetric about the x-axis if replacing θ by $-\theta$ in its equation produces an equivalent equation* (Figure 11.1.11*a*).
>
> (b) *A curve in polar coordinates is symmetric about the y-axis if replacing θ by $\pi - \theta$ in its equation produces an equivalent equation* (Figure 11.1.11*b*).
>
> (c) *A curve in polar coordinates is symmetric about the origin if replacing r by $-r$ in its equation produces an equivalent equation* (Figure 11.1.11*c*).

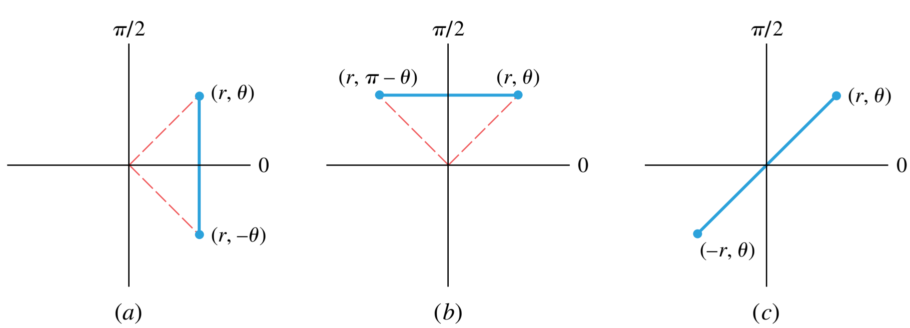

Figure 11.1.11

Example 5 Use Theorem 11.1.1 to confirm that the graph of $r = \cos 2\theta$ in Figure 11.1.10 is symmetric about the x-axis and y-axis.

Solution. To test for symmetry about the x-axis, we replace θ by $-\theta$. This yields

$$r = \cos(-2\theta) = \cos 2\theta$$

Thus, replacing θ by $-\theta$ does not alter the equation.

To test for symmetry about the y-axis, we replace θ by $\pi - \theta$. This yields

$$r = \cos 2(\pi - \theta) = \cos(2\pi - 2\theta) = \cos(-2\theta) = \cos 2\theta$$

Thus, replacing θ by $\pi - \theta$ does not alter the equation. ◀

Example 6 Sketch the graph of $r = a(1 - \cos\theta)$ in polar coordinates, assuming a to be a positive constant.

Solution. Observe first that replacing θ by $-\theta$ does not alter the equation, so we know in advance that the graph is symmetric about the polar axis. Thus, if we graph the upper half of the curve, then we can obtain the lower half by reflection about the polar axis.

As in our previous examples, we will first graph the equation in rectangular coordinates. This graph, which is shown in Figure 11.1.12a, can be obtained by rewriting the given equation as $r = a - a\cos\theta$, from which we see that the graph in rectangular coordinates can be obtained by first reflecting the graph of $r = a\cos\theta$ about the x-axis to obtain the graph of $r = -a\cos\theta$, and then translating that graph up a units to obtain the graph of $r = a - a\cos\theta$. Now we can see that:

- As θ varies from 0 to $\pi/3$, r increases from 0 to $a/2$.
- As θ varies from $\pi/3$ to $\pi/2$, r increases from $a/2$ to a.
- As θ varies from $\pi/2$ to $2\pi/3$, r increases from a to $3a/2$.
- As θ varies from $2\pi/3$ to π, r increases from $3a/2$ to $2a$.

This produces the polar curve shown in Figure 11.1.12b. The rest of the curve can be obtained by continuing the preceding analysis from π to 2π or, as noted above, by reflecting the portion already graphed about the x-axis (Figure 11.1.12c). This heart-shaped curve is called a ***cardioid*** (from the Greek word "kardia" for heart). ◀

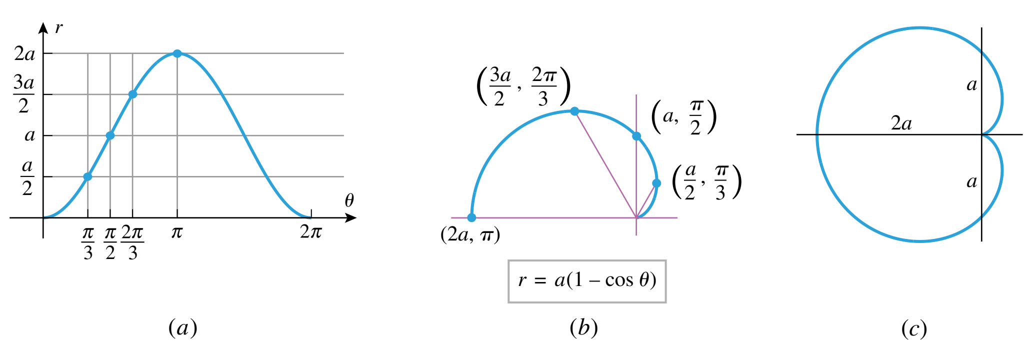

Figure 11.1.12

Example 7 Sketch the curves

(a) $r = 1$ (b) $\theta = \dfrac{\pi}{4}$ (c) $r = \theta \quad (\theta \geq 0)$

in polar coordinates.

Solution (a). For all values of θ, the point $(1, \theta)$ is 1 unit away from the pole. Thus, the graph is the circle of radius 1 centered at the pole (Figure 11.1.13*a*).

Solution (b). For all values of r, the point $(r, \pi/4)$ lies on a line that makes an angle of $\pi/4$ with the polar axis (Figure 11.1.13*b*). Positive values of r correspond to points on the line in the first quadrant and negative values of r to points on the line in the third quadrant. Thus, in absence of any restriction on r, the graph is the entire line. Observe, however, that had we imposed the restriction $r \geq 0$, the graph would have been just the ray in the first quadrant.

Solution (c). Observe that as θ increases, so does r; thus, the graph is a curve that spirals out from the pole as θ increases. A reasonably accurate sketch of the spiral can be obtained by plotting the intersections with the x- and y-axes for values of θ that are multiples of $\pi/2$, keeping in mind that the value of r is always equal to the value of θ (Figure 11.1.13*c*). ◀

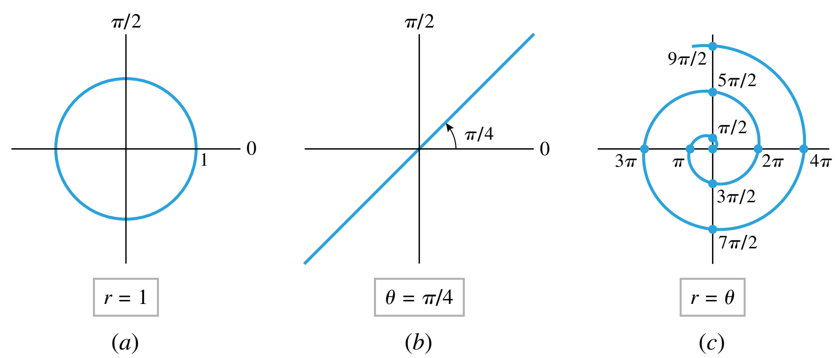

Figure 11.1.13

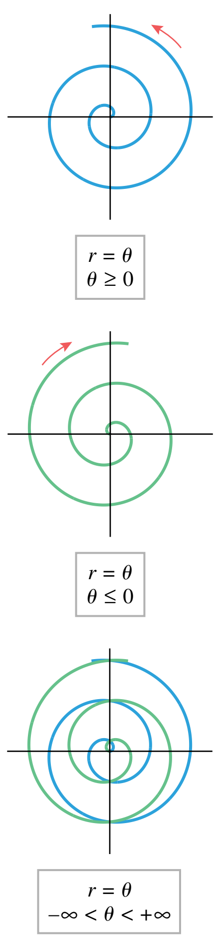

Figure 11.1.14

REMARK. The spiral in Figure 11.1.13*c*, which belongs to the family of ***Archimedean spirals*** $r = a\theta$, coils counterclockwise around the pole because of the restriction $\theta \geq 0$. Had we made the restriction $\theta \leq 0$, the spiral would have coiled clockwise, and had we allowed both positive and negative values of θ, the clockwise and counterclockwise spirals would have been superimposed to form a double Archimedean spiral (Figure 11.1.14).

Example 8 Sketch the graph of $r^2 = 4\cos 2\theta$ in polar coordinates.

Solution. This equation does not express r as a function of θ, since solving for r in terms of θ yields two functions:

$$r = 2\sqrt{\cos 2\theta} \quad \text{and} \quad r = -2\sqrt{\cos 2\theta}$$

Thus, to graph the equation $r^2 = 4\cos 2\theta$ we will have to graph the two functions separately and then combine those graphs.

We will start with the graph of $r = 2\sqrt{\cos 2\theta}$. Observe first that this equation is not changed if we replace θ by $-\theta$ or if we replace θ by $\pi - \theta$. Thus, the graph is symmetric about the x-axis and the y-axis. This means that the entire graph can be obtained by graphing the portion in the first quadrant, reflecting that portion about the y-axis to obtain the portion in the second quadrant and then reflecting those two portions about the x-axis to obtain the portions in the third and fourth quadrants.

To begin the analysis, we will graph the equation $r = 2\sqrt{\cos 2\theta}$ in rectangular coordinates (see Figure 11.1.15*a*). Note that there are gaps in that graph over the intervals $\pi/4 < \theta < 3\pi/4$ and $5\pi/4 < \theta < 7\pi/4$ because $\cos 2\theta$ is negative for those values of θ. From this graph we can see that:

- As θ varies from 0 to $\pi/4$, r decreases from 2 to 0.
- As θ varies from $\pi/4$ to $\pi/2$, no points are generated on the polar graph.

This produces the portion of the graph shown in Figure 11.1.15*b*. As noted above, we can complete the graph by a reflection about the y-axis followed by a reflection about the x-axis (11.1.15*c*). The resulting propeller-shaped graph is called a ***lemniscate*** (from the Greek word "lemniscos" for a looped ribbon resembling the number 8). We leave it for you to verify that the equation $r = 2\sqrt{\cos 2\theta}$ has the same graph as $r = -2\sqrt{\cos 2\theta}$, but traced in a diagonally opposite manner. Thus, the graph of the equation $r^2 = 4\cos 2\theta$ consists of two identical superimposed lemniscates. ◀

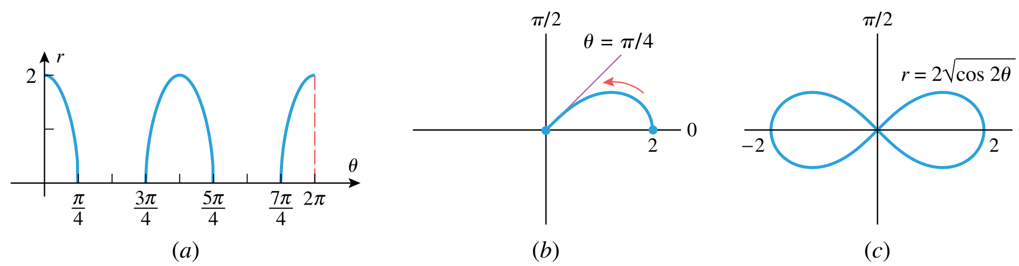

Figure 11.1.15

FAMILIES OF LINES AND RAYS THROUGH THE POLE

If θ_0 is a fixed angle, then for all values of r the point (r, θ_0) lies on the line that makes an angle of $\theta = \theta_0$ with the polar axis; and, conversely, every point on this line has a pair of polar coordinates of the form (r, θ_0). Thus, the equation $\theta = \theta_0$ represents the line that passes through the pole and makes an angle of θ_0 with the polar axis (Figure 11.1.16*a*). If r is restricted to be nonnegative, then the graph of the equation $\theta = \theta_0$ is the ray that emanates from the pole and makes an angle of θ_0 with the polar axis (Figure 11.1.16*b*). Thus, as θ_0 varies, the equation $\theta = \theta_0$ produces either a family of lines through the pole or a family of rays through the pole, depending on the restrictions on r.

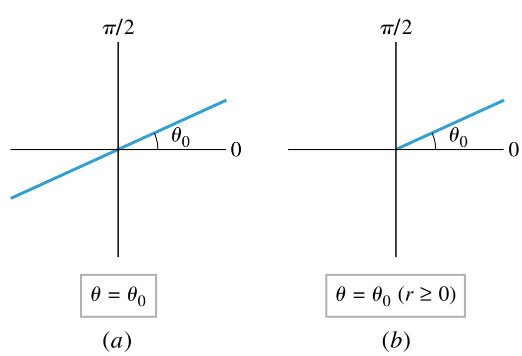

Figure 11.1.16

FAMILIES OF CIRCLES

We will consider three families of circles in which a is assumed to be a positive constant:

$$r = a \qquad r = 2a\cos\theta \qquad r = 2a\sin\theta \tag{3–5}$$

The equation $r = a$ represents a circle of radius a centered at the pole (Figure 11.1.17*a*). Thus, as a varies, this equation produces a family of circles centered at the pole. For families (4) and (5), recall from plane geometry that a triangle that is inscribed in a circle with a diameter of the circle for a side must be a right triangle. Thus, as indicated in Figures 11.1.17*b* and 11.1.17*c*, the equation $r = 2a\cos\theta$ represents a circle of radius a, centered on the x-axis and tangent to the y-axis at the origin; similarly, the equation $r = 2a\sin\theta$ represents a circle

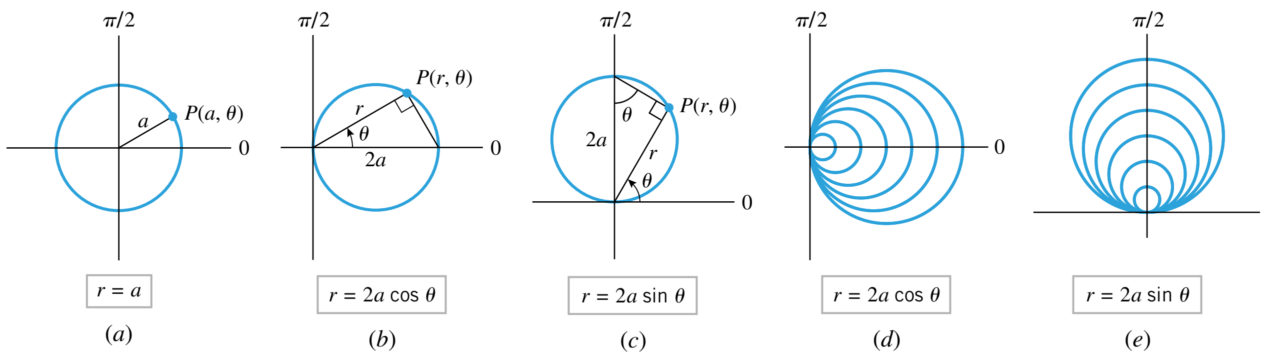

Figure 11.1.17

of radius a, centered on the y-axis and tangent to the x-axis at the origin. Thus, as a varies, Equations (4) and (5) produce the families illustrated in Figures 11.1.17*d* and 11.1.17*e*.

REMARK. Observe that replacing θ by $-\theta$ does not change the equation $r = 2a\cos\theta$, and replacing θ by $\pi - \theta$ does not change the equation $r = 2a\sin\theta$. This explains why the circles in Figure 11.1.17*d* are symmetric about the x-axis and those in Figure 11.1.17*e* are symmetric about the y-axis.

FAMILIES OF ROSE CURVES

In polar coordinates, equations of the form

$$r = a\sin n\theta \qquad r = a\cos n\theta \tag{6–7}$$

in which $a > 0$ and n is a positive integer represent families of flower-shaped curves called ***roses*** (Figure 11.1.18). The rose consists of n equally spaced petals of radius a if n is odd and $2n$ equally spaced petals of radius a if n is even. It can be shown that a rose with an even number of petals is traced out exactly once as θ varies over the interval $0 \le \theta < 2\pi$ and a rose with an odd number of petals is traced out exactly once as θ varies over the interval $0 \le \theta < \pi$ (Exercise 73). A four-petal rose of radius 1 was graphed in Example 4.

ROSE CURVES

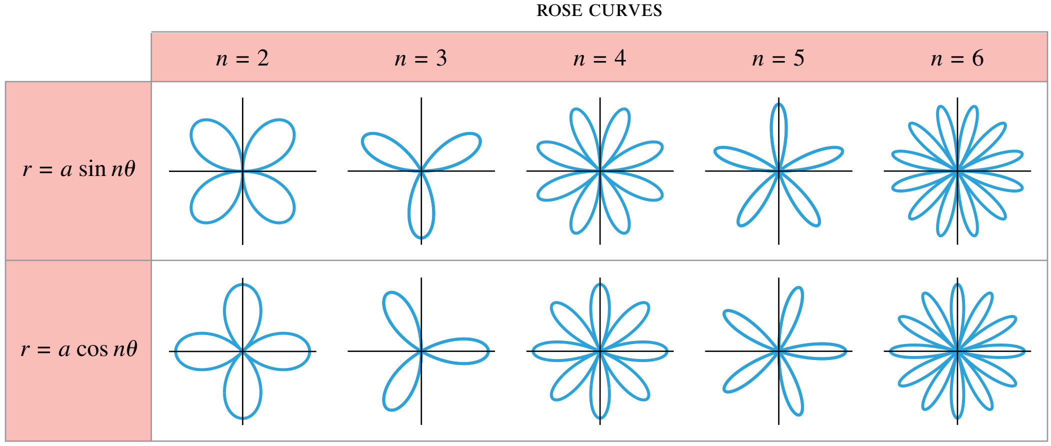

Figure 11.1.18

FOR THE READER. What do the graphs of the one-petal roses look like?

FAMILIES OF CARDIOIDS AND LIMAÇONS

Equations with any of the four forms

$$r = a \pm b\sin\theta \qquad r = a \pm b\cos\theta \tag{8–9}$$

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in which $a > 0$ and $b > 0$ represent polar curves called ***limaçons*** (from the Latin word "limax" for a snail-like creature that is commonly called a slug). There are four possible shapes for a limaçon that are determined by the ratio a/b (Figure 11.1.19). If $a = b$ (the case $a/b = 1$), then the limaçon is called a ***cardioid*** because of its heart-shaped appearance, as noted in Example 6.

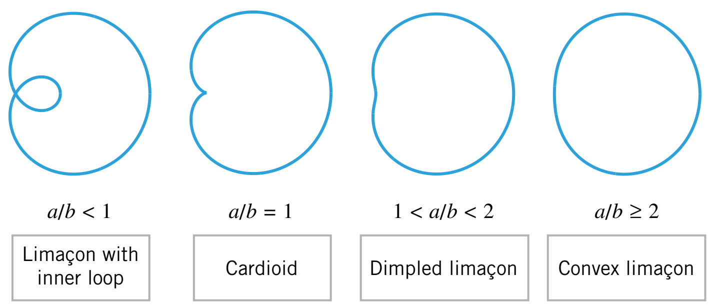


Figure 11.1.19

Example 9 Figure 11.1.20 shows the family of limaçons $r = a + \cos\theta$ with the constant a varying from 0.25 to 2.50 in steps of 0.25. In keeping with Figure 11.1.19, the limaçons evolve from the loop type to the convex type. As a increases from the starting value of 0.25, the loops get smaller and smaller until the cardioid is reached at $a = 1$. As a increases further, the limaçons evolve through the dimpled type into the convex type. ◀

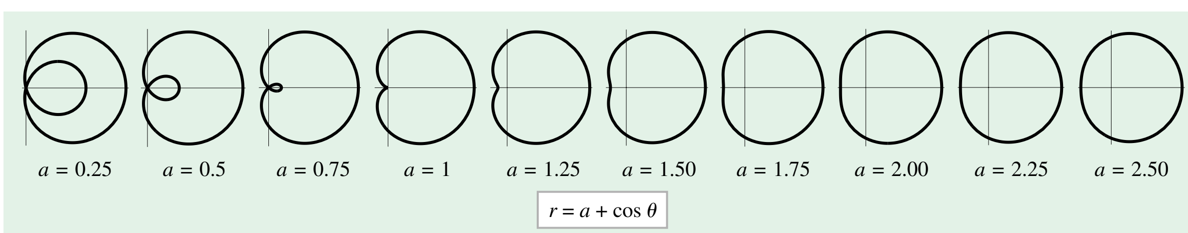


Figure 11.1.20

FAMILIES OF SPIRALS

A ***spiral*** is a curve that coils around a central point. As illustrated in Figure 11.1.14, spirals generally have "left-hand" and "right-hand" versions that coil in opposite directions, depending on the restrictions on the polar angle and the signs of constants that appear in their equations. Some of the more common types of spirals are shown in Figure 11.1.21 for nonnegative values of θ, a, and b.

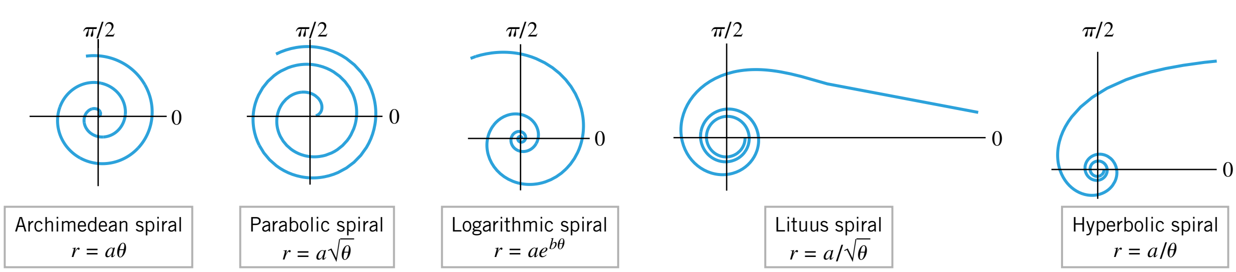


Figure 11.1.21

SPIRALS IN NATURE

Spirals of many kinds occur in nature. For example, the shell of the chambered nautilus (*below*) forms a logarithmic spiral, and a coiled sailor's rope forms an Archimedean spiral. Spirals also occur in flowers, the tusks of certain animals, and in the shapes of galaxies.

The shell of the chambered nautilus reveals a logarithmic spiral. The animal lives in the outermost chamber.

A sailor's coiled rope forms an Archimedean spiral.

GENERATING POLAR CURVES WITH GRAPHING UTILITIES

For polar curves that are too complicated for hand computation, graphing utilities must be used. Although many graphing utilities are capable of graphing polar curves directly, some are not. However, if a graphing utility is capable of graphing parametric equations, then it can be used to graph a polar curve $r = f(\theta)$ by converting this equation to parametric form. This can be done by substituting $f(\theta)$ for r in (1). This yields

$$x = f(\theta)\cos\theta, \quad y = f(\theta)\sin\theta \tag{10}$$

which is a pair of parametric equations for the polar curve in terms of the parameter θ.

Example 10 Express the polar equation

$$r = 2 + \cos\frac{5\theta}{2}$$

parametrically, and generate the polar graph from the parametric equations using a graphing utility.

Solution. Substituting the given expression for r in $x = r\cos\theta$ and $y = r\sin\theta$ yields the parametric equations

$$x = \left[2 + \cos\frac{5\theta}{2}\right]\cos\theta, \quad y = \left[2 + \cos\frac{5\theta}{2}\right]\sin\theta$$

Next, we need to find an interval over which to vary θ to produce the entire graph. To find such an interval, we will look for the smallest number of complete revolutions that must occur until the value of r begins to repeat. Algebraically, this amounts to finding the smallest positive integer n such that

$$2 + \cos\left(\frac{5(\theta + 2n\pi)}{2}\right) = 2 + \cos\frac{5\theta}{2}$$

or

$$\cos\left(\frac{5\theta}{2} + 5n\pi\right) = \cos\frac{5\theta}{2}$$

For this equality to hold, the quantity $5n\pi$ must be an even multiple of π; the smallest n for which this occurs is $n = 2$. Thus, the entire graph will be traced in two revolutions, which means it can be generated from the parametric equations

$$x = \left[2 + \cos\frac{5\theta}{2}\right]\cos\theta, \quad y = \left[2 + \cos\frac{5\theta}{2}\right]\sin\theta \qquad (0 \le \theta \le 4\pi)$$

This yields the graph in Figure 11.1.22. ◀

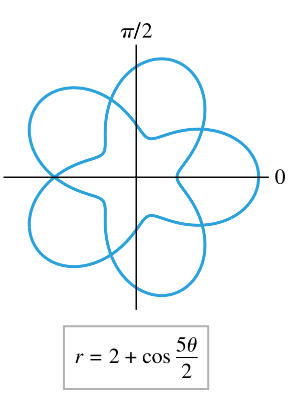


Figure 11.1.22

FOR THE READER. Some graphing utilities require that t be used for the parameter. If this is true of your graphing utility, then you will have to replace θ by t in (10) to generate graphs in polar coordinates. Use a graphing utility to duplicate the curve in Figure 11.1.22.

EXERCISE SET 11.1 Graphing Utility

In Exercises 1 and 2, plot the points in polar coordinates.

1. (a) $(3, \pi/4)$ (b) $(5, 2\pi/3)$ (c) $(1, \pi/2)$ (d) $(4, 7\pi/6)$ (e) $(-6, -\pi)$ (f) $(-1, 9\pi/4)$

2. (a) $(2, -\pi/3)$ (b) $(3/2, -7\pi/4)$ (c) $(-3, 3\pi/2)$ (d) $(-5, -\pi/6)$ (e) $(2, 4\pi/3)$ (f) $(0, \pi)$

In Exercises 3 and 4, find the rectangular coordinates of the points whose polar coordinates are given.

3. (a) $(6, \pi/6)$ (b) $(7, 2\pi/3)$ (c) $(-6, -5\pi/6)$ (d) $(0, -\pi)$ (e) $(7, 17\pi/6)$ (f) $(-5, 0)$

4. (a) $(-8, \pi/4)$ (b) $(7, -\pi/4)$ (c) $(8, 9\pi/4)$ (d) $(5, 0)$ (e) $(-2, -3\pi/2)$ (f) $(0, \pi)$

5. In each part, a point is given in rectangular coordinates. Find two pairs of polar coordinates for the point, one pair satisfying $r \geq 0$ and $0 \leq \theta < 2\pi$, and the second pair satisfying $r \geq 0$ and $-\pi < \theta \leq \pi$.
(a) $(-5, 0)$ (b) $(2\sqrt{3}, -2)$ (c) $(0, -2)$
(d) $(-8, -8)$ (e) $(-3, 3\sqrt{3})$ (f) $(1, 1)$

6. In each part find polar coordinates satisfying the stated conditions for the point whose rectangular coordinates are $(-\sqrt{3}, 1)$.
(a) $r \geq 0$ and $0 \leq \theta < 2\pi$
(b) $r \leq 0$ and $0 \leq \theta < 2\pi$
(c) $r \geq 0$ and $-2\pi < \theta \leq 0$
(d) $r \leq 0$ and $-\pi < \theta \leq \pi$

In Exercises 7 and 8, use a calculating utility, where needed, to approximate the polar coordinates of the points whose rectangular coordinates are given.

7. (a) $(4, 3)$ (b) $(2, -5)$ (c) $(1, \tan^{-1} 1)$

8. (a) $(-3, 4)$ (b) $(-3, 1.7)$ (c) $\left(2, \sin^{-1} \frac{1}{2}\right)$

In Exercises 9 and 10, identify the curve by transforming the given polar equation to rectangular coordinates.

9. (a) $r = 2$ (b) $r \sin\theta = 4$
(c) $r = 3\cos\theta$ (d) $r = \dfrac{6}{3\cos\theta + 2\sin\theta}$

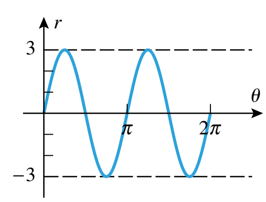
10. (a) $r = 5\sec\theta$ (b) $r = 2\sin\theta$
(c) $r = 4\cos\theta + 4\sin\theta$ (d) $r = \sec\theta\tan\theta$

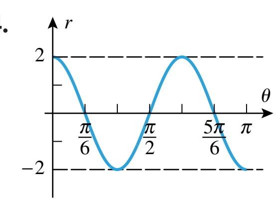
In Exercises 11 and 12, express the given equations in polar coordinates.

11. (a) $x = 7$ (b) $x^2 + y^2 = 9$
(c) $x^2 + y^2 - 6y = 0$ (d) $4xy = 9$

12. (a) $y = -3$ (b) $x^2 + y^2 = 5$
(c) $x^2 + y^2 + 4x = 0$ (d) $x^2(x^2 + y^2) = y^2$

In Exercises 13–16, a graph is given in a rectangular θr-coordinate system. Sketch the corresponding graph in polar coordinates.

13.


14.


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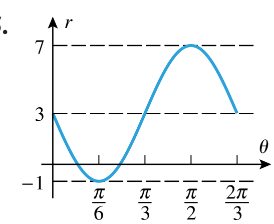
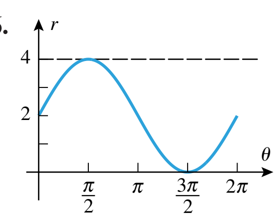
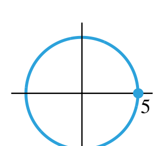
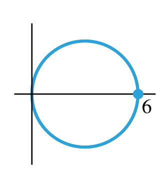


16.


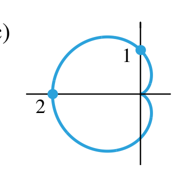
In Exercises 17–20, find an equation for the given polar graph.

17. (a)


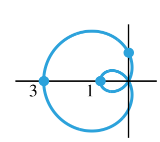
Circle

(b)


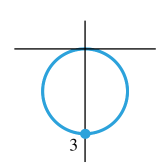
Circle

(c)


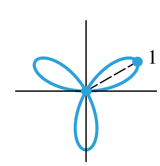
Cardioid

18. (a)


Limaçon

(b)


Circle

(c)


Three-petal rose

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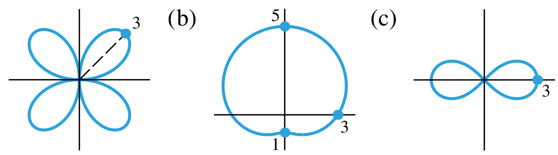
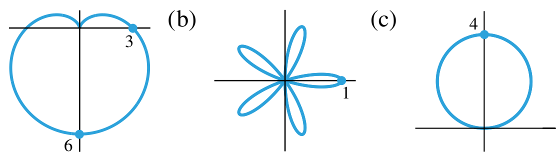
19. (a) (b) (c)



Four-petal rose Limaçon Lemniscate

20. (a) (b) (c)



Cardioid Five-petal rose Circle

In Exercises 21–50, sketch the curve in polar coordinates.

21. $\theta = \dfrac{\pi}{6}$ **22.** $\theta = -\dfrac{3\pi}{4}$

23. $r = 3$ **24.** $r = 4\sin\theta$

25. $r = 6\cos\theta$ **26.** $r = 1 + \sin\theta$

27. $2r = \cos\theta$ **28.** $r - 2 = 2\cos\theta$

29. $r = 3(1 - \sin\theta)$ **30.** $r = -5 + 5\sin\theta$

31. $r = 4 - 4\cos\theta$ **32.** $r = 1 + 2\sin\theta$

33. $r = -1 - \cos\theta$ **34.** $r = 4 + 3\cos\theta$

35. $r = 2 + \sin\theta$ **36.** $r = 3 - \cos\theta$

37. $r = 3 + 4\cos\theta$ **38.** $r - 5 = 3\sin\theta$

39. $r = 5 - 2\cos\theta$ **40.** $r = -3 - 4\sin\theta$

41. $r^2 = 9\cos 2\theta$ **42.** $r^2 = \sin 2\theta$

43. $r^2 = 16\sin 2\theta$ **44.** $r = 4\theta \quad (\theta \ge 0)$

45. $r = 4\theta \quad (\theta \le 0)$ **46.** $r = 4\theta$

47. $r = \cos 2\theta$ **48.** $r = 3\sin 2\theta$

49. $r = 9\sin 4\theta$ **50.** $r = 2\cos 3\theta$

In Exercises 51–55, use a graphing utility to generate the polar graph. Be sure to choose the parameter interval so that a complete graph is generated.

51. $r = \cos\dfrac{\theta}{2}$ **52.** $r = \sin\dfrac{\theta}{2}$

53. $r = 1 + 2\cos\dfrac{\theta}{4}$ **54.** $r = 0.5 + \cos\dfrac{\theta}{3}$

55. $r = \cos\dfrac{\theta}{5}$

56. The accompanying figure shows the graph of the "butterfly curve"

$$r = e^{\cos\theta} - 2\cos 4\theta + \sin^3\frac{\theta}{4}$$

Generate the complete butterfly with a graphing utility, and state the parameter interval you used.

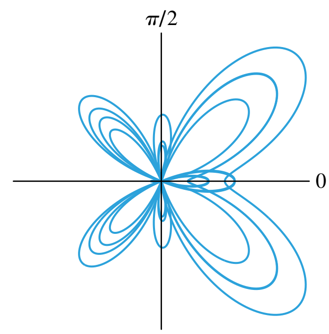
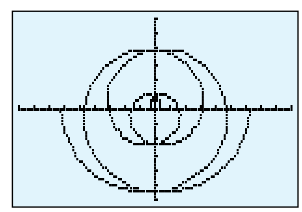


Figure Ex-56

57. The accompanying figure shows the Archimedean spiral $r = \theta/2$ produced with a graphing calculator.
(a) What interval of values for θ do you think was used to generate the graph?
(b) Duplicate the graph with your own graphing utility.



$[-9, 9] \times [-6, 6]$
xScl = 1, yScl = 1

Figure Ex-57

58. The accompanying figure shows graphs of the Archimedean spiral $r = \theta$ and the parabolic spiral $r = \sqrt{\theta}$. Which is which? Explain your reasoning.

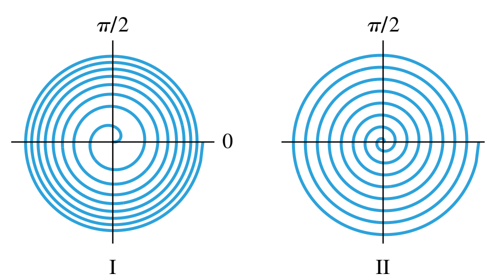


Figure Ex-58

59. (a) Show that if a varies, then the polar equation

$$r = a\sec\theta \quad (-\pi/2 < \theta < \pi/2)$$

describes a family of lines perpendicular to the polar axis.

(b) Show that if b varies, then the polar equation

$$r = b\csc\theta \quad (0 < \theta < \pi)$$

describes a family of lines parallel to the polar axis.

60. Show that if the polar graph of $r = f(\theta)$ is rotated counterclockwise around the origin through an angle α, then $r = f(\theta - \alpha)$ is an equation for the rotated curve. [*Hint:* If (r_0, θ_0) is any point on the original graph, then $(r_0, \theta_0 + \alpha)$ is a point on the rotated graph.]

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61. Use the result in Exercise 60 to find an equation for the cardioid $r = 1 + \cos\theta$ after it has been rotated through the given angle, and check your answer with a graphing utility.

(a) $\dfrac{\pi}{4}$ (b) $\dfrac{\pi}{2}$ (c) π (d) $\dfrac{5\pi}{4}$

62. Use the result in Exercise 60 to find an equation for the lemniscate that results when the lemniscate in Example 8 is rotated counterclockwise through an angle of $\pi/2$.

63. Sketch the polar graph of the equation $(r-1)(\theta-1) = 0$.

64. (a) Show that if A and B are not both zero, then the graph of the polar equation

$$r = A\sin\theta + B\cos\theta$$

is a circle. Find its radius.

(b) Derive Formulas (4) and (5) from the formula given in part (a).

65. Find the highest point on the cardioid $r = 1 + \cos\theta$.

66. Find the leftmost point on the upper half of the cardioid $r = 1 + \cos\theta$.

67. (a) Show that in a polar coordinate system the distance d between the points (r_1, θ_1) and (r_2, θ_2) is

$$d = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}$$

(b) Show that if $0 \le \theta_1 < \theta_2 \le \pi$ and if r_1 and r_2 are positive, then the area A of the triangle with vertices $(0, 0)$, (r_1, θ_1), and (r_2, θ_2) is

$$A = \tfrac{1}{2}r_1r_2\sin(\theta_2 - \theta_1)$$

(c) Find the distance between the points whose polar coordinates are $(3, \pi/6)$ and $(2, \pi/3)$.

(d) Find the area of the triangle whose vertices in polar coordinates are $(0, 0)$, $(1, 5\pi/6)$, and $(2, \pi/3)$.

68. In the late seventeenth century the Italian astronomer Giovanni Domenico Cassini (1625–1712) introduced the family of curves

$$(x^2 + y^2 + a^2)^2 - b^4 - 4a^2x^2 = 0 \quad (a > 0, b > 0)$$

in his studies of the relative motions of the Earth and the Sun. These curves, which are called ***Cassini ovals***, have one of the three basic shapes shown in the accompanying figure.

(a) Show that if $a = b$, then the polar equation of the Cassini oval is $r^2 = 2a^2\cos 2\theta$, which is a lemniscate.

(b) Use the formula in Exercise 67(a) to show that the lemniscate in part (a) is the curve traced by a point that moves in such a way that the product of its distances from the polar points $(a, 0)$ and (a, π) is a^2.

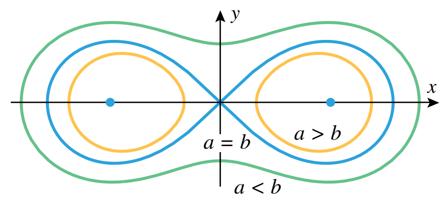

Figure Ex-68

Vertical and horizontal asymptotes of polar curves can often be detected by investigating the behavior of $x = r\cos\theta$ and $y = r\sin\theta$ as θ varies. This idea is used in Exercises 69–72.

69. Show that the ***hyperbolic spiral*** $r = 1/\theta$ $(\theta > 0)$ has a horizontal asymptote at $y = 1$ by showing that $y \to 1$ and $x \to +\infty$ as $\theta \to 0^+$. Confirm this result by generating the spiral with a graphing utility.

70. Show that the spiral $r = 1/\theta^2$ does not have any horizontal asymptotes.

71. (a) Show that the ***kappa curve*** $r = 4\tan\theta$ $(0 \le \theta \le 2\pi)$ has a vertical asymptote at $x = 4$ by showing that $x \to 4$ and $y \to +\infty$ as $\theta \to \pi/2^-$ and that $x \to 4$ and $y \to -\infty$ as $\theta \to \pi/2^+$.

(b) Use the method in part (a) to show that the kappa curve also has a vertical asymptote at $x = -4$.

(c) Confirm the results in parts (a) and (b) by generating the kappa curve with a graphing utility.

72. Use a graphing utility to make a conjecture about the existence of asymptotes for the ***cissoid*** $r = 2\sin\theta\tan\theta$, and then confirm your conjecture by calculating appropriate limits.

73. Prove that a rose with an even number of petals is traced out exactly once as θ varies over the interval $0 \le \theta < 2\pi$ and a rose with an odd number of petals is traced out exactly once as θ varies over the interval $0 \le \theta < \pi$.

11.2 TANGENT LINES AND ARC LENGTH FOR PARAMETRIC AND POLAR CURVES

In this section we will derive the formulas required to find slopes, tangent lines, and arc lengths of parametric and polar curves.

TANGENT LINES TO PARAMETRIC CURVES

We will be concerned in this section with curves that are given by parametric equations

$$x = f(t), \quad y = g(t)$$

in which $f(t)$ and $g(t)$ have continuous first derivatives with respect to t. It can be proved that if $dx/dt \ne 0$, then y is a differentiable function of x, in which case the chain rule implies that

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \tag{1}$$

This formula makes it possible to find dy/dx directly from the parametric equations without eliminating the parameter.

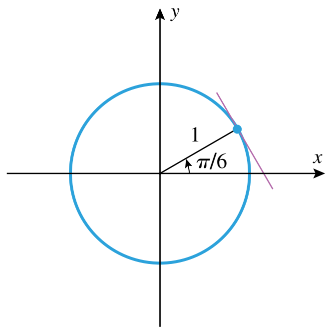

Figure 11.2.1

Example 1 Find the slope of the tangent line to the unit circle

$$x = \cos t, \quad y = \sin t \qquad (0 \le t \le 2\pi)$$

at the point where $t = \pi/6$ (Figure 11.2.1).

Solution. From (1), the slope at a general point on the circle is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos t}{-\sin t} = -\cot t \tag{2}$$

Thus, the slope at $t = \pi/6$ is

$$\left.\frac{dy}{dx}\right|_{t=\pi/6} = -\cot\frac{\pi}{6} = -\sqrt{3}$$ ◀

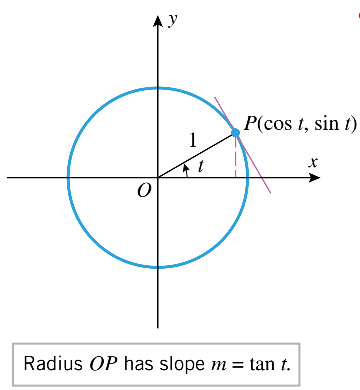

Figure 11.2.2

REMARK. Note that Formula (2) makes sense geometrically because the radius to the point $P(\cos t, \sin t)$ has slope $m = \tan t$; hence, the tangent line at P, being perpendicular to the radius, has slope $-1/m = -1/\tan t = -\cot t$ (Figure 11.2.2).

It follows from Formula (1) that the tangent line to a parametric curve will be horizontal at those points where $dy/dt = 0$ and $dx/dt \neq 0$, since $dy/dx = 0$ at such points. Two different situations occur when $dx/dt = 0$. At points where $dx/dt = 0$ and $dy/dt \neq 0$, the right side of (1) has a nonzero numerator and a zero denominator; we will agree that the curve has ***infinite slope*** and a ***vertical tangent line*** at such points. At points where dx/dt and dy/dt are both zero, the right side of (1) becomes an indeterminate form; we call such points ***singular points***. No general statement can be made about the behavior of parametric curves at singular points; they must be analyzed case by case.

Example 2 In a disastrous first flight, an experimental paper airplane follows the trajectory

$$x = t - 3\sin t, \quad y = 4 - 3\cos t \qquad (t \ge 0)$$

but crashes into a wall at time $t = 10$ (Figure 11.2.3).

(a) At what times was the airplane flying horizontally?

(b) At what times was it flying vertically?

Solution (a). The airplane was flying horizontally at those times when $dy/dt = 0$ and $dx/dt \neq 0$. From the given trajectory we have

$$\frac{dy}{dt} = 3\sin t \quad \text{and} \quad \frac{dx}{dt} = 1 - 3\cos t \tag{3}$$

Setting $dy/dt = 0$ yields the equation $3\sin t = 0$, or, more simply, $\sin t = 0$. This equation has four solutions in the time interval $0 \le t \le 10$:

$$t = 0, \quad t = \pi, \quad t = 2\pi, \quad t = 3\pi$$

Since $dx/dt = 1 - 3\cos t \neq 0$ for these values of t (verify), the airplane was flying horizontally at times

$$t = 0, \quad t = \pi \approx 3.14, \quad t = 2\pi \approx 6.28, \quad \text{and} \quad t = 3\pi \approx 9.42$$

which is consistent with Figure 11.2.3.

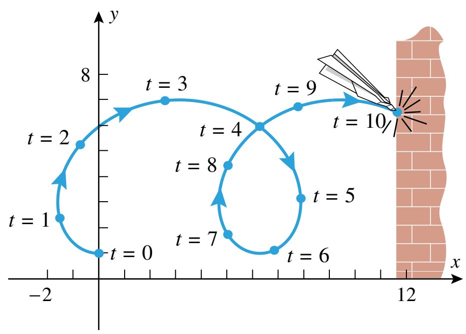

Figure 11.2.3

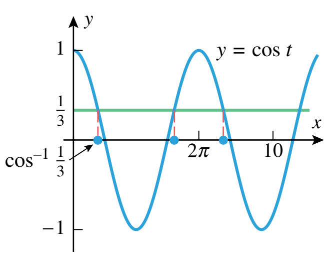

Figure 11.2.4

Solution (b). The airplane was flying vertically at those times when $dx/dt = 0$ and $dy/dt \neq 0$. Setting $dx/dt = 0$ in (3) yields the equation

$$1 - 3\cos t = 0 \quad \text{or} \quad \cos t = \tfrac{1}{3}$$

This equation has three solutions in the time interval $0 \le t \le 10$ (Figure 11.2.4):

$$t = \cos^{-1}\tfrac{1}{3}, \quad t = 2\pi - \cos^{-1}\tfrac{1}{3}, \quad t = 2\pi + \cos^{-1}\tfrac{1}{3}$$

Since $dy/dt = 3\sin t$ is not zero at these points (why?), it follows that the airplane was flying vertically at times

$$t = \cos^{-1}\tfrac{1}{3} \approx 1.23, \quad t \approx 2\pi - 1.23 \approx 5.05, \quad t \approx 2\pi + 1.23 \approx 7.51$$

which again is consistent with Figure 11.2.3. ◀

Example 3 The curve represented by the parametric equations

$$x = t^2, \quad y = t^3 \qquad (-\infty < t < +\infty)$$

is called a ***semicubical parabola***. The parameter t can be eliminated by cubing x and squaring y, from which it follows that $y^2 = x^3$. The graph of this equation, shown in Figure 11.2.5, consists of two branches: an upper branch obtained by graphing $y = x^{3/2}$ and a lower branch obtained by graphing $y = -x^{3/2}$. The two branches meet at the origin, which corresponds to $t = 0$ in the parametric equations. This is a singular point because the derivatives $dx/dt = 2t$ and $dy/dt = 3t^2$ are both zero there. ◀

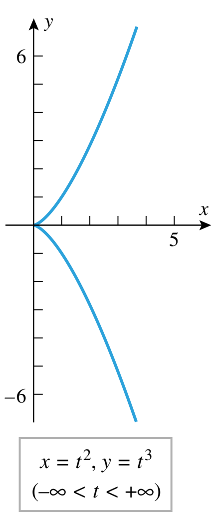

Figure 11.2.5

Example 4 Without eliminating the parameter, find dy/dx and d^2y/dx^2 at the points (1, 1) and (1, −1) on the semicubical parabola given by the parametric equations in Example 3.

Solution. From (1) we have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2}{2t} = \frac{3}{2}t \quad (t \neq 0) \tag{4}$$

and from (1) applied to $y' = dy/dx$ we have

$$\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{dy'/dt}{dx/dt} = \frac{3/2}{2t} = \frac{3}{4t} \tag{5}$$

Since the point (1, 1) on the curve corresponds to $t = 1$ in the parametric equations, it follows from (4) and (5) that

$$\left.\frac{dy}{dx}\right|_{t=1} = \frac{3}{2} \quad \text{and} \quad \left.\frac{d^2y}{dx^2}\right|_{t=1} = \frac{3}{4}$$

Similarly, the point (1, −1) corresponds to $t = -1$ in the parametric equations, so applying (4) and (5) again yields

$$\left.\frac{dy}{dx}\right|_{t=-1} = -\frac{3}{2} \quad \text{and} \quad \left.\frac{d^2y}{dx^2}\right|_{t=-1} = -\frac{3}{4}$$

Note that the values we obtained for the first and second derivatives are consistent with the graph in Figure 11.2.5, since at (1, 1) on the upper branch the tangent line has positive slope and the curve is concave up, and at $(1, -1)$ on the lower branch the tangent line has negative slope and the curve is concave down.

Finally, observe that we were able to apply Formulas (4) and (5) for both $t = 1$ and $t = -1$, even though the points $(1, 1)$ and $(1, -1)$ lie on different branches. In contrast, had we chosen to perform the same computations by eliminating the parameter, we would have had to obtain separate derivative formulas for $y = x^{3/2}$ and $y = -x^{3/2}$. ◀

TANGENT LINES TO POLAR CURVES

Our next objective is to find a method for obtaining slopes of tangent lines to polar curves of the form $r = f(\theta)$ in which r is a differentiable function of θ. We showed in the last section that a curve of this form can be expressed parametrically in terms of the parameter θ by substituting $f(\theta)$ for r in the equations $x = r\cos\theta$ and $y = r\sin\theta$. This yields

$$x = f(\theta)\cos\theta, \qquad y = f(\theta)\sin\theta$$

from which we obtain

$$\begin{aligned} \frac{dx}{d\theta} &= -f(\theta)\sin\theta + f'(\theta)\cos\theta = -r\sin\theta + \frac{dr}{d\theta}\cos\theta \\ \frac{dy}{d\theta} &= f(\theta)\cos\theta + f'(\theta)\sin\theta = r\cos\theta + \frac{dr}{d\theta}\sin\theta \end{aligned} \tag{6}$$

Thus, if $dx/d\theta$ and $dy/d\theta$ are continuous and if $dx/d\theta \neq 0$, then y is a differentiable function of x, and Formula (1) with θ in place of t yields

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r\cos\theta + \sin\theta\,\dfrac{dr}{d\theta}}{-r\sin\theta + \cos\theta\,\dfrac{dr}{d\theta}} \tag{7}$$

Example 5 Find the slope of the tangent line to the circle $r = 4\cos\theta$ at the point where $\theta = \pi/4$.

Solution. From (7) with $r = 4\cos\theta$ we obtain (verify)

$$\frac{dy}{dx} = \frac{4\cos^2\theta - 4\sin^2\theta}{-8\sin\theta\cos\theta} = \frac{4\cos 2\theta}{-4\sin 2\theta} = -\cot 2\theta$$

Thus, at the point where $\theta = \pi/4$ the slope of the tangent line is

$$m = \left.\frac{dy}{dx}\right|_{\theta=\pi/4} = -\cot\frac{\pi}{2} = 0$$

which implies that the circle has a horizontal tangent line at the point where $\theta = \pi/4$ (Figure 11.2.6). ◀

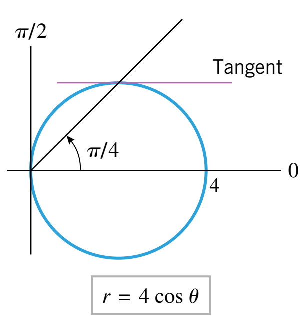

Figure 11.2.6

Example 6 Find the points on the cardioid $r = 1 - \cos\theta$ at which there is a horizontal tangent line, a vertical tangent line, or a singular point.

Solution. A horizontal tangent line will occur where $dy/d\theta = 0$ and $dx/d\theta \neq 0$, a vertical tangent line where $dy/d\theta \neq 0$ and $dx/d\theta = 0$, and a singular point where $dy/d\theta = 0$ and $dx/d\theta = 0$. We could find these derivatives from the formulas in (6). However, an alternative approach is to go back to basic principles and express the cardioid parametrically by substituting $r = 1 - \cos\theta$ in the conversion formulas $x = r\cos\theta$ and $y = r\sin\theta$. This yields

$$x = (1 - \cos\theta)\cos\theta, \quad y = (1 - \cos\theta)\sin\theta \qquad (0 \le \theta \le 2\pi)$$

Differentiating these equations with respect to θ and then simplifying yields (verify)

$$\frac{dx}{d\theta} = \sin\theta(2\cos\theta - 1), \qquad \frac{dy}{d\theta} = (1 - \cos\theta)(1 + 2\cos\theta)$$

Thus, $dx/d\theta = 0$ if $\sin\theta = 0$ or $\cos\theta = \frac{1}{2}$, and $dy/d\theta = 0$ if $\cos\theta = 1$ or $\cos\theta = -\frac{1}{2}$. We leave it for you to solve these equations and show that the solutions of $dx/d\theta = 0$ on the interval $0 \le \theta \le 2\pi$ are

$$\frac{dx}{d\theta} = 0: \quad \theta = 0,\ \frac{\pi}{3},\ \pi,\ \frac{5\pi}{3},\ 2\pi$$

and the solutions of $dy/d\theta = 0$ on the interval $0 \le \theta \le 2\pi$ are

$$\frac{dy}{d\theta} = 0: \quad \theta = 0,\ \frac{2\pi}{3},\ \frac{4\pi}{3},\ 2\pi$$

Thus, horizontal tangent lines occur at $\theta = 2\pi/3$ and $\theta = 4\pi/3$; vertical tangent lines occur at $\theta = \pi/3$, π, and $5\pi/3$; and singular points occur at $\theta = 0$ and $\theta = 2\pi$ (Figure 11.2.7). Note, however, that $r = 0$ at both singular points, so there is really only one singular point on the cardioid—the pole. ◀

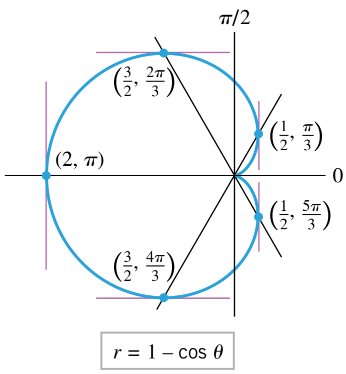

Figure 11.2.7

TANGENT LINES TO POLAR CURVES AT THE ORIGIN

Formula (7) reveals some useful information about the behavior of a polar curve $r = f(\theta)$ that passes through the origin. If we assume that $r = 0$ and $dr/d\theta \ne 0$ when $\theta = \theta_0$, then it follows from Formula (7) that the slope of the tangent line to the curve at $\theta = \theta_0$ is

$$\frac{dy}{dx} = \frac{0 + \sin\theta_0 \dfrac{dr}{d\theta}}{0 + \cos\theta_0 \dfrac{dr}{d\theta}} = \frac{\sin\theta_0}{\cos\theta_0} = \tan\theta_0$$

(Figure 11.2.8). However, $\tan\theta_0$ is also the slope of the line $\theta = \theta_0$, so we can conclude that this line is tangent to the curve at the origin. Thus, we have established the following result.

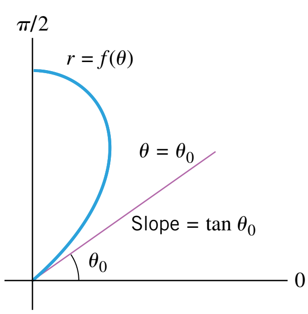

Figure 11.2.8

11.2.1 THEOREM. *If the polar curve $r = f(\theta)$ passes through the origin at $\theta = \theta_0$, and if $dr/d\theta \ne 0$ at $\theta = \theta_0$, then the line $\theta = \theta_0$ is tangent to the curve at the origin.*

This theorem tells us that equations of the tangent lines at the origin to the curve $r = f(\theta)$ can be obtained by solving the equation $f(\theta) = 0$. It is important to keep in mind, however, that $r = f(\theta)$ may be zero for more than one value of θ, so there may be more than one tangent line at the origin. This is illustrated in the next example.

Example 7 The three-petal rose $r = \sin 3\theta$ in Figure 11.2.9 has three tangent lines at the origin, which can be found by solving the equation

$$\sin 3\theta = 0$$

It was shown in Exercise 73 of Section 11.1 that the complete rose is traced once as θ varies over the interval $0 \le \theta < \pi$, so we need only look for solutions in this interval. We leave it for you to confirm that these solutions are

$$\theta = 0, \quad \theta = \frac{\pi}{3}, \quad \text{and} \quad \theta = \frac{2\pi}{3}$$

Since $dr/d\theta = 3\cos 3\theta \ne 0$ for these values of θ, these three lines are tangent to the rose at the origin, which is consistent with the figure. ◀

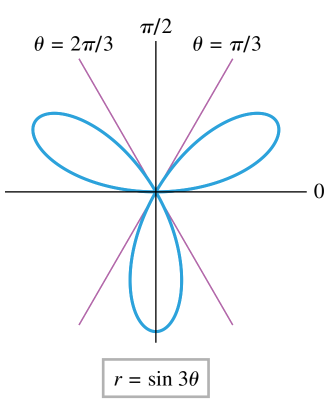

Figure 11.2.9

ARC LENGTH OF A POLAR CURVE

A formula for the arc length of a polar curve $r = f(\theta)$ can be derived by expressing the curve in parametric form and applying Formula (6) of Section 6.4 for the arc length of a parametric curve. We leave it as an exercise to show the following.

11.2.2 ARC LENGTH FORMULA FOR POLAR CURVES. If no segment of the polar curve $r = f(\theta)$ is traced more than once as θ increases from α to β, and if $dr/d\theta$ is continuous for $\alpha \le \theta \le \beta$, then the arc length L from $\theta = \alpha$ to $\theta = \beta$ is

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \tag{8}$$

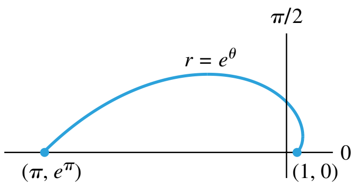

Figure 11.2.10

Example 8 Find the arc length of the spiral $r = e^{\theta}$ in Figure 11.2.10 between $\theta = 0$ and $\theta = \pi$.

Solution.

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{\pi} \sqrt{(e^{\theta})^2 + (e^{\theta})^2}\, d\theta$$

$$= \int_0^{\pi} \sqrt{2}\, e^{\theta}\, d\theta = \sqrt{2}\, e^{\theta}\Big]_0^{\pi} = \sqrt{2}(e^{\pi} - 1) \approx 31.3 \qquad \blacktriangleleft$$

Example 9 Find the total arc length of the cardioid $r = 1 + \cos\theta$.

Solution. The cardioid is traced out once as θ varies from $\theta = 0$ to $\theta = 2\pi$. Thus,

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{(1 + \cos\theta)^2 + (-\sin\theta)^2}\, d\theta$$

$$= \sqrt{2} \int_0^{2\pi} \sqrt{1 + \cos\theta}\, d\theta$$

$$= 2 \int_0^{2\pi} \sqrt{\cos^2 \frac{1}{2}\theta}\, d\theta$$

Identity (45) of Appendix E

$$= 2 \int_0^{2\pi} \left|\cos \frac{1}{2}\theta\right| d\theta$$

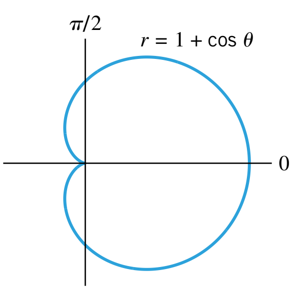

Figure 11.2.11

Since $\cos \frac{1}{2}\theta$ changes sign at π, we must split the last integral into the sum of two integrals: the integral from 0 to π plus the integral from π to 2π. However, the integral from π to 2π is equal to the integral from 0 to π, since the cardioid is symmetric about the polar axis (Figure 11.2.11). Thus,

$$L = 2\int_0^{2\pi} \left|\cos \frac{1}{2}\theta\right| d\theta = 4\int_0^{\pi} \cos \frac{1}{2}\theta\, d\theta = 8 \sin \frac{1}{2}\theta\Big]_0^{\pi} = 8 \qquad \blacktriangleleft$$

EXERCISE SET 11.2 Graphing Utility

1. (a) Find the slope of the tangent line to the parametric curve $x = t^2 + 1$, $y = t/2$ at $t = -1$ and at $t = 1$ without eliminating the parameter.
   (b) Check your answers in part (a) by eliminating the parameter and differentiating an appropriate function of x.
2. (a) Find the slope of the tangent line to the parametric curve $x = 3\cos t$, $y = 4\sin t$ at $t = \pi/4$ and at $t = 7\pi/4$ without eliminating the parameter.
   (b) Check your answers in part (a) by eliminating the parameter and differentiating an appropriate function of x.
3. For the parametric curve in Exercise 1, make a conjecture about the sign of d^2y/dx^2 at $t = -1$ and at $t = 1$, and confirm your conjecture without eliminating the parameter.
4. For the parametric curve in Exercise 2, make a conjecture about the sign of d^2y/dx^2 at $t = \pi/4$ and at $t = 7\pi/4$, and confirm your conjecture without eliminating the parameter.

In Exercises 5–10, find dy/dx and d^2y/dx^2 at the given point without eliminating the parameter.

5. $x = \sqrt{t}$, $y = 2t + 4$; $t = 1$
6. $x = \frac{1}{2}t^2$, $y = \frac{1}{3}t^3$; $t = 2$
7. $x = \sec t$, $y = \tan t$; $t = \pi/3$
8. $x = \sinh t$, $y = \cosh t$; $t = 0$
9. $x = 2\theta + \cos\theta$, $y = 1 - \sin\theta$; $\theta = \pi/3$
10. $x = \cos\phi$, $y = 3\sin\phi$; $\phi = 5\pi/6$
11. (a) Find the equation of the tangent line to the curve

$$x = e^t, \quad y = e^{-t}$$

   at $t = 1$ without eliminating the parameter.
   (b) Check your answer in part (a) by eliminating the parameter.

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12. (a) Find the equation of the tangent line to the curve

$$x = 2t + 4, \quad y = 8t^2 - 2t + 4$$

at $t = 1$ without eliminating the parameter.

(b) Check your answer in part (a) by eliminating the parameter.

> In Exercises 13 and 14, find all values of t at which the parametric curve has (a) a horizontal tangent line and (b) a vertical tangent line.

13. $x = 2\cos t, \;\; y = 4\sin t \quad (0 \le t \le 2\pi)$

14. $x = 2t^3 - 15t^2 + 24t + 7, \;\; y = t^2 + t + 1$

15. As shown in the accompanying figure, the Lissajous curve

$$x = \sin t, \quad y = \sin 2t \qquad (0 \le t \le 2\pi)$$

crosses itself at the origin. Find equations for the two tangent lines at the origin.

16. As shown in the accompanying figure, the ***prolate cycloid***

$$x = 2 - \pi\cos t, \quad y = 2t - \pi\sin t \qquad (-\pi \le t \le \pi)$$

crosses itself at a point on the x-axis. Find equations for the two tangent lines at that point.

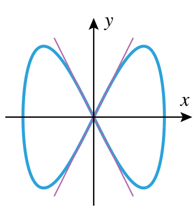

Figure Ex-15

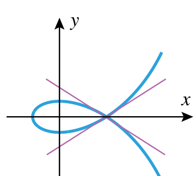

Figure Ex-16

17. Show that the curve $x = t^3 - 4t$, $y = t^2$ intersects itself at the point $(0, 4)$, and find equations for the two tangent lines to the curve at the point of intersection.

18. Show that the curve with parametric equations

$$x = t^2 - 3t + 5, \quad y = t^3 + t^2 - 10t + 9$$

intersects itself at the point $(3, 1)$, and find equations for the two tangent lines to the curve at the point of intersection.

19. (a) Use a graphing utility to generate the graph of the parametric curve

$$x = \cos^3 t, \quad y = \sin^3 t \qquad (0 \le t \le 2\pi)$$

and make a conjecture about the values of t at which singular points occur.

(b) Confirm your conjecture in part (a) by calculating appropriate derivatives.

20. (a) At what values of θ would you expect the cycloid in Figure 1.8.13 to have singular points?

(b) Confirm your answer in part (a) by calculating appropriate derivatives.

> In Exercises 21–26, find the slope of the tangent line to the polar curve for the given value of θ.

21. $r = 2\cos\theta;\;\; \theta = \pi/3$

22. $r = 1 + \sin\theta;\;\; \theta = \pi/4$

23. $r = 1/\theta;\;\; \theta = 2$

24. $r = a\sec 2\theta;\;\; \theta = \pi/6$

25. $r = \cos 3\theta;\;\; \theta = 3\pi/4$

26. $r = 4 - 3\sin\theta;\;\; \theta = \pi$

> In Exercises 27 and 28, calculate the slopes of the tangent lines indicated in the accompanying figures.

27. $r = 2 + 2\sin\theta$

28. $r = 1 - 2\sin\theta$

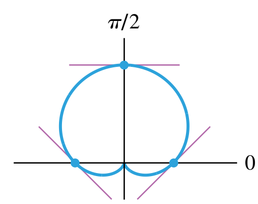

Figure Ex-27

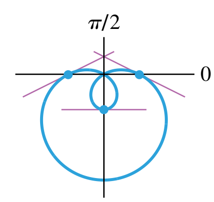

Figure Ex-28

> In Exercises 29 and 30, find polar coordinates of all points at which the polar curve has a horizontal or a vertical tangent line.

29. $r = a(1 + \cos\theta)$

30. $r = a\sin\theta$

> In Exercises 31 and 32, use a graphing utility to make a conjecture about the number of points on the polar curve at which there is a horizontal tangent line, and confirm your conjecture by finding appropriate derivatives.

31. $r = \sin\theta\cos^2\theta$

32. $r = 1 - 2\sin\theta$

> In Exercises 33–38, sketch the polar curve and find polar equations of the tangent lines to the curve at the pole.

33. $r = 2\cos 3\theta$

34. $r = 4\cos\theta$

35. $r = 4\sqrt{\cos 2\theta}$

36. $r = \sin 2\theta$

37. $r = 1 + 2\cos\theta$

38. $r = 2\theta$

> In Exercises 39–44, use Formula (8) to calculate the arc length of the polar curve.

39. The entire circle $r = a$

40. The entire circle $r = 2a\cos\theta$

41. The entire cardioid $r = a(1 - \cos\theta)$

42. $r = \sin^2(\theta/2)$ from $\theta = 0$ to $\theta = \pi$

43. $r = e^{3\theta}$ from $\theta = 0$ to $\theta = 2$

44. $r = \sin^3(\theta/3)$ from $\theta = 0$ to $\theta = \pi/2$

45. (a) What is the slope of the tangent line at time t to the trajectory of the paper airplane in Example 2?

(b) What was the airplane's approximate angle of inclination when it crashed into the wall?

46. Suppose that a bee follows the trajectory

$$x = t - 2\sin t, \quad y = 2 - 2\cos t \qquad (t \geq 0)$$

but lands on a wall at time $t = 10$.
(a) At what times was the bee flying horizontally?
(b) At what times was the bee flying vertically?

47. (a) Show that the arc length of one petal of the rose $r = \cos n\theta$ is given by

$$2\int_0^{\pi/(2n)} \sqrt{1 + (n^2 - 1)\sin^2 n\theta}\, d\theta$$

(b) Use the numerical integration capability of a calculating utility to approximate the arc length of one petal of the four-petal rose $r = \cos 2\theta$.
(c) Use the numerical integration capability of a calculating utility to approximate the arc length of one petal of the n-petal rose $r = \cos n\theta$ for $n = 2, 3, 4, \ldots, 20$; then make a conjecture about the limit of these arc lengths as $n \to +\infty$.

48. (a) Sketch the spiral $r = e^{-\theta}$ $(0 \leq \theta < +\infty)$.
(b) Find an improper integral for the total arc length of the spiral.
(c) Show that the integral converges and find the total arc length of the spiral.

Exercises 49–54 require the formulas developed in the following discussion: If $f'(t)$ and $g'(t)$ are continuous functions and if no segment of the curve

$$x = f(t), \quad y = g(t) \qquad (a \leq t \leq b)$$

is traced more than once, then it can be shown that the area of the surface generated by revolving this curve about the x-axis is

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

and the area of the surface generated by revolving the curve about the y-axis is

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

[The derivations are similar to those used to obtain Formulas (4) and (5) in Section 6.5.]

49. Find the area of the surface generated by revolving $x = t^2$, $y = 2t$ $(0 \leq t \leq 4)$ about the x-axis.

50. Find the area of the surface generated by revolving the curve $x = e^t \cos t$, $y = e^t \sin t$ $(0 \leq t \leq \pi/2)$ about the x-axis.

51. Find the area of the surface generated by revolving the curve $x = \cos^2 t$, $y = \sin^2 t$ $(0 \leq t \leq \pi/2)$ about the y-axis.

52. Find the area of the surface generated by revolving $x = t$, $y = 2t^2$ $(0 \leq t \leq 1)$ about the y-axis.

53. By revolving the semicircle

$$x = r\cos t, \quad y = r\sin t \qquad (0 \leq t \leq \pi)$$

about the x-axis, show that the surface area of a sphere of radius r is $4\pi r^2$.

54. The equations

$$x = a\phi - a\sin\phi, \quad y = a - a\cos\phi \quad (0 \leq \phi \leq 2\pi)$$

represent one arch of a cycloid. Show that the surface area generated by revolving this curve about the x-axis is given by $S = 64\pi a^2/3$.

55. As illustrated in the accompanying figure, suppose that a rod with one end fixed at the pole of a polar coordinate system rotates counterclockwise at the constant rate of 1 rad/s. At time $t = 0$ a bug on the rod is 10 mm from the pole and is moving outward along the rod at the constant speed of 2 mm/s.
(a) Find an equation of the form $r = f(\theta)$ for the path of motion of the bug, assuming that $\theta = 0$ when $t = 0$.
(b) Find the distance the bug travels along the path in part (a) during the first 5 seconds. Round your answer to the nearest tenth of a millimeter.

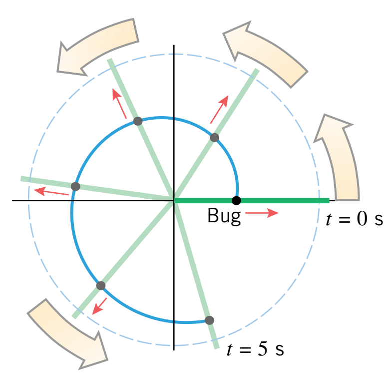

Figure Ex-55

56. Use Formula (6) of Section 6.4 to derive Formula (8).

11.3 AREA IN POLAR COORDINATES

In this section we will show how to find areas of regions that are bounded by polar curves.

AREA IN POLAR COORDINATES

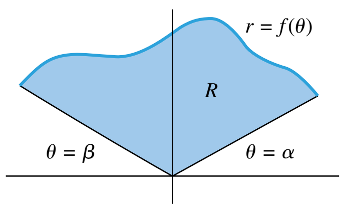

Figure 11.3.1

11.3.1 AREA PROBLEM IN POLAR COORDINATES. Suppose that α and β are angles that satisfy the condition

$$\alpha < \beta \leq \alpha + 2\pi$$

and suppose that $f(\theta)$ is continuous for $\alpha \leq \theta \leq \beta$. Find the area of the region R enclosed by the polar curve $r = f(\theta)$ and the rays $\theta = \alpha$ and $\theta = \beta$ (Figure 11.3.1).

In rectangular coordinates we solved Area Problem 5.1.1 by dividing the region into an increasing number of vertical strips, approximating the strips by rectangles, and taking a limit. In polar coordinates rectangles are clumsy to work with, and it is better to divide the region into ***wedges*** by using rays

$$\theta = \theta_1, \; \theta = \theta_2, \ldots, \; \theta = \theta_{n-1}$$

such that

$$\alpha < \theta_1 < \theta_2 < \cdots < \theta_{n-1} < \beta$$

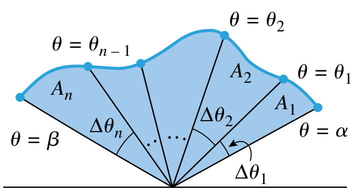

Figure 11.3.2

(Figure 11.3.2). As shown in that figure, the rays divide the region R into n wedges with areas $A_1, A_2, \ldots, A_n$ and central angles $\Delta\theta_1, \Delta\theta_2, \ldots, \Delta\theta_n$. The area of the entire region can be written as

$$A = A_1 + A_2 + \cdots + A_n = \sum_{k=1}^{n} A_k \tag{1}$$

If $\Delta\theta_k$ is small, and if we assume for simplicity that $f(\theta)$ is nonnegative, then we can approximate the area A_k of the kth wedge by the area of a sector with central angle $\Delta\theta_k$ and radius $f(\theta_k^*)$, where $\theta = \theta_k^*$ is any ray that lies in the kth wedge (Figure 11.3.3). Thus, from (1) and Formula (5) of Appendix E for the area of a sector, we obtain

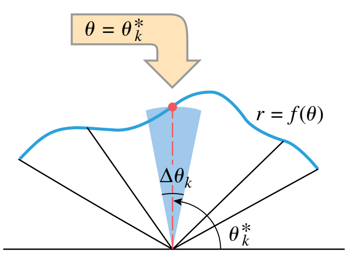

Figure 11.3.3

$$A = \sum_{k=1}^{n} A_k \approx \sum_{k=1}^{n} \frac{1}{2}[f(\theta_k^*)]^2 \Delta\theta_k \tag{2}$$

If we now increase n in such a way that $\max \Delta\theta_k \to 0$, then the sectors will become better and better approximations of the wedges and it is reasonable to expect that (2) will approach the exact value of the area A (Figure 11.3.4); that is,

$$A = \lim_{\max \Delta\theta_k \to 0} \sum_{k=1}^{n} \frac{1}{2}[f(\theta_k^*)]^2 \Delta\theta_k = \int_{\alpha}^{\beta} \frac{1}{2}[f(\theta)]^2 \, d\theta$$

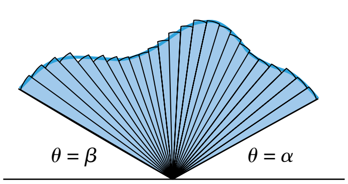

Figure 11.3.4

Thus, we have the following solution of Area Problem 11.3.1.

11.3.2 AREA IN POLAR COORDINATES. If α and β are angles that satisfy the condition

$$\alpha < \beta \leq \alpha + 2\pi$$

and if $f(\theta)$ is continuous for $\alpha \leq \theta \leq \beta$, then the area A of the region R enclosed by the polar curve $r = f(\theta)$ and the rays $\theta = \alpha$ and $\theta = \beta$ is

$$A = \int_{\alpha}^{\beta} \frac{1}{2}[f(\theta)]^2 \, d\theta = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta \tag{3}$$

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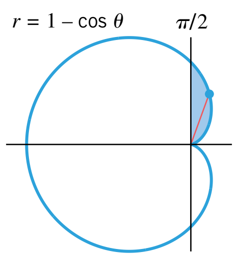
The hardest part of applying (3) is determining the limits of integration. This can be done as follows:

Step 1. Sketch the region R whose area is to be determined.

Step 2. Draw an arbitrary "radial line" from the pole to the boundary curve $r = f(\theta)$.

Step 3. Ask, "Over what interval of values must θ vary in order for the radial line to sweep out the region R?"

Step 4. Your answer in Step 3 will determine the lower and upper limits of integration.



The shaded region is swept out by the radial line as θ varies from 0 to $\pi/2$.

Figure 11.3.5

Example 1 Find the area of the region in the first quadrant that is within the cardioid $r = 1 - \cos\theta$.

Solution. The region and a typical radial line are shown in Figure 11.3.5. For the radial line to sweep out the region, θ must vary from 0 to $\pi/2$. Thus, from (3) with $\alpha = 0$ and $\beta = \pi/2$, we obtain

$$A = \int_0^{\pi/2} \frac{1}{2}r^2\,d\theta = \frac{1}{2}\int_0^{\pi/2} (1-\cos\theta)^2\,d\theta = \frac{1}{2}\int_0^{\pi/2} (1 - 2\cos\theta + \cos^2\theta)\,d\theta$$

With the help of the identity $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$, this can be rewritten as

$$A = \frac{1}{2}\int_0^{\pi/2}\left(\frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos 2\theta\right)d\theta = \frac{1}{2}\left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{\pi/2} = \frac{3}{8}\pi - 1 \quad \blacktriangleleft$$

Example 2 Find the entire area within the cardioid of Example 1.

Solution. For the radial line to sweep out the entire cardioid, θ must vary from 0 to 2π. Thus, from (3) with $\alpha = 0$ and $\beta = 2\pi$,

$$A = \int_0^{2\pi} \frac{1}{2}r^2\,d\theta = \frac{1}{2}\int_0^{2\pi} (1-\cos\theta)^2\,d\theta$$

If we proceed as in Example 1, this reduces to

$$A = \frac{1}{2}\int_0^{2\pi}\left(\frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos 2\theta\right)d\theta = \frac{3\pi}{2}$$

Alternative Solution. Since the cardioid is symmetric about the x-axis, we can calculate the portion of the area above the x-axis and double the result. In the portion of the cardioid above the x-axis, θ ranges from 0 to π, so that

$$A = 2\int_0^{\pi} \frac{1}{2}r^2\,d\theta = \int_0^{\pi} (1-\cos\theta)^2\,d\theta = \frac{3\pi}{2} \quad \blacktriangleleft$$

USING SYMMETRY

Although Formula (3) is applicable if $r = f(\theta)$ is negative, area computations can sometimes be simplified by using symmetry to restrict the limits of integration to intervals where $r \geq 0$. This is illustrated in the next example.

Example 3 Find the area of the region enclosed by the rose curve $r = \cos 2\theta$.

Solution. Referring to Figure 11.1.10 and using symmetry, the area in the first quadrant that is swept out for $0 \leq \theta \leq \pi/4$ is one-eighth of the total area inside the rose. Thus, from

Formula (3)

$$A = 8\int_0^{\pi/4} \frac{1}{2}r^2\,d\theta = 4\int_0^{\pi/4} \cos^2 2\theta\,d\theta$$
$$= 4\int_0^{\pi/4} \frac{1}{2}(1+\cos 4\theta)\,d\theta = 2\int_0^{\pi/4}(1+\cos 4\theta)\,d\theta$$
$$= 2\theta + \frac{1}{2}\sin 4\theta\Big]_0^{\pi/4} = \frac{\pi}{2}$$ ◀

Sometimes the most natural way to satisfy the restriction $\alpha < \beta \le \alpha + 2\pi$ required by Formula (3) is to use a negative value for α. For example, suppose that we are interested in finding the area of the shaded region in Figure 11.3.6*a*. The first step would be to determine the intersections of the cardioid $r = 4 + 4\cos\theta$ and the circle $r = 6$, since this information is needed for the limits of integration. To find the points of intersection, we can equate the two expressions for r. This yields

$$4 + 4\cos\theta = 6 \quad \text{or} \quad \cos\theta = \frac{1}{2}$$

which is satisfied by the positive angles

$$\theta = \frac{\pi}{3} \quad \text{and} \quad \theta = \frac{5\pi}{3}$$

However, there is a problem here because the radial lines to the circle and cardioid do not sweep through the shaded region shown in Figure 11.3.6*b* as θ varies over the interval $\pi/3 \le \theta \le 5\pi/3$. There are two ways to circumvent this problem—one is to take advantage of the symmetry by integrating over the interval $0 \le \theta \le \pi/3$ and doubling the result, and the second is to use a negative lower limit of integration and integrate over the interval $-\pi/3 \le \theta \le \pi/3$ (Figure 11.3.6*c*). The two methods are illustrated in the next example.

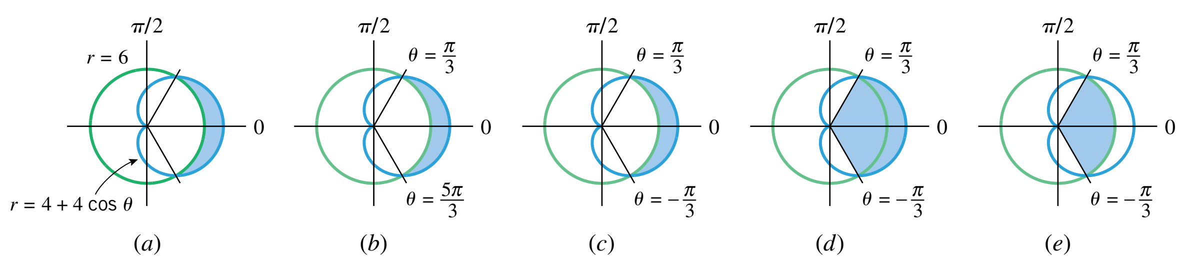


Figure 11.3.6

Example 4 Find the area of the region that is inside of the cardioid $r = 4 + 4\cos\theta$ and outside of the circle $r = 6$.

Solution Using a Negative Angle. The area of the region can be obtained by subtracting the areas in Figures 11.3.6*d* and 11.3.6*e*:

$$A = \int_{-\pi/3}^{\pi/3} \frac{1}{2}(4+4\cos\theta)^2\,d\theta - \int_{-\pi/3}^{\pi/3}\frac{1}{2}(6)^2\,d\theta$$ Area inside cardioid minus area inside circle.

$$= \int_{-\pi/3}^{\pi/3}\frac{1}{2}[(4+4\cos\theta)^2 - 36]\,d\theta = \int_{-\pi/3}^{\pi/3}(16\cos\theta + 8\cos^2\theta - 10)\,d\theta$$
$$= \big[16\sin\theta + (4\theta + 2\sin 2\theta) - 10\,\theta\big]_{-\pi/3}^{\pi/3} = 18\sqrt{3} - 4\pi$$

Solution Using Symmetry. Using symmetry, we can calculate the area above the polar axis and double it. This yields (verify)

$$A = 2\int_0^{\pi/3}\frac{1}{2}[(4+4\cos\theta)^2 - 36]\,d\theta = 2(9\sqrt{3} - 2\pi) = 18\sqrt{3} - 4\pi$$

which agrees with the preceding result. ◀

INTERSECTIONS OF POLAR GRAPHS

In the last example we found the intersections of the cardioid and circle by equating their expressions for r and solving for θ. However, because a point can be represented in different ways in polar coordinates, this procedure will not always produce all of the intersections. For example, the cardioids

$$r = 1 - \cos\theta \quad \text{and} \quad r = 1 + \cos\theta \tag{4}$$

intersect at three points: the pole, the point $(1, \pi/2)$, and the point $(1, 3\pi/2)$ (Figure 11.3.7). Equating the right-hand sides of the equations in (4) yields $1 - \cos\theta = 1 + \cos\theta$ or $\cos\theta = 0$, so

$$\theta = \frac{\pi}{2} + k\pi, \quad k = 0, \pm 1, \pm 2, \ldots$$

Substituting any of these values in (4) yields $r = 1$, so that we have found only two distinct points of intersection, $(1, \pi/2)$ and $(1, 3\pi/2)$; the pole has been missed. This problem occurs because the two cardioids pass through the pole at different values of θ—the cardioid $r = 1 - \cos\theta$ passes through the pole at $\theta = 0$, and the cardioid $r = 1 + \cos\theta$ passes through the pole at $\theta = \pi$.

The situation with the cardioids is analogous to two satellites circling the Earth in intersecting orbits (Figure 11.3.8). The satellites will not collide unless they reach the same point at the same time. In general, when looking for intersections of polar curves, it is a good idea to graph the curves to determine how many intersections there should be.

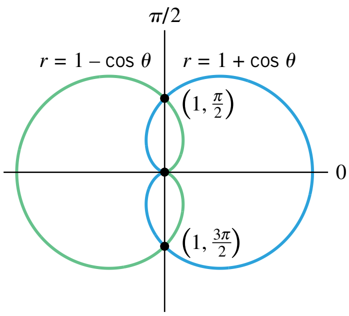
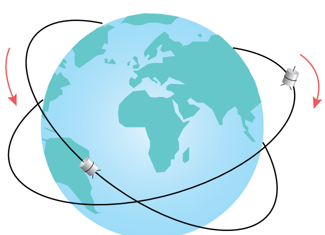


Figure 11.3.7



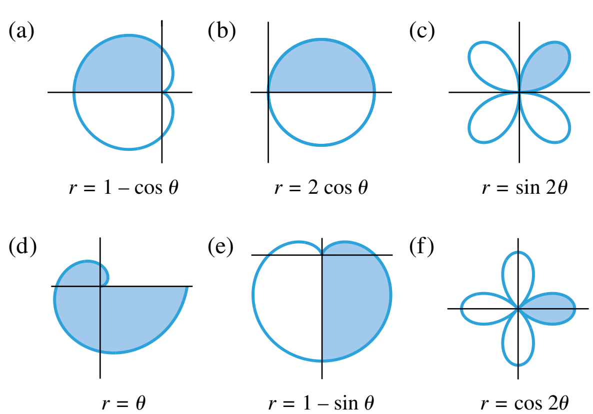
The orbits intersect, but the satellites do not collide.

Figure 11.3.8

Exercise Set 11.3

Graphing Utility   C CAS

1. Write down, but do not evaluate, an integral for the area of each shaded region.



2. Evaluate the integrals you obtained in Exercise 1.

3. In each part, find the area of the circle by integration.
   (a) $r = a$ (b) $r = 2a\sin\theta$ (c) $r = 2a\cos\theta$

4. (a) Show that $r = \sin\theta + \cos\theta$ is a circle.
   (b) Find the area of the circle using a geometric formula and then by integration.

In Exercises 5–10, find the area of the region described.

5. The region that is enclosed by the cardioid $r = 2 + 2\cos\theta$.

6. The region in the first quadrant within the cardioid $r = 1 + \sin\theta$.

7. The region enclosed by the rose $r = 4\cos 3\theta$.

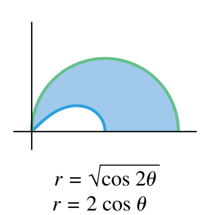
8. The region enclosed by the rose $r = 2\sin 2\theta$.

9. The region enclosed by the inner loop of the limaçon $r = 1 + 2\cos\theta$. [*Hint:* $r \le 0$ over the interval of integration.]

10. The region swept out by a radial line from the pole to the curve $r = 2/\theta$ as θ varies over the interval $1 \le \theta \le 3$.

In Exercises 11–14, find the area of the shaded region.

11.



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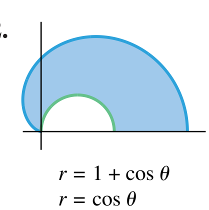

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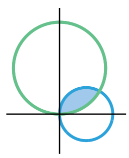
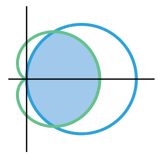


$r = 4\cos t$
$r = 4\sqrt{3}\sin t$

14.



$r = 1 + \cos t$
$r = 3\cos t$

In Exercises 15–22, find the area of the region described.

15. The region inside the circle $r = 5\sin\theta$ and outside the limaçon $r = 2 + \sin\theta$.

16. The region outside the cardioid $r = 2 - 2\cos\theta$ and inside the circle $r = 4$.

17. The region inside the cardioid $r = 2 + 2\cos\theta$ and outside the circle $r = 3$.

18. The region that is common to the circles $r = 4\cos\theta$ and $r = 4\sin\theta$.

19. The region between the loops of the limaçon $r = \frac{1}{2} + \cos\theta$.

20. The region inside the cardioid $r = 2 + 2\cos\theta$ and to the right of the line $r\cos\theta = \frac{3}{2}$.

21. The region inside the circle $r = 10$ and to the right of the line $r = 6\sec\theta$.

22. The region inside the rose $r = 2a\cos 2\theta$ and outside the circle $r = a\sqrt{2}$.

23. (a) Find the error: The area that is inside the lemniscate $r^2 = a^2\cos 2\theta$ is

$$A = \int_0^{2\pi} \frac{1}{2}r^2\,d\theta = \int_0^{2\pi} \frac{1}{2}a^2\cos 2\theta\,d\theta$$
$$= \frac{1}{4}a^2\sin 2\theta\Big]_0^{2\pi} = 0$$

(b) Find the correct area.
(c) Find the area inside the lemniscate $r^2 = 4\cos 2\theta$ and outside the circle $r = \sqrt{2}$.

24. Find the area inside the curve $r^2 = \sin 2\theta$.

25. A radial line is drawn from the origin to the spiral $r = a\theta$ ($a > 0$ and $\theta \geq 0$). Find the area swept out during the second revolution of the radial line that was not swept out during the first revolution.

26. (a) In the discussion associated with Exercises 49–54 of Section 11.2, formulas were given for the area of the surface of revolution that is generated by revolving a parametric curve about the x-axis or y-axis. Use those formulas to derive the following formulas for the areas of the surfaces of revolution that are generated by revolving the portion of the polar curve $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ about the polar axis and about the line $\theta = \pi/2$:

$$S = \int_\alpha^\beta 2\pi r\sin\theta\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta \quad \boxed{\text{About } \theta = 0}$$

$$S = \int_\alpha^\beta 2\pi r\cos\theta\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta \quad \boxed{\text{About } \theta = \pi/2}$$

(b) State conditions under which these formulas hold.

In Exercises 27–30, sketch the surface, and use the formulas in Exercise 26 to find the surface area.

27. The surface generated by revolving the circle $r = \cos\theta$ about the line $\theta = \pi/2$.

28. The surface generated by revolving the spiral $r = e^\theta$ ($0 \leq \theta \leq \pi/2$) about the line $\theta = \pi/2$.

29. The "apple" generated by revolving the upper half of the cardioid $r = 1 - \cos\theta$ ($0 \leq \theta \leq \pi$) about the polar axis.

30. The sphere of radius a generated by revolving the semicircle $r = a$ in the upper half-plane about the polar axis.

C **31.** (a) Show that the Folium of Descartes $x^3 - 3xy + y^3 = 0$ can be expressed in polar coordinates as

$$r = \frac{3\sin\theta\cos\theta}{\cos^3\theta + \sin^3\theta}$$

(b) Use a CAS to show that the area inside of the loop is $\frac{3}{2}$ (Figure 3.6.2).

C **32.** (a) What is the area that is enclosed by one petal of the rose $r = a\cos n\theta$ if n is an even integer?
(b) What is the area that is enclosed by one petal of the rose $r = a\cos n\theta$ if n is an odd integer?
(c) Use a CAS to show that the total area enclosed by the rose $r = a\cos n\theta$ is $\pi a^2/2$ if the number of petals is even. [*Hint:* See Exercise 73 of Section 11.1.]
(d) Use a CAS to show that the total area enclosed by the rose $r = a\cos n\theta$ is $\pi a^2/4$ if the number of petals is odd.

33. One of the most famous problems in Greek antiquity was "squaring the circle"; that is, using a straightedge and compass to construct a square whose area is equal to that of a given circle. It was proved in the nineteenth century that no such construction is possible. However, show that the shaded areas in the accompanying figure are equal, thereby "squaring the crescent."

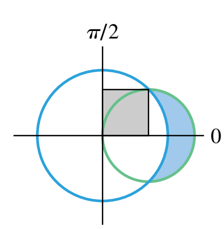


Figure Ex-33

34. Use a graphing utility to generate the polar graph of the equation $r = \cos 3\theta + 2$, and find the area that it encloses.

35. Use a graphing utility to generate the graph of the ***bifolium*** $r = 2\cos\theta\sin^2\theta$, and find the area of the upper loop.

11.4 CONIC SECTIONS IN CALCULUS

In this section we will discuss some of the basic geometric properties of parabolas, ellipses, and hyperbolas. These curves play an important role in calculus and also arise naturally in a broad range of applications in such fields as planetary motion, design of telescopes and antennas, geodetic positioning, and medicine, to name a few.

Some students may already be familiar with the material in this section, in which case it can be treated as a review. Instructors who want to spend some additional time on precalculus review may want to allocate more than one lecture on this material.

CONIC SECTIONS

Circles, ellipses, parabolas, and hyperbolas are called ***conic sections*** or ***conics*** because they can be obtained as intersections of a plane with a double-napped circular cone (Figure 11.4.1). If the plane passes through the vertex of the double-napped cone, then the intersection is a point, a pair of intersecting lines, or a single line. These are called ***degenerate conic sections***.

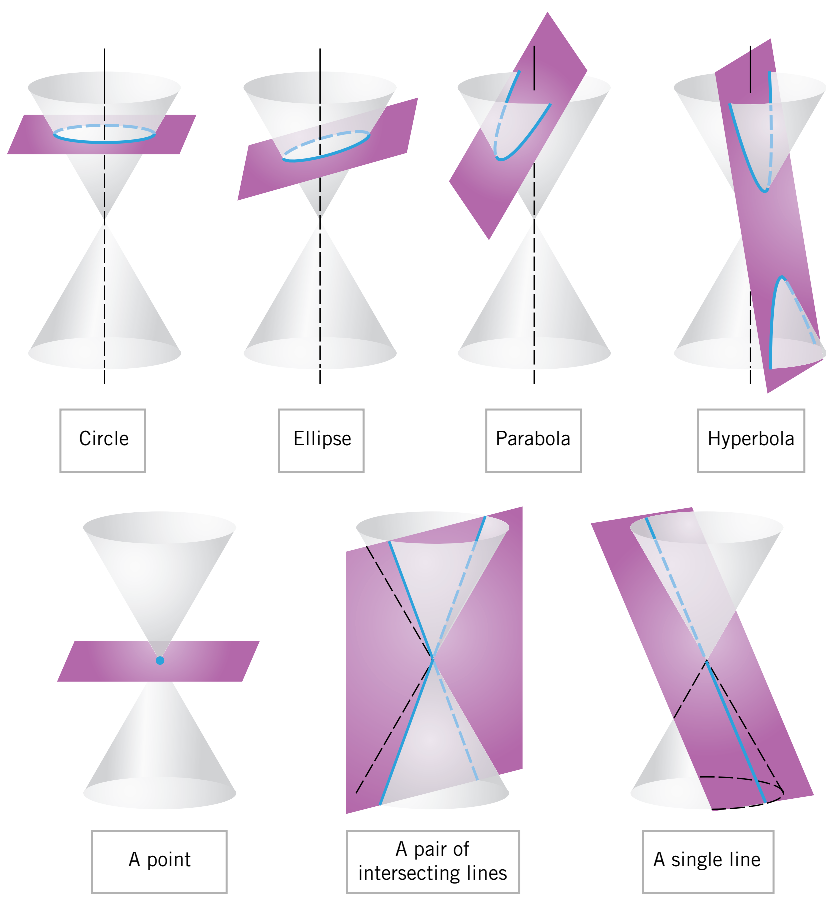


Figure 11.4.1

DEFINITIONS OF THE CONIC SECTIONS

Although we could derive properties of parabolas, ellipses, and hyperbolas by defining them as intersections with a double-napped cone, it will be better suited to calculus if we begin with equivalent definitions that are based on their geometric properties.

11.4.1 DEFINITION. A ***parabola*** is the set of all points in the plane that are equidistant from a fixed line and a fixed point not on the line.

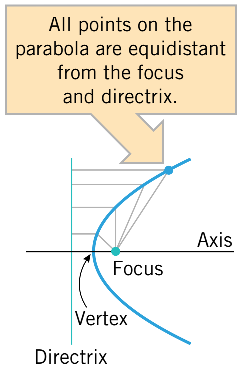


Figure 11.4.2

The line is called the ***directrix*** of the parabola, and the point is called the ***focus*** (Figure 11.4.2). A parabola is symmetric about the line that passes through the focus at right angles to the directrix. This line, called the ***axis*** or the ***axis of symmetry*** of the parabola, intersects the parabola at a point called the ***vertex***.

11.4.2 DEFINITION. An ***ellipse*** is the set of all points in the plane, the sum of whose distances from two fixed points is a given positive constant that is greater than the distance between the fixed points.

The two fixed points are called the ***foci*** (plural of "focus") of the ellipse, and the midpoint of the line segment joining the foci is called the ***center*** (Figure 11.4.3*a*). To help visualize Definition 11.4.2, imagine that two ends of a string are tacked to the foci and a pencil traces a curve as it is held tight against the string (Figure 11.4.3*b*). The resulting curve will be an ellipse since the sum of the distances to the foci is a constant, namely the total length of the string. Note that if the foci coincide, the ellipse reduces to a circle. For ellipses other than circles, the line segment through the foci and across the ellipse is called the ***major axis*** (Figure 11.4.3*c*), and the line segment across the ellipse, through the center, and perpendicular to the major axis is called the ***minor axis***. The endpoints of the major axis are called ***vertices***.

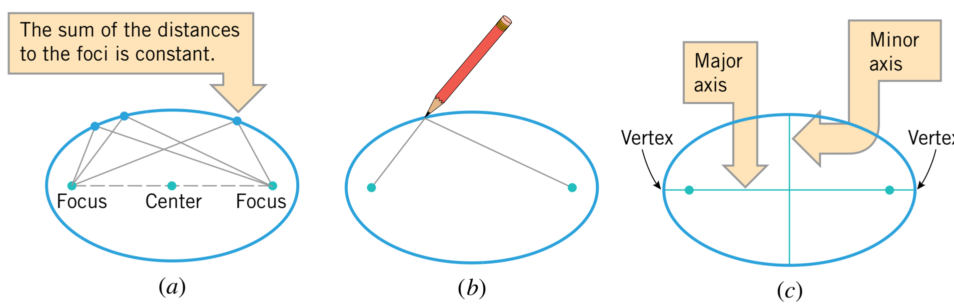


Figure 11.4.3

11.4.3 DEFINITION. A ***hyperbola*** is the set of all points in the plane, the difference of whose distances from two fixed distinct points is a given positive constant that is less than the distance between the fixed points.

The two fixed points are called the ***foci*** of the hyperbola, and the term "difference" that is used in the definition is understood to mean the distance to the farther focus minus the distance to the closer focus. As a result, the points on the hyperbola form two ***branches***, each "wrapping around" the closer focus (Figure 11.4.4*a*). The midpoint of the line segment joining the foci is called the ***center*** of the hyperbola, the line through the foci is called the ***focal axis***, and the line through the center that is perpendicular to the focal axis is called the ***conjugate axis***. The hyperbola intersects the focal axis at two points called the ***vertices***.

Associated with every hyperbola is a pair of lines, called the ***asymptotes*** of the hyperbola. These lines intersect at the center of the hyperbola and have the property that as a point P moves along the hyperbola away from the center, the distance between P and one of the asymptotes approaches zero (Figure 11.4.4*b*).

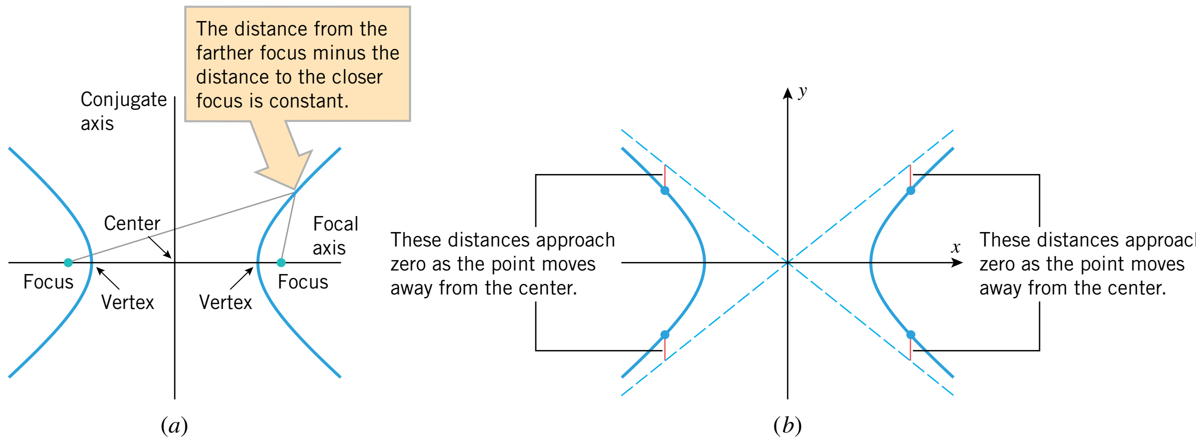


Figure 11.4.4

EQUATIONS OF PARABOLAS IN STANDARD POSITION

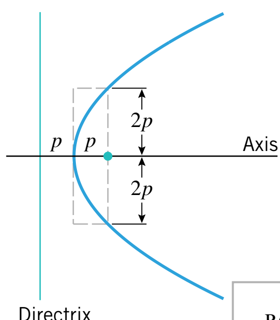


Figure 11.4.5

It is traditional in the study of parabolas to denote the distance between the focus and the vertex by p. The vertex is equidistant from the focus and the directrix, so the distance between the vertex and the directrix is also p; consequently, the distance between the focus and the directrix is $2p$ (Figure 11.4.5). As illustrated in that figure, the parabola passes through two of the corners of a box that extends from the vertex to the focus along the axis of symmetry and extends $2p$ units above and $2p$ units below the axis of symmetry.

The equation of a parabola is simplest if the vertex is the origin and the axis of symmetry is along the x-axis or y-axis. The four possible such orientations are shown in Figure 11.4.6. These are called the ***standard positions*** of a parabola, and the resulting equations are called the ***standard equations*** of a parabola.

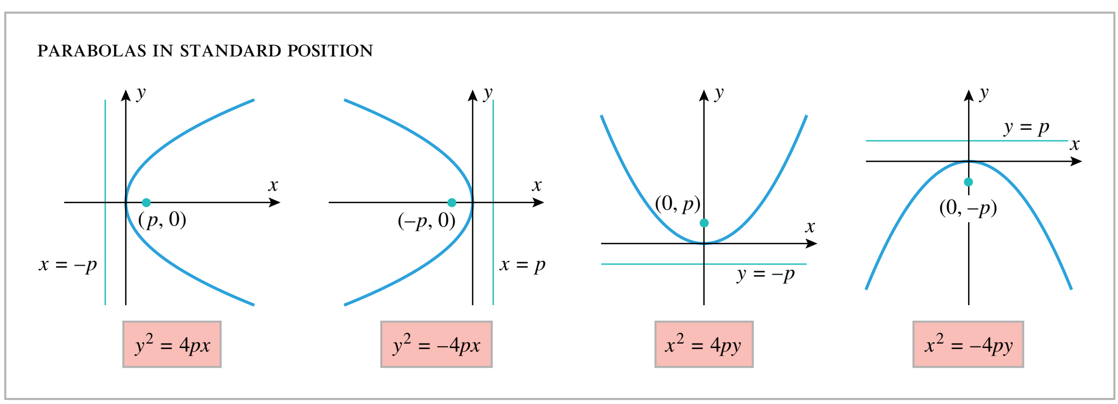


Figure 11.4.6

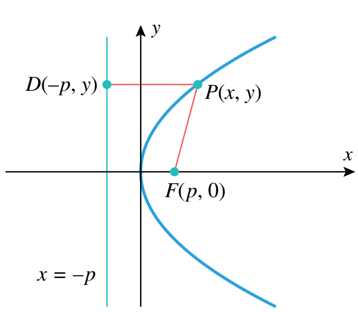


Figure 11.4.7

To illustrate how the equations in Figure 11.4.6 are obtained, we will derive the equation for the parabola with focus $(p, 0)$ and directrix $x = -p$. Let $P(x, y)$ be any point on the parabola. Since P is equidistant from the focus and directrix, the distances PF and PD in Figure 11.4.7 are equal; that is,

$$PF = PD \tag{1}$$

where $D(-p, y)$ is the foot of the perpendicular from P to the directrix. From the distance formula, the distances PF and PD are

$$PF = \sqrt{(x-p)^2 + y^2} \quad \text{and} \quad PD = \sqrt{(x+p)^2} \tag{2}$$

Substituting in (1) and squaring yields

$$(x-p)^2 + y^2 = (x+p)^2 \tag{3}$$

and after simplifying

$$y^2 = 4px \tag{4}$$

The derivations of the other equations in Figure 11.4.6 are similar.

A TECHNIQUE FOR SKETCHING PARABOLAS

Parabolas can be sketched from their *standard equations* using four basic steps:

- Determine whether the axis of symmetry is along the x-axis or the y-axis. Referring to Figure 11.4.6, the axis of symmetry is along the x-axis if the equation has a y^2-term, and it is along the y-axis if it has an x^2-term.
- Determine which way the parabola opens. If the axis of symmetry is along the x-axis, then the parabola opens to the right if the coefficient of x is positive, and it opens to the left if the coefficient is negative. If the axis of symmetry is along the y-axis, then the parabola opens up if the coefficient of y is positive, and it opens down if the coefficient is negative.
- Determine the value of p and draw a box extending p units from the origin along the axis of symmetry in the direction in which the parabola opens and extending $2p$ units on each side of the axis of symmetry.
- Using the box as a guide, sketch the parabola so that its vertex is at the origin and it passes through the corners of the box (Figure 11.4.8).

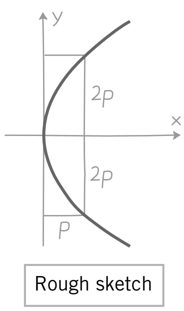


Figure 11.4.8

Example 1 Sketch the graphs of the parabolas

(a) $x^2 = 12y$ (b) $y^2 + 8x = 0$

and show the focus and directrix of each.

Solution (a). This equation involves x^2, so the axis of symmetry is along the y-axis, and the coefficient of y is positive, so the parabola opens upward. From the coefficient of y, we obtain $4p = 12$ or $p = 3$. Drawing a box extending $p = 3$ units up from the origin and $2p = 6$ units to the left and $2p = 6$ units to the right of the y-axis, then using corners of the box as a guide, yields the graph in Figure 11.4.9.

The focus is $p = 3$ units from the vertex along the axis of symmetry in the direction in which the parabola opens, so its coordinates are $(0, 3)$. The directrix is perpendicular to the axis of symmetry at a distance of $p = 3$ units from the vertex on the opposite side from the focus, so its equation is $y = -3$.

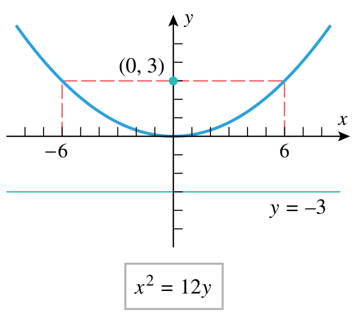


Figure 11.4.9

Solution (b). We first rewrite the equation in the standard form

$$y^2 = -8x$$

This equation involves y^2, so the axis of symmetry is along the x-axis, and the coefficient of x is negative, so the parabola opens to the left. From the coefficient of x we obtain $4p = 8$, so $p = 2$. Drawing a box extending $p = 2$ units left from the origin and $2p = 4$ units above and $2p = 4$ units below the x-axis, then using corners of the box as a guide, yields the graph in Figure 11.4.10. ◀

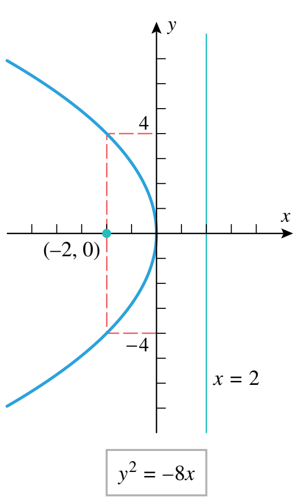


Figure 11.4.10

Example 2 Find an equation of the parabola that is symmetric about the y-axis, has its vertex at the origin, and passes through the point $(5, 2)$.

Solution. Since the parabola is symmetric about the y-axis and has its vertex at the origin, the equation is of the form

$$x^2 = 4py \quad \text{or} \quad x^2 = -4py$$

where the sign depends on whether the parabola opens up or down. But the parabola must open up, since it passes through the point $(5, 2)$, which lies in the first quadrant. Thus, the equation is of the form

$$x^2 = 4py \tag{5}$$

Since the parabola passes through (5, 2), we must have $5^2 = 4p \cdot 2$ or $4p = \frac{25}{2}$. Therefore, (5) becomes

$$x^2 = \tfrac{25}{2}y$$ ◀

EQUATIONS OF ELLIPSES IN STANDARD POSITION

It is traditional in the study of ellipses to denote the length of the major axis by $2a$, the length of the minor axis by $2b$, and the distance between the foci by $2c$ (Figure 11.4.11). The number a is called the ***semimajor axis*** and the number b the ***semiminor axis*** (standard but odd terminology, since a and b are numbers, not geometric axes).

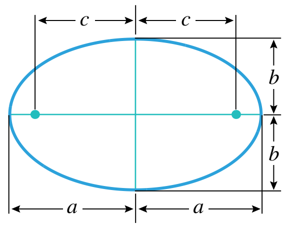


Figure 11.4.11

There is a basic relationship between the numbers a, b, and c that can be obtained by examining the sum of the distances to the foci from a point P at the end of the major axis and from a point Q at the end of the minor axis (Figure 11.4.12). From Definition 11.4.2, these sums must be equal, so we obtain

$$2\sqrt{b^2+c^2} = (a-c) + (a+c)$$

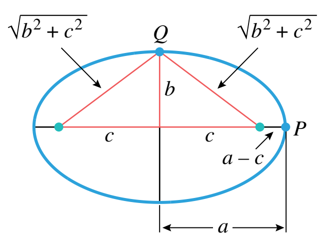


Figure 11.4.12

from which it follows that

$$a = \sqrt{b^2+c^2} \tag{6}$$

or, equivalently,

$$c = \sqrt{a^2-b^2} \tag{7}$$

From (6), the distance from a focus to an end of the minor axis is a (Figure 11.4.13), which implies that for *all* points on the ellipse the sum of the distances to the foci is $2a$.

It also follows from (6) that $a \ge b$ with the equality holding only when $c = 0$. Geometrically, this means that the major axis of an ellipse is at least as large as the minor axis and that the two axes have equal length only when the foci coincide, in which case the ellipse is a circle.

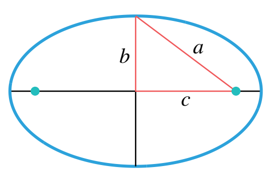


Figure 11.4.13

The equation of an ellipse is simplest if the center of the ellipse is at the origin and the foci are on the x-axis or y-axis. The two possible such orientations are shown in Figure 11.4.14. These are called the ***standard positions*** of an ellipse, and the resulting equations are called the ***standard equations*** of an ellipse.

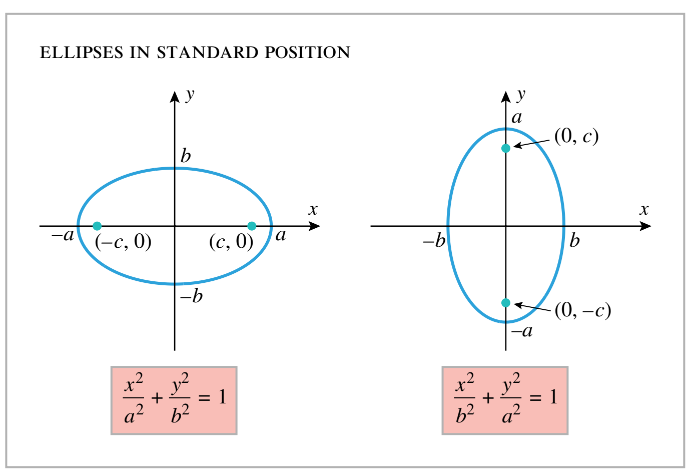


Figure 11.4.14

To illustrate how the equations in Figure 11.4.14 are obtained, we will derive the equation for the ellipse with foci on the x-axis. Let $P(x, y)$ be any point on that ellipse. Since the sum of the distances from P to the foci is $2a$, it follows (Figure 11.4.15) that

$$PF' + PF = 2a$$

so

$$\sqrt{(x+c)^2+y^2} + \sqrt{(x-c)^2+y^2} = 2a$$

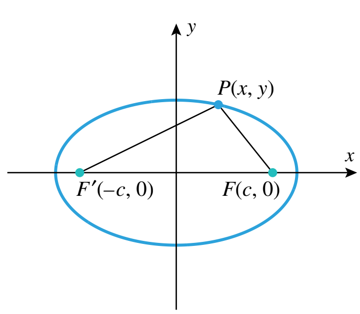


Figure 11.4.15

Transposing the second radical to the right side of the equation and squaring yields

$$(x+c)^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2$$

and, on simplifying,

$$\sqrt{(x-c)^2 + y^2} = a - \frac{c}{a}x \tag{8}$$

Squaring again and simplifying yields

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$$

which, by virtue of (6), can be written as

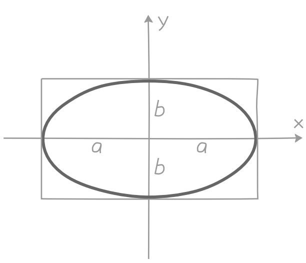
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{9}$$

Conversely, it can be shown that any point whose coordinates satisfy (9) has $2a$ as the sum of its distances from the foci, so that such a point is on the ellipse.

A TECHNIQUE FOR SKETCHING ELLIPSES

Ellipses can be sketched from their *standard equations* using three basic steps:

- Determine whether the major axis is on the x-axis or the y-axis. This can be ascertained from the sizes of the denominators in the equation. Referring to Figure 11.4.14, and keeping in mind that $a^2 > b^2$ (since $a > b$), the major axis is along the x-axis if x^2 has the larger denominator, and it is along the y-axis if y^2 has the larger denominator. If the denominators are equal, the ellipse is a circle.
- Determine the values of a and b and draw a box extending a units on each side of the center along the major axis and b units on each side of the center along the minor axis.
- Using the box as a guide, sketch the ellipse so that its center is at the origin and it touches the sides of the box where the sides intersect the coordinate axes (Figure 11.4.16).



Rough sketch

Figure 11.4.16

Example 3 Sketch the graphs of the ellipses

$$\text{(a)}\ \frac{x^2}{9} + \frac{y^2}{16} = 1 \qquad \text{(b)}\ x^2 + 2y^2 = 4$$

showing the foci of each.

Solution (a). Since y^2 has the larger denominator, the major axis is along the y-axis. Moreover, since $a^2 > b^2$, we must have $a^2 = 16$ and $b^2 = 9$, so

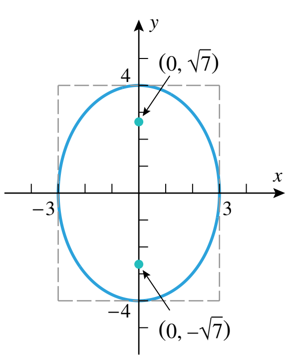
$$a = 4 \quad \text{and} \quad b = 3$$

Drawing a box extending 4 units on each side of the origin along the y-axis and 3 units on each side of the origin along the x-axis as a guide yields the graph in Figure 11.4.17.

The foci lie c units on each side of the center along the major axis, where c is given by (7). From the values of a^2 and b^2 above, we obtain

$$c = \sqrt{a^2 - b^2} = \sqrt{16 - 9} = \sqrt{7} \approx 2.6$$

Thus, the coordinates of the foci are $(0, \sqrt{7})$ and $(0, -\sqrt{7})$, since they lie on the y-axis.



$\frac{x^2}{9} + \frac{y^2}{16} = 1$

Figure 11.4.17

Solution (b). We first rewrite the equation in the standard form

$$\frac{x^2}{4} + \frac{y^2}{2} = 1$$

Since x^2 has the larger denominator, the major axis lies along the x-axis, and we have $a^2 = 4$ and $b^2 = 2$. Drawing a box extending $a = 2$ on each side of the origin along the x-axis and extending $b = \sqrt{2} \approx 1.4$ units on each side of the origin along the y-axis as a guide yields the graph in Figure 11.4.18.

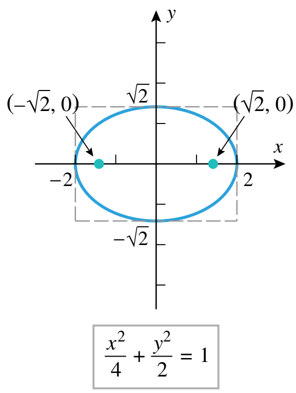


Figure 11.4.18

From (7), we obtain

$$c = \sqrt{a^2 - b^2} = \sqrt{2} \approx 1.4$$

Thus, the coordinates of the foci are $(\sqrt{2}, 0)$ and $(-\sqrt{2}, 0)$, since they lie on the x-axis. ◀

Example 4 Find an equation for the ellipse with foci $(0, \pm 2)$ and major axis with endpoints $(0, \pm 4)$.

Solution. From Figure 11.4.14, the equation has the form

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

and from the given information, $a = 4$ and $c = 2$. It follows from (6) that

$$b^2 = a^2 - c^2 = 16 - 4 = 12$$

so the equation of the ellipse is

$$\frac{x^2}{12} + \frac{y^2}{16} = 1$$ ◀

EQUATIONS OF HYPERBOLAS IN STANDARD POSITION

It is traditional in the study of hyperbolas to denote the distance between the vertices by $2a$, the distance between the foci by $2c$ (Figure 11.4.19), and to define the quantity b as

$$b = \sqrt{c^2 - a^2} \tag{10}$$

This relationship, which can also be expressed as

$$c = \sqrt{a^2 + b^2} \tag{11}$$

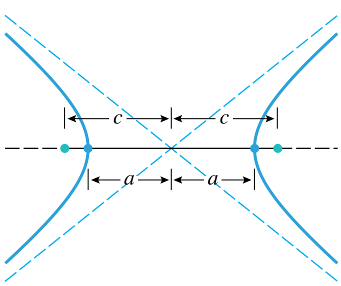


Figure 11.4.19

is pictured geometrically in Figure 11.4.20. As illustrated in that figure, and as we will show later in this section, the asymptotes pass through the corners of a box extending b units on each side of the center along the conjugate axis and a units on each side of the center along the focal axis. The number a is called the ***semifocal axis*** of the hyperbola and the number b the ***semiconjugate axis***. (As with the semimajor and semiminor axes of an ellipse, these are numbers, not geometric axes).

If V is one vertex of a hyperbola, then, as illustrated in Figure 11.4.21, the distance from V to the farther focus minus the distance from V to the closer focus is

$$[(c - a) + 2a] - (c - a) = 2a$$

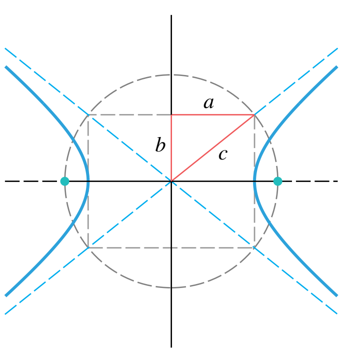


Figure 11.4.20

Thus, for *all* points on a hyperbola, the distance to the farther focus minus the distance to the closer focus is $2a$.

The equation of a hyperbola is simplest if the center of the hyperbola is at the origin and the foci are on the x-axis or y-axis. The two possible such orientations are shown in Figure 11.4.22. These are called the ***standard positions*** of a hyperbola, and the resulting equations are called the ***standard equations*** of a hyperbola.

The derivations of these equations are similar to those already given for parabolas and ellipses, so we will leave them as exercises. However, to illustrate how the equations of the asymptotes are derived, we will derive those equations for the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

We can rewrite this equation as

$$y^2 = \frac{b^2}{a^2}(x^2 - a^2)$$

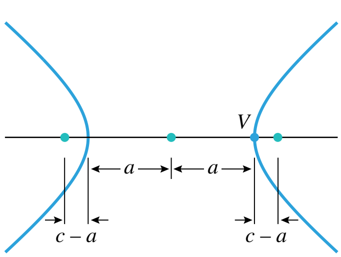


Figure 11.4.21

which is equivalent to the pair of equations

$$y = \frac{b}{a}\sqrt{x^2 - a^2} \quad \text{and} \quad y = -\frac{b}{a}\sqrt{x^2 - a^2}$$

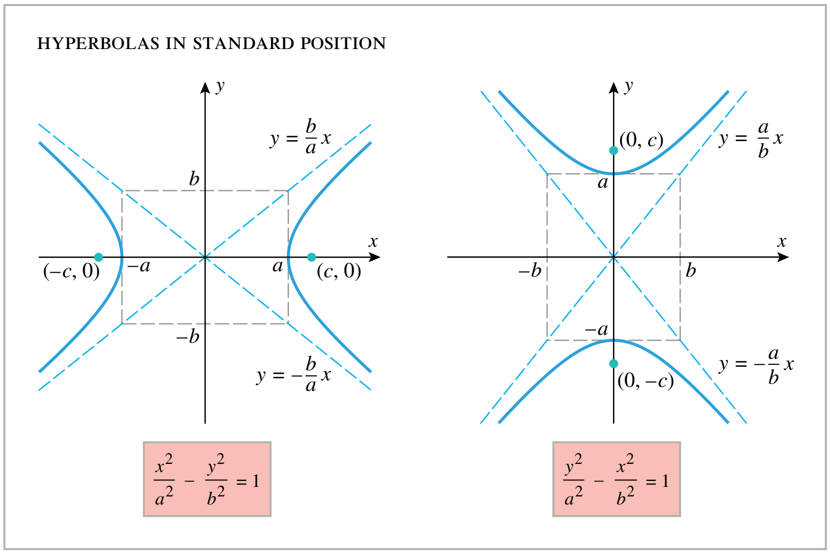


Figure 11.4.22

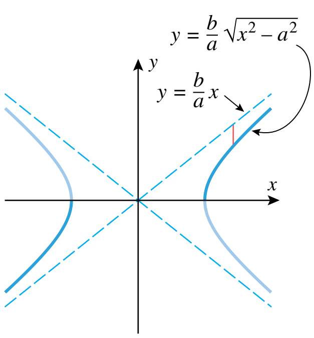


Figure 11.4.23

Thus, in the first quadrant, the vertical distance between the line $y = (b/a)x$ and the hyperbola can be written (Figure 11.4.23) as

$$\frac{b}{a}x - \frac{b}{a}\sqrt{x^2 - a^2}$$

But this distance tends to zero as $x \to +\infty$ since

$$\begin{aligned}
\lim_{x\to+\infty}\left(\frac{b}{a}x - \frac{b}{a}\sqrt{x^2 - a^2}\right) &= \lim_{x\to+\infty}\frac{b}{a}(x - \sqrt{x^2 - a^2}\,) \\
&= \lim_{x\to+\infty}\frac{b}{a}\frac{(x - \sqrt{x^2 - a^2}\,)(x + \sqrt{x^2 - a^2}\,)}{x + \sqrt{x^2 - a^2}} \\
&= \lim_{x\to+\infty}\frac{ab}{x + \sqrt{x^2 - a^2}} = 0
\end{aligned}$$

The analysis in the remaining quadrants is similar.

A QUICK WAY TO FIND ASYMPTOTES

There is a trick that can be used to avoid memorizing the equations of the asymptotes of a hyperbola. They can be obtained, when needed, by substituting 0 for the 1 on the right side of the hyperbola equation, and then solving for y in terms of x. For example, for the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

we would write

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0 \quad \text{or} \quad y^2 = \frac{b^2}{a^2}x^2 \quad \text{or} \quad y = \pm\frac{b}{a}x$$

which are the equations for the asymptotes.

A TECHNIQUE FOR SKETCHING HYPERBOLAS

Hyperbolas can be sketched from their *standard equations* using four basic steps:

- Determine whether the focal axis is on the x-axis or the y-axis. This can be ascertained from the location of the minus sign in the equation. Referring to Figure 11.4.22, the focal axis is along the x-axis when the minus sign precedes the y^2-term, and it is along the y-axis when the minus sign precedes the x^2-term.

- Determine the values of a and b and draw a box extending a units on either side of the center along the focal axis and b units on either side of the center along the conjugate axis. (The squares of a and b can be read directly from the equation.)
- Draw the asymptotes along the diagonals of the box.
- Using the box and the asymptotes as a guide, sketch the graph of the hyperbola (Figure 11.4.24).

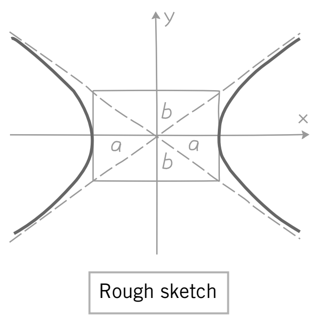


Figure 11.4.24

Example 5 Sketch the graphs of the hyperbolas

$$\text{(a)}\ \frac{x^2}{4}-\frac{y^2}{9}=1 \qquad \text{(b)}\ y^2-x^2=1$$

showing their vertices, foci, and asymptotes.

Solution (a). The minus sign precedes the y^2-term, so the focal axis is along the x-axis. From the denominators in the equation we obtain

$$a^2=4 \quad \text{and} \quad b^2=9$$

Since a and b are positive, we must have $a=2$ and $b=3$. Recalling that the vertices lie a units on each side of the center on the focal axis, it follows that their coordinates in this case are $(2, 0)$ and $(-2, 0)$. Drawing a box extending $a=2$ units along the x-axis on each side of the origin and $b=3$ units on each side of the origin along the y-axis, then drawing the asymptotes along the diagonals of the box as a guide, yields the graph in Figure 11.4.25.

To obtain equations for the asymptotes, we substitute 0 for 1 in the given equation; this yields

$$\frac{x^2}{4}-\frac{y^2}{9}=0 \quad \text{or} \quad y=\pm\frac{3}{2}x$$

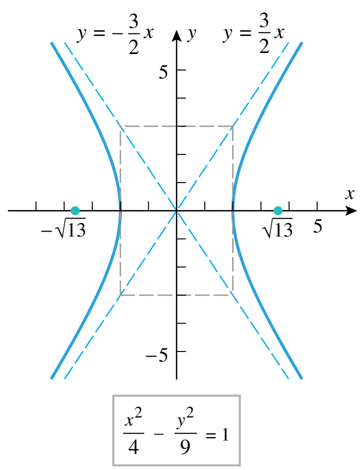


Figure 11.4.25

The foci lie c units on each side of the center along the focal axis, where c is given by (11). From the values of a^2 and b^2 above we obtain

$$c=\sqrt{a^2+b^2}=\sqrt{4+9}=\sqrt{13}\approx 3.6$$

Since the foci lie on the x-axis in this case, their coordinates are $(\sqrt{13}, 0)$ and $(-\sqrt{13}, 0)$.

Solution (b). The minus sign precedes the x^2-term, so the focal axis is along the y-axis. From the denominators in the equation we obtain $a^2=1$ and $b^2=1$, from which it follows that

$$a=1 \quad \text{and} \quad b=1$$

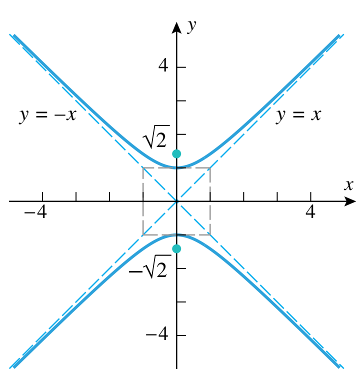


Figure 11.4.26

Thus, the vertices are at $(0, -1)$ and $(0, 1)$. Drawing a box extending $a=1$ unit on either side of the origin along the y-axis and $b=1$ unit on either side of the origin along the x-axis, then drawing the asymptotes, yields the graph in Figure 11.4.26. Since the box is actually a square, the asymptotes are perpendicular and have equations $y=\pm x$. This can also be seen by substituting 0 for 1 in the given equation, which yields $y^2-x^2=0$ or $y=\pm x$. Also,

$$c=\sqrt{a^2+b^2}=\sqrt{1+1}=\sqrt{2}$$

so the foci, which lie on the y-axis, are $(0, -\sqrt{2})$ and $(0, \sqrt{2})$. ◀

REMARK. A hyperbola in which $a=b$, as in part (b) of this example, is called an ***equilateral hyperbola***. Such hyperbolas always have perpendicular asymptotes.

Example 6 Find the equation of the hyperbola with vertices $(0, \pm 8)$ and asymptotes $y=\pm\frac{4}{3}x$.

Solution. Since the vertices are on the y-axis, the equation of the hyperbolas has the form $(y^2/a^2) - (x^2/b^2) = 1$ and the asymptotes are

$$y = \pm\frac{a}{b}x$$

From the locations of the vertices we have $a = 8$, so the given equations of the asymptotes yield

$$y = \pm\frac{a}{b}x = \pm\frac{8}{b}x = \pm\frac{4}{3}x$$

from which it follows that $b = 6$. Thus, the hyperbola has the equation

$$\frac{y^2}{64} - \frac{x^2}{36} = 1$$ ◀

TRANSLATED CONICS

Equations of conics that are translated from their standard positions can be obtained by replacing x by $x - h$ and y by $y - k$ in their standard equations. For a parabola, this translates the vertex from the origin to the point (h, k); and for ellipses and hyperbolas, this translates the center from the origin to the point (h, k).

Parabolas with vertex (h, k) and axis parallel to x-axis

$$(y-k)^2 = 4p(x-h) \qquad \text{[Opens right]} \tag{12}$$

$$(y-k)^2 = -4p(x-h) \qquad \text{[Opens left]} \tag{13}$$

Parabolas with vertex (h, k) and axis parallel to y-axis

$$(x-h)^2 = 4p(y-k) \qquad \text{[Opens up]} \tag{14}$$

$$(x-h)^2 = -4p(y-k) \qquad \text{[Opens down]} \tag{15}$$

Ellipse with center (h, k) and major axis parallel to x-axis

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \quad [b \le a] \tag{16}$$

Ellipse with center (h, k) and major axis parallel to y-axis

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1 \quad [b \le a] \tag{17}$$

Hyperbola with center (h, k) and focal axis parallel to x-axis

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \tag{18}$$

Hyperbola with center (h, k) and focal axis parallel to y-axis

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1 \tag{19}$$

Example 7 Find an equation for the parabola that has its vertex at $(1, 2)$ and its focus at $(4, 2)$.

Solution. Since the focus and vertex are on a horizontal line, and since the focus is to the right of the vertex, the parabola opens to the right and its equation has the form

$$(y-k)^2 = 4p(x-h)$$

Since the vertex and focus are 3 units apart, we have $p = 3$, and since the vertex is at $(h, k) = (1, 2)$, we obtain

$$(y-2)^2 = 12(x-1)$$ ◀

Sometimes the equations of translated conics occur in expanded form, in which case we are faced with the problem of identifying the graph of a quadratic equation in x and y:

$$Ax^2 + Cy^2 + Dx + Ey + F = 0 \tag{20}$$

The basic procedure for determining the nature of such a graph is to complete the squares of the quadratic terms and then try to match up the resulting equation with one of the forms of a translated conic.

Example 8 Describe the graph of the equation

$$y^2 - 8x - 6y - 23 = 0$$

Solution. The equation involves quadratic terms in y but none in x, so we first take all of the y-terms to one side:

$$y^2 - 6y = 8x + 23$$

Next, we complete the square on the y-terms by adding 9 to both sides:

$$(y-3)^2 = 8x + 32$$

Finally, we factor out the coefficient of the x-term to obtain

$$(y-3)^2 = 8(x+4)$$

This equation is of form (12) with $h = -4$, $k = 3$, and $p = 2$, so the graph is a parabola with vertex $(-4, 3)$ opening to the right. Since $p = 2$, the focus is 2 units to the right of the vertex, which places it at the point $(-2, 3)$; and the directrix is 2 units to the left of the vertex, which means that its equation is $x = -6$. The parabola is shown in Figure 11.4.27. ◀

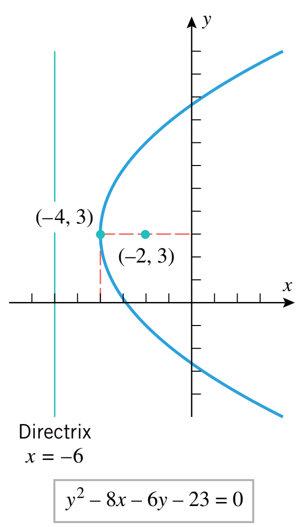


Figure 11.4.27

Example 9 Describe the graph of the equation

$$16x^2 + 9y^2 - 64x - 54y + 1 = 0$$

Solution. This equation involves quadratic terms in both x and y, so we will group the x-terms and the y-terms on one side and put the constant on the other:

$$(16x^2 - 64x) + (9y^2 - 54y) = -1$$

Next, factor out the coefficients of x^2 and y^2 and complete the squares:

$$16(x^2 - 4x + 4) + 9(y^2 - 6y + 9) = -1 + 64 + 81$$

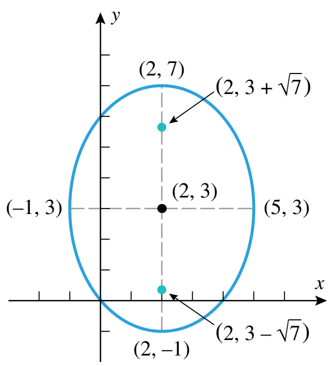
or

$$16(x-2)^2 + 9(y-3)^2 = 144$$

Finally, divide through by 144 to introduce a 1 on the right side:

$$\frac{(x-2)^2}{9} + \frac{(y-3)^2}{16} = 1$$

This is an equation of form (17), with $h = 2$, $k = 3$, $a^2 = 16$, and $b^2 = 9$. Thus, the graph



$16x^2 + 9y^2 - 64x - 54y + 1 = 0$

Figure 11.4.28

of the equation is an ellipse with center (2, 3) and major axis parallel to the y-axis. Since $a = 4$, the major axis extends 4 units above and 4 units below the center, so its endpoints are (2, 7) and (2, −1) (Figure 11.4.28). Since $b = 3$, the minor axis extends 3 units to the left and 3 units to the right of the center, so its endpoints are (−1, 3) and (5, 3). Since

$$c = \sqrt{a^2 - b^2} = \sqrt{16 - 9} = \sqrt{7}$$

the foci lie $\sqrt{7}$ units above and below the center, placing them at the points $(2, 3 + \sqrt{7})$ and $(2, 3 - \sqrt{7})$. ◀

Example 10 Describe the graph of the equation

$$x^2 - y^2 - 4x + 8y - 21 = 0$$

Solution. This equation involves quadratic terms in both x and y, so we will group the x-terms and the y-terms on one side and put the constant on the other:

$$(x^2 - 4x) - (y^2 - 8y) = 21$$

We leave it for you to verify by completing the squares that this equation can be written as

$$\frac{(x-2)^2}{9} - \frac{(y-4)^2}{9} = 1 \tag{21}$$

This is an equation of form (18) with $h = 2$, $k = 4$, $a^2 = 9$, and $b^2 = 9$. Thus, the equation represents a hyperbola with center (2, 4) and focal axis parallel to the x-axis. Since $a = 3$, the vertices are located 3 units to the left and 3 units to the right of the center, or at the points (−1, 4) and (5, 4). From (11), $c = \sqrt{a^2 + b^2} = \sqrt{9 + 9} = 3\sqrt{2}$, so the foci are located $3\sqrt{2}$ units to the left and right of the center, or at the points $(2 - 3\sqrt{2}, 4)$ and $(2 + 3\sqrt{2}, 4)$.

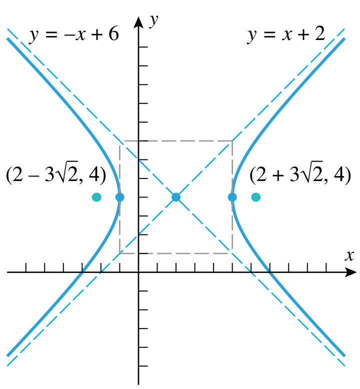
The equations of the asymptotes may be found using the trick of substituting 0 for 1 in (21) to obtain

$$\frac{(x-2)^2}{9} - \frac{(y-4)^2}{9} = 0$$

This can be written as $y - 4 = \pm(x - 2)$, which yields the asymptotes

$$y = x + 2 \quad \text{and} \quad y = -x + 6$$

With the aid of a box extending $a = 3$ units left and right of the center and $b = 3$ units above and below the center, we obtain the sketch in Figure 11.4.29. ◀



$x^2 - y^2 - 4x + 8y - 21 = 0$

Figure 11.4.29

REFLECTION PROPERTIES OF THE CONIC SECTIONS

Parabolas, ellipses, and hyperbolas have certain reflection properties that make them extremely valuable in various applications. In the exercises we will ask you to prove the following results.

> **11.4.4** THEOREM (***Reflection Property of Parabolas***). *The tangent line at a point P on a parabola makes equal angles with the line through P parallel to the axis of symmetry and the line through P and the focus* (Figure 11.4.30*a*).

> **11.4.5** THEOREM (***Reflection Property of Ellipses***). *A line tangent to an ellipse at a point P makes equal angles with the lines joining P to the foci* (Figure 11.4.30*b*).

> **11.4.6** THEOREM (***Reflection Property of Hyperbolas***). *A line tangent to a hyperbola at a point P makes equal angles with the lines joining P to the foci* (Figure 11.4.30*c*).

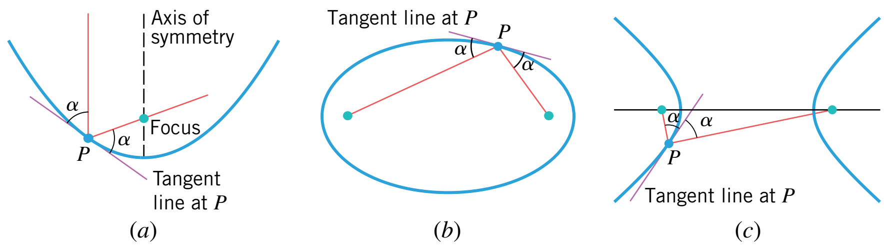


Figure 11.4.30

APPLICATIONS OF THE CONIC SECTIONS

Incoming signals are reflected by the parabolic antenna to the receiver at the focus.

Fermat's principle in optics states that light reflects off of a surface at an angle equal to its angle of incidence. (See Exercise 61 in Section 4.6.) In particular, if a reflecting surface is generated by revolving a parabola about its axis of symmetry, it follows from Theorem 11.4.4 that all light rays entering parallel to the axis will be reflected to the focus (Figure 11.4.31*a*); conversely, if a light source is located at the focus, then the reflected rays will all be parallel to the axis (Figure 11.4.31*b*). This principle is used in certain telescopes to reflect the approximately parallel rays of light from the stars and planets off of a parabolic mirror to an eyepiece at the focus; and the parabolic reflectors in flashlights and automobile headlights utilize this principle to form a parallel beam of light rays from a bulb placed at the focus. The same optical principles apply to radar signals and sound waves, which explains the parabolic shape of many antennas.

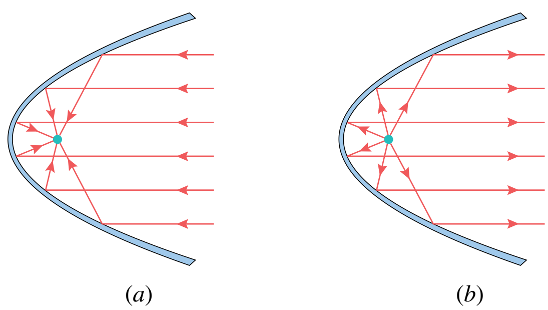


Figure 11.4.31

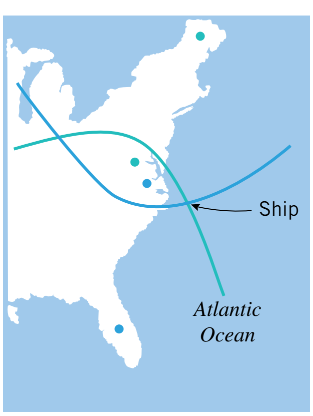


Figure 11.4.32

Visitors to various rooms in the United States Capitol Building and in St. Paul's Cathedral in Rome are often astonished by the "whispering gallery" effect in which two people at opposite ends of the room can hear one another's whispers very clearly. Such rooms have ceilings with elliptical cross sections and common foci. Thus, when the two people stand at the foci, their whispers are reflected directly to one another off of the elliptical ceiling.

Hyperbolic navigation systems, which were developed in World War II as navigational aids to ships, are based on the definition of a hyperbola. With these systems the ship receives synchronized radio signals from two widely spaced transmitters with known positions. The ship's electronic receiver measures the difference in reception times between the signals and then uses that difference to compute the difference $2a$ in its distance between the two transmitters. This information places the ship somewhere on the hyperbola whose foci are at the transmitters and whose points have $2a$ as the difference in their distances from the foci. By repeating the process with a second set of transmitters, the position of the ship can be approximated as the intersection of two hyperbolas (Figure 11.4.32).

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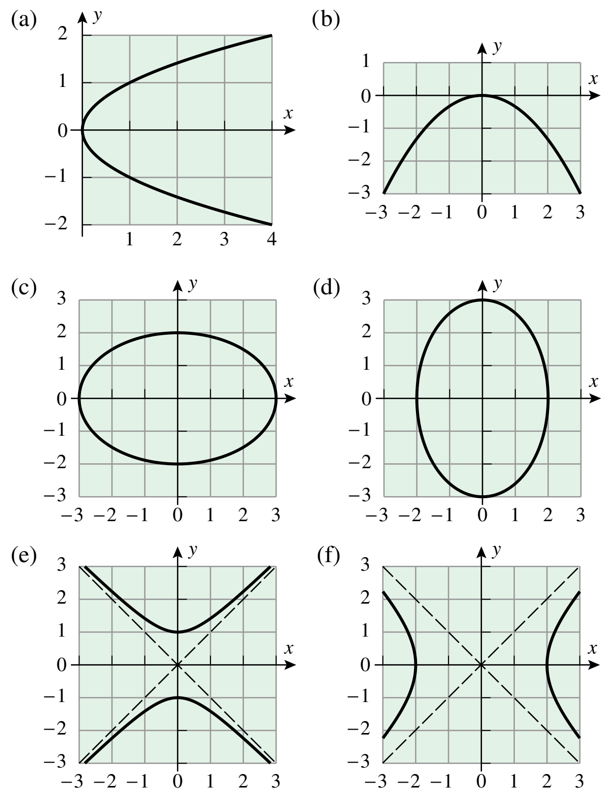
EXERCISE SET 11.4 Graphing Utility C CAS

1. In each part, find the equation of the conic.



2. (a) Find the focus and directrix for each parabola in Exercise 1.
(b) Find the foci of the ellipses in Exercise 1.
(c) Find the foci and the equations of the asymptotes of the hyperbolas in Exercise 1.

In Exercises 3–8, sketch the parabola, and label the focus, vertex, and directrix.

3. (a) $y^2 = 6x$ (b) $x^2 = -9y$

4. (a) $y^2 = -10x$ (b) $x^2 = 4y$

5. (a) $(y-3)^2 = 6(x-2)$ (b) $(x+2)^2 = -(y+2)$

6. (a) $(y+1)^2 = -7(x-4)$ (b) $\left(x-\frac{1}{2}\right)^2 = 2(y-1)$

7. (a) $x^2 - 4x + 2y = 1$ (b) $x = y^2 - 4y + 2$

8. (a) $y^2 - 6y - 2x + 1 = 0$ (b) $y = 4x^2 + 8x + 5$

In Exercises 9–14, sketch the ellipse, and label the foci, the vertices, and the ends of the minor axis.

9. (a) $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$ (b) $9x^2 + y^2 = 9$

10. (a) $\dfrac{x^2}{4} + \dfrac{y^2}{25} = 1$ (b) $4x^2 + 9y^2 = 36$

11. (a) $9(x-1)^2 + 16(y-3)^2 = 144$
(b) $3(x+2)^2 + 4(y+1)^2 = 12$

12. (a) $(x+3)^2 + 4(y-5)^2 = 16$
(b) $\frac{1}{4}x^2 + \frac{1}{9}(y+2)^2 - 1 = 0$

13. (a) $x^2 + 9y^2 + 2x - 18y + 1 = 0$
(b) $4x^2 + y^2 + 8x - 10y = -13$

14. (a) $9x^2 + 4y^2 + 18x - 24y + 9 = 0$
(b) $5x^2 + 9y^2 - 20x + 54y = -56$

In Exercises 15–20, sketch the hyperbola, and label the vertices, foci, and asymptotes.

15. (a) $\dfrac{x^2}{16} - \dfrac{y^2}{4} = 1$ (b) $9y^2 - 4x^2 = 36$

16. (a) $\dfrac{y^2}{9} - \dfrac{x^2}{25} = 1$ (b) $16x^2 - 25y^2 = 400$

17. (a) $\dfrac{(x-2)^2}{9} - \dfrac{(y-4)^2}{4} = 1$
(b) $(y+3)^2 - 9(x+2)^2 = 36$

18. (a) $\dfrac{(y+4)^2}{3} - \dfrac{(x-2)^2}{5} = 1$
(b) $16(x+1)^2 - 8(y-3)^2 = 16$

19. (a) $x^2 - 4y^2 + 2x + 8y - 7 = 0$
(b) $16x^2 - y^2 - 32x - 6y = 57$

20. (a) $4x^2 - 9y^2 + 16x + 54y - 29 = 0$
(b) $4y^2 - x^2 + 40y - 4x = -60$

In Exercises 21–26, find an equation for the parabola that satisfies the given conditions.

21. (a) Vertex $(0, 0)$; focus $(3, 0)$.
(b) Vertex $(0, 0)$; directrix $x = 7$.

22. (a) Vertex $(0, 0)$; focus $(0, -4)$.
(b) Vertex $(0, 0)$; directrix $y = \frac{1}{2}$.

23. (a) Focus $(0, -3)$; directrix $y = 3$.
(b) Vertex $(1, 1)$; directrix $y = -2$.

24. (a) Focus $(6, 0)$; directrix $x = -6$.
(b) Focus $(-1, 4)$; directrix $x = 5$.

25. Axis $y = 0$; passes through $(3, 2)$ and $(2, -3)$.

26. Vertex $(5, -3)$; axis parallel to the y-axis; passes through $(9, 5)$.

In Exercises 27–32, find an equation for the ellipse that satisfies the given conditions.

27. (a) Ends of major axis $(\pm 3, 0)$; ends of minor axis $(0, \pm 2)$.
(b) Length of major axis 26; foci $(\pm 5, 0)$.

28. (a) Ends of major axis $(0, \pm\sqrt{5})$; ends of minor axis $(\pm 1, 0)$.
(b) Length of minor axis 16; foci $(0, \pm 6)$.

29. (a) Foci $(\pm 1, 0)$; $b = \sqrt{2}$.
(b) $c = 2\sqrt{3}$; $a = 4$; center at the origin; foci on a coordinate axis (two answers).

30. (a) Foci $(\pm 3, 0)$; $a = 4$.
(b) $b = 3$; $c = 4$; center at the origin; foci on a coordinate axis (two answers).

31. (a) Ends of major axis $(\pm 6, 0)$; passes through $(2, 3)$.
(b) Foci $(1, 2)$ and $(1, 4)$; minor axis of length 2.

32. (a) Center at $(0, 0)$; major and minor axes along the coordinate axes; passes through $(3, 2)$ and $(1, 6)$.
(b) Foci $(2, 1)$ and $(2, -3)$; major axis of length 6.

In Exercises 33–38, find an equation for a hyperbola that satisfies the given conditions. (In some cases there may be more than one hyperbola.)

33. (a) Vertices $(\pm 2, 0)$; foci $(\pm 3, 0)$.
(b) Vertices $(\pm 1, 0)$; asymptotes $y = \pm 2x$.

34. (a) Vertices $(0, \pm 3)$; foci $(0, \pm 5)$.
(b) Vertices $(0, \pm 3)$; asymptotes $y = \pm x$.

35. (a) Asymptotes $y = \pm\frac{3}{2}x$; $b = 4$.
(b) Foci $(0, \pm 5)$; asymptotes $y = \pm 2x$.

36. (a) Asymptotes $y = \pm\frac{3}{4}x$; $c = 5$.
(b) Foci $(\pm 3, 0)$; asymptotes $y = \pm 2x$.

37. (a) Vertices $(2, 4)$ and $(10, 4)$; foci 10 units apart.
(b) Asymptotes $y = 2x + 1$ and $y = -2x + 3$; passes through the origin.

38. (a) Foci $(1, 8)$ and $(1, -12)$; vertices 4 units apart.
(b) Vertices $(-3, -1)$ and $(5, -1)$; $b = 4$.

39. (a) As illustrated in the accompanying figure, a parabolic arch spans a road 40 feet wide. How high is the arch if a center section of the road 20 feet wide has a minimum clearance of 12 feet?
(b) How high would the center be if the arch were the upper half of an ellipse?

40. (a) Find an equation for the parabolic arch with base b and height h, shown in the accompanying figure.
(b) Find the area under the arch.

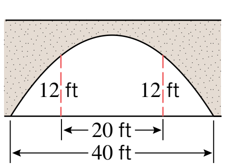


Figure Ex-39

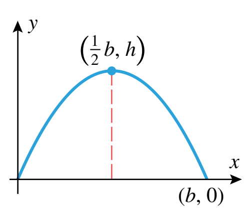


Figure Ex-40

41. Show that the vertex is the closest point on a parabola to the focus. [*Suggestion:* Introduce a convenient coordinate system and use Definition 11.4.1.]

42. As illustrated in the accompanying figure, suppose that a comet moves in a parabolic orbit with the Sun at its focus and that the line from the Sun to the comet makes an angle of 60° with the axis of the parabola when the comet is 40 million miles from the center of the Sun. Use the result in Exercise 41 to determine how close the comet will come to the center of the Sun.

43. For the parabolic reflector in the accompanying figure, how far from the vertex should the light source be placed to produce a beam of parallel rays?

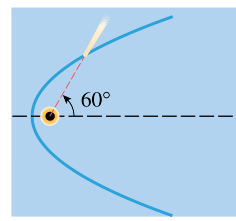


Figure Ex-42

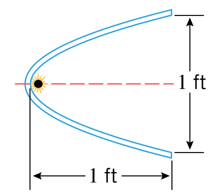
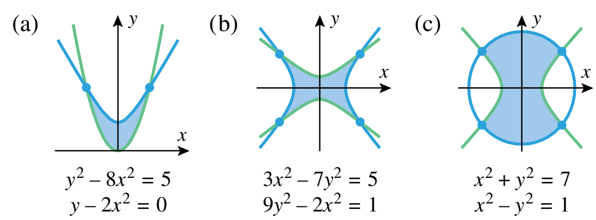


Figure Ex-43

44. In each part, find the shaded area in the figure.



45. (a) The accompanying figure shows an ellipse with semimajor axis a and semiminor axis b. Express the coordinates of the points P, Q, and R in terms of t.
(b) How does the geometric interpretation of the parameter t differ between a circle

$$x = a\cos t, \quad y = a\sin t$$

and an ellipse

$$x = a\cos t, \quad y = b\sin t?$$

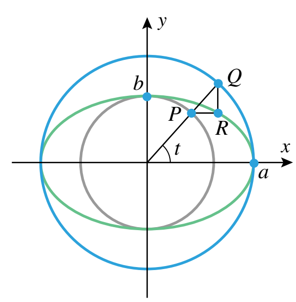


Figure Ex-45

46. (a) Show that the right and left branches of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

can be represented parametrically as

$$x = a\cosh t, \quad y = b\sinh t \quad (-\infty < t < +\infty)$$
$$x = -a\cosh t, \quad y = b\sinh t \quad (-\infty < t < +\infty)$$

(b) Use a graphing utility to generate both branches of the hyperbola $x^2 - y^2 = 1$ on the same screen.

47. (a) Show that the right and left branches of the hyperbola
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$
can be represented parametrically as
$$x = a\sec t, \quad y = b\tan t \qquad (-\pi/2 < t < \pi/2)$$
$$x = -a\sec t, \quad y = b\tan t \qquad (-\pi/2 < t < \pi/2)$$
(b) Use a graphing utility to generate both branches of the hyperbola $x^2 - y^2 = 1$ on the same screen.

48. Find an equation of the parabola traced by a point that moves so that its distance from $(-1, 4)$ is the same as its distance to $y = 1$.

49. Find an equation of the ellipse traced by a point that moves so that the sum of its distances to $(4, 1)$ and $(4, 5)$ is 12.

50. Find the equation of the hyperbola traced by a point that moves so that the difference between its distances to $(0, 0)$ and $(1, 1)$ is 1.

51. Suppose that the base of a solid is elliptical with a major axis of length 9 and a minor axis of length 4. Find the volume of the solid if the cross sections perpendicular to the major axis are squares (see the accompanying figure).

52. Suppose that the base of a solid is elliptical with a major axis of length 9 and a minor axis of length 4. Find the volume of the solid if the cross sections perpendicular to the minor axis are equilateral triangles (see the accompanying figure).

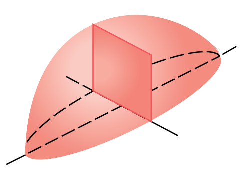


Figure Ex-51

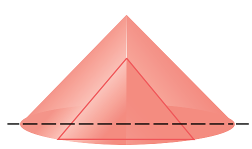


Figure Ex-52

53. Show that an ellipse with semimajor axis a and semiminor axis b has area $A = \pi ab$.

54. (a) Show that the ellipsoid that results when an ellipse with semimajor axis a and semiminor axis b is revolved about the major axis has volume $V = \frac{4}{3}\pi ab^2$.
(b) Show that the ellipsoid that results when an ellipse with semimajor axis a and semiminor axis b is revolved about the minor axis has volume $V = \frac{4}{3}\pi a^2 b$.

55. Show that the ellipsoid that results when an ellipse with semimajor axis a and semiminor axis b is revolved about the major axis has surface area
$$S = 2\pi ab\left(\frac{b}{a} + \frac{a}{c}\sin^{-1}\frac{c}{a}\right)$$
where $c = \sqrt{a^2 - b^2}$.

56. Show that the ellipsoid that results when an ellipse with semimajor axis a and semiminor axis b is revolved about the minor axis has surface area
$$S = 2\pi ab\left(\frac{a}{b} + \frac{b}{c}\ln\frac{a+c}{b}\right)$$
where $c = \sqrt{a^2 - b^2}$.

57. Suppose that you want to draw an ellipse that has given values for the lengths of the major and minor axes by using the method shown in Figure 11.4.3*b*. Assuming that the axes are drawn, explain how a compass can be used to locate the positions for the tacks.

58. The accompanying figure shows Kepler's method for constructing a parabola: a piece of string the length of the left edge of the drafting triangle is tacked to the vertex Q of the triangle and the other end to a fixed point F. A pencil holds the string taut against the base of the triangle as the edge opposite Q slides along a horizontal line L below F. Show that the pencil traces an arc of a parabola with focus F and directrix L.

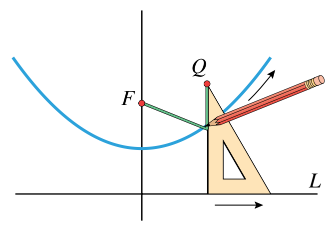


Figure Ex-58

59. The accompanying figure shows a method for constructing a hyperbola: a corner of a ruler is pinned to a fixed point F_1 and the ruler is free to rotate about that point. A piece of string whose length is less than that of the ruler is tacked to a point F_2 and to the free corner Q of the ruler on the same edge as F_1. A pencil holds the string taut against the top edge of the ruler as the ruler rotates about the point F_1. Show that the pencil traces an arc of a hyperbola with foci F_1 and F_2.

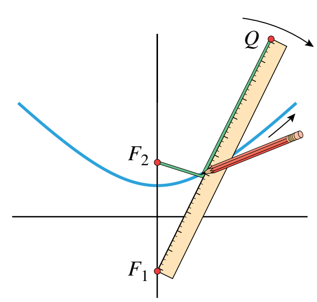


Figure Ex-59

60. Show that if a plane is not parallel to the axis of a right circular cylinder, then the intersection of the plane and cylinder is an ellipse (possibly a circle). [*Hint:* Let θ be the angle shown in Figure Ex-60 (next page), introduce coordinate axes as shown, and express x' and y' in terms of x and y.]

61. As illustrated in the accompanying figure, a carpenter needs to cut an elliptical hole in a sloped roof through which a circular vent pipe of diameter D is to be inserted vertically. The carpenter wants to draw the outline of the hole on the roof using a pencil, two tacks, and a piece of string (as in Figure 11.4.3*b*). The center point of the ellipse is known,

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and common sense suggests that its major axis must be perpendicular to the drip line of the roof. The carpenter needs to determine the length L of the string and the distance T between a tack and the center point. The architect's plans show that the pitch of the roof is p (pitch = rise over run; see the accompanying figure). Find T and L in terms of D and p. [*Note:* This exercise is based on an article by William H. Enos, which appeared in the *Mathematics Teacher*, Feb. 1991, p. 148.]

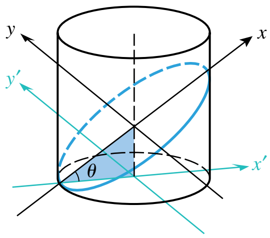


Figure Ex-60

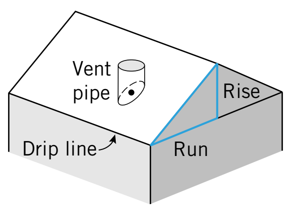


Figure Ex-61

62. Prove: The line tangent to the parabola $x^2 = 4py$ at the point (x_0, y_0) is $x_0x = 2p(y + y_0)$.

63. Prove: The line tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the point (x_0, y_0) has the equation

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1$$

64. Prove: The line tangent to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

at the point (x_0, y_0) has the equation

$$\frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1$$

65. Use the results in Exercises 63 and 64 to show that if an ellipse and a hyperbola have the same foci, then at each point of intersection their tangent lines are perpendicular.

66. Find two values of k such that the line $x + 2y = k$ is tangent to the ellipse $x^2 + 4y^2 = 8$. Find the points of tangency.

67. Find the coordinates of all points on the hyperbola

$$4x^2 - y^2 = 4$$

where the two lines that pass through the point and the foci are perpendicular.

68. A line tangent to the hyperbola $4x^2 - y^2 = 36$ intersects the y-axis at the point $(0, 4)$. Find the point(s) of tangency.

69. As illustrated in the accompanying figure, suppose that two observers are stationed at the points $F_1(c, 0)$ and $F_2(-c, 0)$ in an xy-coordinate system. Suppose also that the sound of an explosion in the xy-plane is heard by the F_1 observer t seconds before it is heard by the F_2 observer. Assuming that the speed of sound is a constant v, show that the explosion occurred somewhere on the hyperbola

$$\frac{x^2}{v^2t^2/4} - \frac{y^2}{c^2 - (v^2t^2/4)} = 1$$

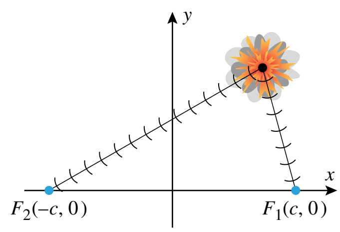


Figure Ex-69

70. As illustrated in the accompanying figure, suppose that two transmitting stations are positioned 100 km apart at points $F_1(50, 0)$ and $F_2(-50, 0)$ on a straight shoreline in an xy-coordinate system. Suppose also that a ship is traveling parallel to the shoreline but 200 km at sea. Find the coordinates of the ship if the stations transmit a pulse simultaneously, but the pulse from station F_1 is received by the ship 0.1 microsecond sooner than the pulse from station F_2. [*Hint:* Use the formula obtained in Exercise 69, assuming that the pulses travel at the speed of light (299,792,458 m/s).]

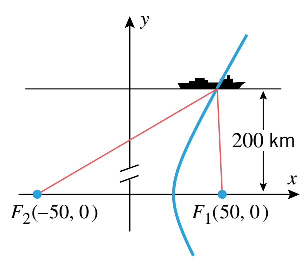


Figure Ex-70

C **71.** As illustrated in the accompanying figure, the tank of an oil truck is 18 feet long and has elliptical cross sections that are 6 feet wide and 4 feet high.

(a) Show that the volume V of oil in the tank (in cubic feet) when it is filled to a depth of h feet is

$$V = 27\left[4\sin^{-1}\frac{h-2}{2} + (h-2)\sqrt{4h - h^2} + 2\pi\right]$$

(b) Use the numerical root-finding capability of a CAS to determine how many inches from the bottom of a dipstick the calibration marks should be placed to indicate when the tank is $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$ full.

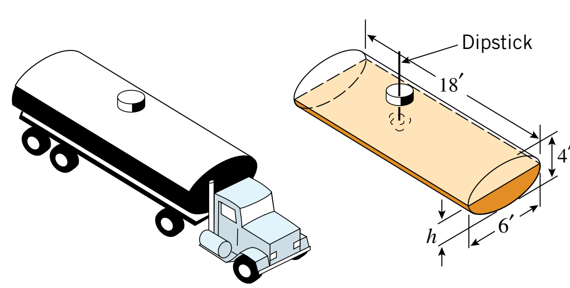


Figure Ex-71

72. Consider the second-degree equation
$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$
where A and C are not both 0. Show by completing the square:
(a) If $AC > 0$, then the equation represents an ellipse, a circle, a point, or has no graph.
(b) If $AC < 0$, then the equation represents a hyperbola or a pair of intersecting lines.
(c) If $AC = 0$, then the equation represents a parabola, a pair of parallel lines, or has no graph.

73. In each part, use the result in Exercise 72 to make a statement about the graph of the equation, and then check your conclusion by completing the square and identifying the graph.
(a) $x^2 - 5y^2 - 2x - 10y - 9 = 0$
(b) $x^2 - 3y^2 - 6y - 3 = 0$
(c) $4x^2 + 8y^2 + 16x + 16y + 20 = 0$
(d) $3x^2 + y^2 + 12x + 2y + 13 = 0$
(e) $x^2 + 8x + 2y + 14 = 0$
(f) $5x^2 + 40x + 2y + 94 = 0$

74. Derive the equation $x^2 = 4py$ in Figure 11.4.6.

75. Derive the equation $(x^2/b^2) + (y^2/a^2) = 1$ given in Figure 11.4.14.

76. Derive the equation $(x^2/a^2) - (y^2/b^2) = 1$ given in Figure 11.4.22.

77. Prove Theorem 11.4.4. [*Hint:* Choose coordinate axes so that the parabola has the equation $x^2 = 4py$. Show that the tangent line at $P(x_0, y_0)$ intersects the y-axis at $Q(0, -y_0)$ and that the triangle whose three vertices are at P, Q, and the focus is isosceles.]

78. Given two intersecting lines, let L_2 be the line with the larger angle of inclination ϕ_2, and let L_1 be the line with the smaller angle of inclination ϕ_1. We define the ***angle θ between L_1 and L_2*** by $\theta = \phi_2 - \phi_1$. (See the accompanying figure.)
(a) Prove: If L_1 and L_2 are not perpendicular, then
$$\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}$$
where L_1 and L_2 have slopes m_1 and m_2.
(b) Prove Theorem 11.4.5. [*Hint:* Introduce coordinate axes so that the ellipse has the equation $x^2/a^2 + y^2/b^2 = 1$, and use part (a).]
(c) Prove Theorem 11.4.6. [*Hint:* Introduce coordinate axes so that the hyperbola has the equation $x^2/a^2 - y^2/b^2 = 1$, and use part (a).]

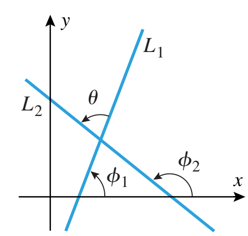


Figure Ex-78

11.5 ROTATION OF AXES; SECOND-DEGREE EQUATIONS

In the preceding section we obtained equations of conic sections with axes parallel to the coordinate axes. In this section we will study the equations of conics that are "tilted" relative to the coordinate axes. This will lead us to investigate rotations of coordinate axes.

QUADRATIC EQUATIONS IN *x* AND *y*

We saw in Examples 8–10 of the preceding section that equations of the form
$$Ax^2 + Cy^2 + Dx + Ey + F = 0 \tag{1}$$
can represent conic sections. Equation (1) is a special case of the more general equation
$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{2}$$
which, if A, B, and C are not all zero, is called a ***second-degree equation*** or ***quadratic equation*** in x and y. We will show later in this section that the graph of any second-degree equation is a conic section (possibly a degenerate conic section). If $B = 0$, then (2) reduces to (1) and the conic section has its axis or axes parallel to the coordinate axes. However, if $B \neq 0$, then (2) contains a "cross-product" term Bxy, and the graph of the conic section represented by the equation has its axis or axes "tilted" relative to the coordinate axes. As an illustration, consider the ellipse with foci $F_1(1, 2)$ and $F_2(-1, -2)$ and such that the sum of the distances from each point $P(x, y)$ on the ellipse to the foci is 6 units. Expressing this condition as an equation, we obtain (Figure 11.5.1)
$$\sqrt{(x-1)^2 + (y-2)^2} + \sqrt{(x+1)^2 + (y+2)^2} = 6$$

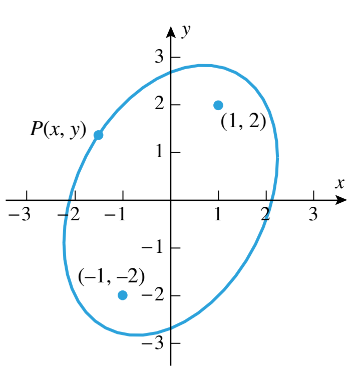


Figure 11.5.1

Squaring both sides, then isolating the remaining radical, then squaring again ultimately yields

$$8x^2 - 4xy + 5y^2 = 36$$

as the equation of the ellipse. This is of form (2) with $A = 8$, $B = -4$, $C = 5$, $D = 0$, $E = 0$, $F = -36$.

ROTATION OF AXES

To study conics that are tilted relative to the coordinate axes it is frequently helpful to rotate the coordinate axes, so that the rotated coordinate axes are parallel to the axes of the conic. Before we can discuss the details, we need to develop some ideas about rotation of coordinate axes.

In Figure 11.5.2*a* the axes of an xy-coordinate system have been rotated about the origin through an angle θ to produce a new $x'y'$-coordinate system. As shown in the figure, each point P in the plane has coordinates (x', y') as well as coordinates (x, y). To see how the two are related, let r be the distance from the common origin to the point P, and let α be the angle shown in Figure 11.5.2*b*. It follows that

$$x = r\cos(\theta + \alpha), \quad y = r\sin(\theta + \alpha) \tag{3}$$

and

$$x' = r\cos\alpha, \quad y' = r\sin\alpha \tag{4}$$

Using familiar trigonometric identities, the relationships in (3) can be written as

$$x = r\cos\theta\cos\alpha - r\sin\theta\sin\alpha$$
$$y = r\sin\theta\cos\alpha + r\cos\theta\sin\alpha$$

and on substituting (4) in these equations we obtain the following relationships called the ***rotation equations***:

$$\begin{aligned} x &= x'\cos\theta - y'\sin\theta \\ y &= x'\sin\theta + y'\cos\theta \end{aligned} \tag{5}$$

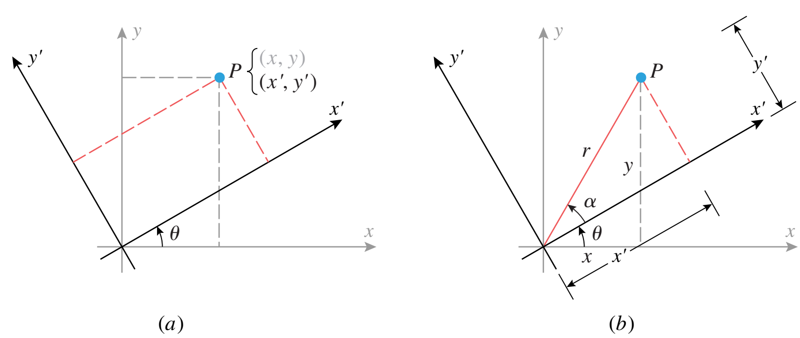


Figure 11.5.2

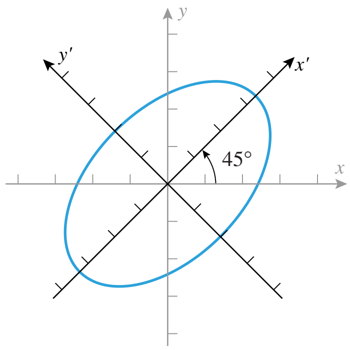
Example 1 Suppose that the axes of an xy-coordinate system are rotated through an angle of $\theta = 45°$ to obtain an $x'y'$-coordinate system. Find the equation of the curve

$$x^2 - xy + y^2 - 6 = 0$$

in $x'y'$-coordinates.

Solution. Substituting $\sin\theta = \sin 45° = 1/\sqrt{2}$ and $\cos\theta = \cos 45° = 1/\sqrt{2}$ in (5) yields the rotation equations

$$x = \frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}} \quad \text{and} \quad y = \frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}}$$



$x^2 - xy + y^2 - 6 = 0$

Figure 11.5.3

Substituting these into the given equation yields

$$\left(\frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}}\right)^2 - \left(\frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}}\right)\left(\frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}}\right) + \left(\frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}}\right)^2 - 6 = 0$$

or

$$\frac{x'^2 - 2x'y' + y'^2 - x'^2 + y'^2 + x'^2 + 2x'y' + y'^2}{2} = 6$$

or

$$\frac{x'^2}{12} + \frac{y'^2}{4} = 1$$

which is the equation of an ellipse (Figure 11.5.3). ◀

If the rotation equations (5) are solved for x' and y' in terms of x and y, one obtains (Exercise 14):

$$\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta \end{aligned} \tag{6}$$

Example 2 Find the new coordinates of the point (2, 4) if the coordinate axes are rotated through an angle of $\theta = 30^\circ$.

Solution. Using the rotation equations in (6) with $x = 2$, $y = 4$, $\cos\theta = \cos 30^\circ = \sqrt{3}/2$, and $\sin\theta = \sin 30^\circ = 1/2$, we obtain

$$\begin{aligned} x' &= 2(\sqrt{3}/2) + 4(1/2) = \sqrt{3} + 2 \\ y' &= -2(1/2) + 4(\sqrt{3}/2) = -1 + 2\sqrt{3} \end{aligned}$$

Thus, the new coordinates are $(\sqrt{3} + 2, -1 + 2\sqrt{3})$. ◀

ELIMINATING THE CROSS-PRODUCT TERM

In Example 1 we were able to identify the curve $x^2 - xy + y^2 - 6 = 0$ as an ellipse because the rotation of axes eliminated the xy-term, thereby reducing the equation to a familiar form. This occurred because the new $x'y'$-axes were aligned with the axes of the ellipse. The following theorem tells how to determine an appropriate rotation of axes to eliminate the cross-product term of a second-degree equation in x and y.

11.5.1 THEOREM. *If the equation*

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{7}$$

is such that $B \neq 0$, and if an $x'y'$-coordinate system is obtained by rotating the xy-axes through an angle θ satisfying

$$\cot 2\theta = \frac{A - C}{B} \tag{8}$$

then, in $x'y'$-coordinates, Equation (7) will have the form

$$A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0$$

Proof. Substituting (5) into (7) and simplifying yields

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0$$

where

$$\begin{aligned} A' &= A\cos^2\theta + B\cos\theta\sin\theta + C\sin^2\theta \\ B' &= B(\cos^2\theta - \sin^2\theta) + 2(C-A)\sin\theta\cos\theta \\ C' &= A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta \\ D' &= D\cos\theta + E\sin\theta \\ E' &= -D\sin\theta + E\cos\theta \\ F' &= F \end{aligned} \tag{9}$$

(Verify.) To complete the proof we must show that $B' = 0$ if

$$\cot 2\theta = \frac{A-C}{B}$$

or equivalently,

$$\frac{\cos 2\theta}{\sin 2\theta} = \frac{A-C}{B} \tag{10}$$

However, by using the trigonometric double-angle formulas, we can rewrite B' in the form

$$B' = B\cos 2\theta - (A-C)\sin 2\theta$$

Thus, $B' = 0$ if θ satisfies (10). ∎

REMARK. It is always possible to satisfy (8) with an angle θ in the range $0 < \theta < \pi/2$. We will always use such a value of θ.

Example 3 Identify and sketch the curve $xy = 1$.

Solution. As a first step, we will rotate the coordinate axes to eliminate the cross-product term. Comparing the given equation to (7), we have

$$A = 0, \quad B = 1, \quad C = 0$$

Thus, the desired angle of rotation must satisfy

$$\cot 2\theta = \frac{A-C}{B} = \frac{0-0}{1} = 0$$

This condition can be met by taking $2\theta = \pi/2$ or $\theta = \pi/4 = 45^\circ$. Substituting $\cos\theta = \cos 45^\circ = 1/\sqrt{2}$ and $\sin\theta = \sin 45^\circ = 1/\sqrt{2}$ in (5) yields

$$x = \frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}} \quad \text{and} \quad y = \frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}}$$

Substituting these in the equation $xy = 1$ yields

$$\left(\frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}}\right)\left(\frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}}\right) = 1 \quad \text{and} \quad \frac{x'^2}{2} - \frac{y'^2}{2} = 1$$

which is the equation in the $x'y'$-coordinate system of an equilateral hyperbola with vertices at $(\sqrt{2}, 0)$ and $(-\sqrt{2}, 0)$ in that coordinate system (Figure 11.5.4). ◀

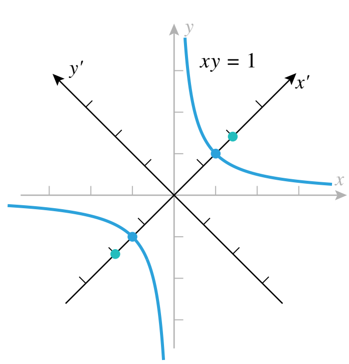


Figure 11.5.4

In problems where it is inconvenient to solve

$$\cot 2\theta = \frac{A-C}{B}$$

for θ, the values of $\sin\theta$ and $\cos\theta$ needed for the rotation equations can be obtained by first calculating $\cos 2\theta$ and then computing $\sin\theta$ and $\cos\theta$ from the identities

$$\sin\theta = \sqrt{\frac{1-\cos 2\theta}{2}} \quad \text{and} \quad \cos\theta = \sqrt{\frac{1+\cos 2\theta}{2}}$$

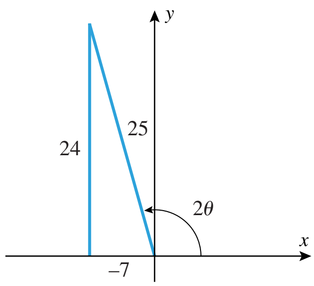


Figure 11.5.5

Example 4 Identify and sketch the curve

$$153x^2 - 192xy + 97y^2 - 30x - 40y - 200 = 0$$

Solution. We have $A = 153$, $B = -192$, and $C = 97$, so

$$\cot 2\theta = \frac{A - C}{B} = -\frac{56}{192} = -\frac{7}{24}$$

Since θ is to be chosen in the range $0 < \theta < \pi/2$, this relationship is represented by the triangle in Figure 11.5.5. From that triangle we obtain $\cos 2\theta = -\frac{7}{25}$, which implies that

$$\cos\theta = \sqrt{\frac{1 + \cos 2\theta}{2}} = \sqrt{\frac{1 - \frac{7}{25}}{2}} = \frac{3}{5}$$

$$\sin\theta = \sqrt{\frac{1 - \cos 2\theta}{2}} = \sqrt{\frac{1 + \frac{7}{25}}{2}} = \frac{4}{5}$$

Substituting these values in (5) yields the rotation equations

$$x = \tfrac{3}{5}x' - \tfrac{4}{5}y' \quad \text{and} \quad y = \tfrac{4}{5}x' + \tfrac{3}{5}y'$$

and substituting these in turn in the given equation yields

$$\tfrac{153}{25}(3x' - 4y')^2 - \tfrac{192}{25}(3x' - 4y')(4x' + 3y') + \tfrac{97}{25}(4x' + 3y')^2$$
$$- \tfrac{30}{5}(3x' - 4y') - \tfrac{40}{5}(4x' + 3y') - 200 = 0$$

which simplifies to

$$25x'^2 + 225y'^2 - 50x' - 200 = 0$$

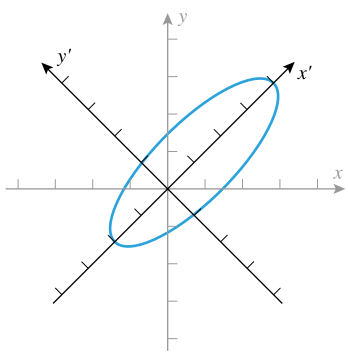
or

$$x'^2 + 9y'^2 - 2x' - 8 = 0$$

Completing the square yields

$$\frac{(x' - 1)^2}{9} + y'^2 = 1$$

which is the equation in the $x'y'$-coordinate system of an ellipse with center $(1, 0)$ in that coordinate system and semiaxes $a = 3$ and $b = 1$ (Figure 11.5.6). ◀



$$\boxed{\frac{(x' - 1)^2}{9} + y'^2 = 1}$$

Figure 11.5.6

THE DISCRIMINANT

It is possible to describe the graph of a second-degree equation without rotating coordinate axes.

> **11.5.2 THEOREM.** *Consider a second-degree equation*
>
> $$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{11}$$
>
> (a) *If* $B^2 - 4AC < 0$, *the equation represents an ellipse, a circle, a point, or else has no graph.*
>
> (b) *If* $B^2 - 4AC > 0$, *the equation represents a hyperbola or a pair of intersecting lines.*
>
> (c) *If* $B^2 - 4AC = 0$, *the equation represents a parabola, a line, a pair of parallel lines, or else has no graph.*

The quantity $B^2 - 4AC$ in this theorem is called the ***discriminant*** of the quadratic equation. To see why Theorem 11.5.2 is true, we need a fact about the discriminant. It can be shown (Exercise 19) that if the coordinate axes are rotated through any angle θ, and if

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0 \tag{12}$$

is the equation resulting from (11) after rotation, then

$$B^2 - 4AC = B'^2 - 4A'C' \tag{13}$$

In other words, the discriminant of a quadratic equation is not altered by rotating the coordinate axes. For this reason the discriminant is said to be ***invariant*** under a rotation of coordinate axes. In particular, if we choose the angle of rotation to eliminate the cross-product term, then (12) becomes

$$A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0 \tag{14}$$

and since $B' = 0$, (13) tells us that

$$B^2 - 4AC = -4A'C' \tag{15}$$

Proof of Theorem 11.5.2(a). If $B^2 - 4AC < 0$, then from (15), $A'C' > 0$, so (14) can be divided through by $A'C'$ and written in the form

$$\frac{1}{C'}\left(x'^2 + \frac{D'}{A'}x'\right) + \frac{1}{A'}\left(y'^2 + \frac{E'}{C'}y'\right) = -\frac{F'}{A'C'}$$

Since $A'C' > 0$, the numbers A' and C' have the same sign. We assume that this sign is positive, since Equation (14) can be multiplied through by -1 to achieve this, if necessary. By completing the squares, we can rewrite the last equation in the form

$$\frac{(x'-h)^2}{(\sqrt{C'})^2} + \frac{(y'-k)^2}{(\sqrt{A'})^2} = K$$

There are three possibilities: $K > 0$, in which case the graph is either a circle or an ellipse, depending on whether or not the denominators are equal; $K < 0$, in which case there is no graph, since the left side is nonnegative for all x' and y'; or $K = 0$, in which case the graph is the single point (h, k), since the equation is satisfied only by $x' = h$ and $y' = k$. The proofs of parts (b) and (c) require a similar kind of analysis. ∎

Example 5 Use the discriminant to identify the graph of

$$8x^2 - 3xy + 5y^2 - 7x + 6 = 0$$

Solution. We have

$$B^2 - 4AC = (-3)^2 - 4(8)(5) = -151$$

Since the discriminant is negative, the equation represents an ellipse, a point, or else has no graph. (Why can't the graph be a circle?) ◀

In cases where a quadratic equation represents a point, a line, a pair of parallel lines, a pair of intersecting lines, or has no graph, we say that equation represents a ***degenerate conic section***. Thus, if we allow for possible degeneracy, it follows from Theorem 11.5.2 that *every quadratic equation has a conic section as its graph*.

EXERCISE SET 11.5 C CAS

1. Let an $x'y'$-coordinate system be obtained by rotating an xy-coordinate system through an angle of $\theta = 60°$.
 (a) Find the $x'y'$-coordinates of the point whose xy-coordinates are $(-2, 6)$.
 (b) Find an equation of the curve $\sqrt{3}xy + y^2 = 6$ in $x'y'$-coordinates.
 (c) Sketch the curve in part (b), showing both xy-axes and $x'y'$-axes.

2. Let an $x'y'$-coordinate system be obtained by rotating an xy-coordinate system through an angle of $\theta = 30°$.
 (a) Find the $x'y'$-coordinates of the point whose xy-coordinates are $(1, -\sqrt{3})$.
 (b) Find an equation of the curve $2x^2 + 2\sqrt{3}xy = 3$ in $x'y'$-coordinates.
 (c) Sketch the curve in part (b), showing both xy-axes and $x'y'$-axes.

In Exercises 3–12, rotate the coordinate axes to remove the xy-term. Then name the conic and sketch its graph.

3. $xy = -9$

4. $x^2 - xy + y^2 - 2 = 0$

5. $x^2 + 4xy - 2y^2 - 6 = 0$

6. $31x^2 + 10\sqrt{3}xy + 21y^2 - 144 = 0$

7. $x^2 + 2\sqrt{3}xy + 3y^2 + 2\sqrt{3}x - 2y = 0$

8. $34x^2 - 24xy + 41y^2 - 25 = 0$

9. $9x^2 - 24xy + 16y^2 - 80x - 60y + 100 = 0$

10. $5x^2 - 6xy + 5y^2 - 8\sqrt{2}x + 8\sqrt{2}y = 8$

11. $52x^2 - 72xy + 73y^2 + 40x + 30y - 75 = 0$

12. $6x^2 + 24xy - y^2 - 12x + 26y + 11 = 0$

13. Let an $x'y'$-coordinate system be obtained by rotating an xy-coordinate system through an angle θ. Prove: For every value of θ, the equation $x^2 + y^2 = r^2$ becomes $x'^2 + y'^2 = r^2$. Give a geometric explanation.

14. Derive (6) by solving the rotation equations in (5) for x' and y' in terms of x and y.

15. Let an $x'y'$-coordinate system be obtained by rotating an xy-coordinate system through an angle of $45°$. Use (6) to find an equation of the curve $3x'^2 + y'^2 = 6$ in xy-coordinates.

16. Let an $x'y'$-coordinate system be obtained by rotating an xy-coordinate system through an angle of $30°$. Use (5) to find an equation in $x'y'$-coordinates of the curve $y = x^2$.

17. Show that the graph of the equation

$$\sqrt{x} + \sqrt{y} = 1$$

is a portion of a parabola. [*Hint:* First rationalize the equation and then perform a rotation of axes.]

18. Derive the expression for B' in (9).

19. Use (9) to prove that $B^2 - 4AC = B'^2 - 4A'C'$ for all values of θ.

20. Use (9) to prove that $A + C = A' + C'$ for all values of θ.

21. Prove: If $A = C$ in (7), then the cross-product term can be eliminated by rotating through $45°$.

22. Prove: If $B \neq 0$, then the graph of $x^2 + Bxy + F = 0$ is a hyperbola if $F \neq 0$ and two intersecting lines if $F = 0$.

In Exercises 23–27, use the discriminant to identify the graph of the given equation.

23. $x^2 - xy + y^2 - 2 = 0$

24. $x^2 + 4xy - 2y^2 - 6 = 0$

25. $x^2 + 2\sqrt{3}xy + 3y^2 + 2\sqrt{3}x - 2y = 0$

26. $6x^2 + 24xy - y^2 - 12x + 26y + 11 = 0$

27. $34x^2 - 24xy + 41y^2 - 25 = 0$

28. Each of the following represents a degenerate conic section. Where possible, sketch the graph.
(a) $x^2 - y^2 = 0$
(b) $x^2 + 3y^2 + 7 = 0$
(c) $8x^2 + 7y^2 = 0$
(d) $x^2 - 2xy + y^2 = 0$
(e) $9x^2 + 12xy + 4y^2 - 36 = 0$
(f) $x^2 + y^2 - 2x - 4y = -5$

29. Prove parts (*b*) and (*c*) of Theorem 11.5.2.

C **30.** Consider the conic whose equation is

$$x^2 + xy + 2y^2 - x + 3y + 1 = 0$$

(a) Use the discriminant to identify the conic.
(b) Graph the equation by solving for y in terms of x and graphing both solutions.
(c) Your CAS may be able to graph the equation in the form given. If so, graph the equation in this way.

C **31.** Consider the conic whose equation is

$$2x^2 + 9xy + y^2 - 6x + y - 4 = 0$$

(a) Use the discriminant to identify the conic.
(b) Graph the equation by solving for y in terms of x and graphing both solutions.
(c) Your CAS may be able to graph the equation in the form given. If so, graph the equation in this way.

11.6 CONIC SECTIONS IN POLAR COORDINATES

It will be shown later in the text that if an object moves in a gravitational field that is directed toward a fixed point (such as the center of the Sun), then the path of that object must be a conic section with the fixed point at a focus. For example, planets in our solar system move along elliptical paths with the Sun at a focus, and the comets move along parabolic, elliptical, or hyperbolic paths with the Sun at a focus, depending on the conditions under which they were born. For applications of this type it is usually desirable to express the equations of the conic sections in polar coordinates with the pole at a focus. In this section we will show how to do this.

THE FOCUS–DIRECTRIX CHARACTERIZATION OF CONICS

To obtain polar equations for the conic sections we will need the following theorem.

> **11.6.1** THEOREM (***Focus–Directrix Property of Conics***). *Suppose that a point P moves in the plane determined by a fixed point (called the* ***focus****) and a fixed line (called the* ***directrix****), where the focus does not lie on the directrix. If the point moves in such a way that its distance to the focus divided by its distance to the directrix is some constant e (called the* ***eccentricity****), then the curve traced by the point is a conic section. Moreover, the conic is a parabola if $e = 1$, an ellipse if $0 < e < 1$, and a hyperbola if $e > 1$.*

REMARK. It is an unfortunate historical accident that the letter e is used for the base of the natural logarithms and the eccentricity of conic sections. However, the appropriate interpretation will usually be clear from the context in which the letter is used.

We will not give a formal proof of this theorem; rather, we will use the specific cases in Figure 11.6.1 to illustrate the basic ideas. For the parabola, we will take the directrix to be $x = -p$, as usual; and for the ellipse and the hyperbola we will take the directrix to be $x = a^2/c$. We want to show in all three cases that if P is a point on the graph, F is the focus, and D is the directrix, then the ratio PF/PD is some constant e, where $e = 1$ for the parabola, $0 < e < 1$ for the ellipse, and $e > 1$ for the hyperbola. We will give the arguments for the parabola and ellipse and leave the argument for the hyperbola as an exercise.

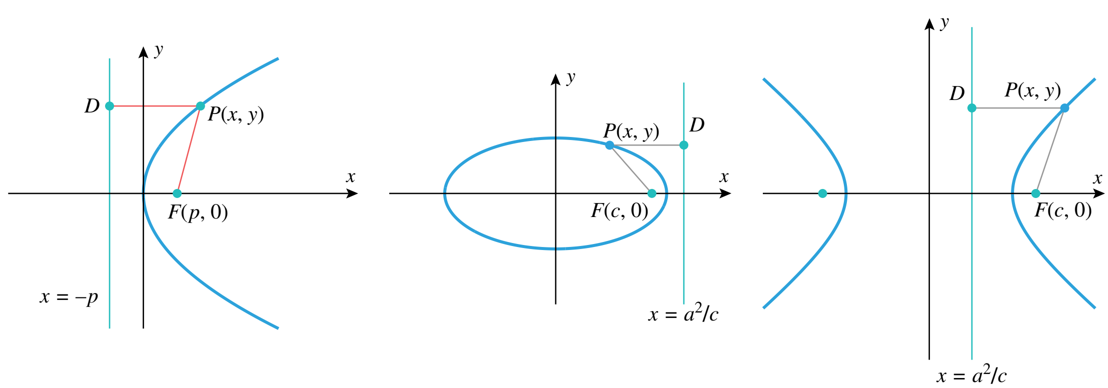


Figure 11.6.1

For the parabola, the distance PF to the focus is equal to the distance PD to the directrix, so that $PF/PD = 1$, which is what we wanted to show. For the ellipse, we rewrite Equation (8) of Section 11.4 as

$$\sqrt{(x-c)^2 + y^2} = a - \frac{c}{a}x = \frac{c}{a}\left(\frac{a^2}{c} - x\right)$$

But the expression on the left side is the distance PF, and the expression in the parentheses on the right side is the distance PD, so we have shown that

$$PF = \frac{c}{a}PD$$

Thus, PF/PD is constant, and the eccentricity is

$$e = \frac{c}{a} \tag{1}$$

If we rule out the degenerate case where $a = 0$ or $c = 0$, then it follows from Formula (7) of Section 11.4 that $0 < c < a$, so $0 < e < 1$, which is what we wanted to show.

We will leave it as an exercise to show that the eccentricity of the hyperbola in Figure 11.6.1 is also given by Formula (1), but in this case it follows from Formula (11) of Section 11.4 that $c > a$, so $e > 1$.

ECCENTRICITY OF AN ELLIPSE AS A MEASURE OF FLATNESS

The eccentricity of an ellipse can be viewed as a measure of its flatness—as e approaches 0 the ellipses become more and more circular, and as e approaches 1 they become more and more flat (Figure 11.6.2). Table 11.6.1 shows the orbital eccentricities of various celestial objects. Note that most of the planets actually have fairly circular orbits.

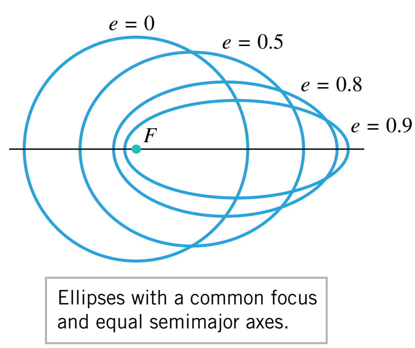


Figure 11.6.2

Table 11.6.1

CELESTIAL BODY	ECCENTRICITY
Mercury	0.206
Venus	0.007
Earth	0.017
Mars	0.093
Jupiter	0.048
Saturn	0.056
Uranus	0.046
Neptune	0.010
Pluto	0.249
Halley's comet	0.970

POLAR EQUATIONS OF CONICS

Our next objective is to derive polar equations for the conic sections from their focus–directrix characterizations. We will assume that the focus is at the pole and the directrix is either parallel or perpendicular to the polar axis. If the directrix is parallel to the polar axis, then it can be above or below the pole; and if the directrix is perpendicular to the polar axis, then it can be to the left or right of the pole. Thus, there are four cases to consider. We will derive the formulas for the case in which the directrix is perpendicular to the polar axis and to the right of the pole.

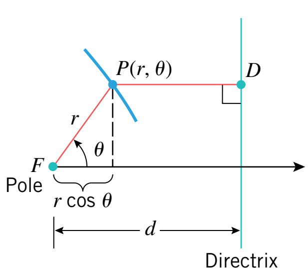


Figure 11.6.3

As illustrated in Figure 11.6.3, let us assume that the directrix is perpendicular to the polar axis and d units to the right of the pole, where the constant d is known. If P is a point on the conic and if the eccentricity of the conic is e, then it follows from Theorem 11.6.1 that $PF/PD = e$ or, equivalently, that

$$PF = ePD \tag{2}$$

However, it is evident from Figure 11.6.3 that $PF = r$ and $PD = d - r\cos\theta$. Thus, (2) can be written as

$$r = e(d - r\cos\theta)$$

which can be solved for r and expressed as

$$r = \frac{ed}{1 + e\cos\theta}$$

(verify). Observe that this single polar equation can represent a parabola, an ellipse, or a hyperbola, depending on the value of e. In contrast, the rectangular equations for these conics all have different forms. The derivations in the other three cases are similar.

11.6.2 THEOREM. *If a conic section with eccentricity e is positioned in a polar coordinate system so that its focus is at the pole and the corresponding directrix is d units from the pole and is either parallel or perpendicular to the polar axis, then the equation of the conic has one of four possible forms, depending on its orientation*:

$$r = \frac{ed}{1 + e\cos\theta} \qquad r = \frac{ed}{1 - e\cos\theta} \tag{3–4}$$

Directrix right of pole — Directrix left of pole

$$r = \frac{ed}{1 + e\sin\theta} \qquad r = \frac{ed}{1 - e\sin\theta} \tag{5–6}$$

Directrix above pole — Directrix below pole

SKETCHING CONICS IN POLAR COORDINATES

Precise graphs of conic sections in polar coordinates can be generated with graphing utilities. However, it is often useful to be able to make quick sketches of these graphs that show their orientations and give some sense of their dimensions. The orientation of a conic relative to the polar axis can be deduced by matching its equation with one of the four forms in Theorem 11.6.2. The key dimensions of a parabola are determined by the constant p (Figure 11.4.5) and those of ellipses and hyperbolas by the constants a, b, and c (Figures 11.4.11 and 11.4.20). Thus, we need to show how these constants can be obtained from the polar equations.

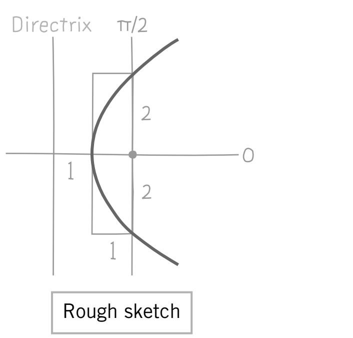


Figure 11.6.4

Example 1 Sketch the graph of $r = \dfrac{2}{1-\cos\theta}$ in polar coordinates.

Solution. The equation is an exact match to (4) with $d = 2$ and $e = 1$. Thus, the graph is a parabola with the focus at the pole and the directrix 2 units to the left of the pole. This tells us that the parabola opens to the right along the polar axis and $p = 1$. Thus, the parabola looks roughly like that sketched in Figure 11.6.4. ◀

All of the important geometric information about an ellipse can be obtained from the values of a, b, and c in Figure 11.6.5. One way to find these values from the polar equation of an ellipse is based on finding the distances from the focus to the vertices. As shown in the figure, let r_0 be the distance from the focus to the closest vertex and r_1 the distance to the farthest vertex. Thus,

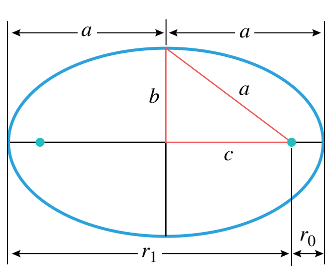


Figure 11.6.5

$$r_0 = a - c \quad \text{and} \quad r_1 = a + c \tag{7}$$

from which it follows that

$$a = \tfrac{1}{2}(r_1 + r_0) \tag{8}$$

and

$$c = \tfrac{1}{2}(r_1 - r_0) \tag{9}$$

Moreover, it also follows from (7) that

$$r_0 r_1 = a^2 - c^2 = b^2$$

Thus,

$$b = \sqrt{r_0 r_1} \tag{10}$$

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REMARK. In words, Formula (8) states that a is the ***arithmetic average*** (also called the ***arithmetic mean***) of r_0 and r_1, and Formula (10) states that b is the ***geometric mean*** of r_0 and r_1.

Example 2 Sketch the graph of $r = \dfrac{6}{2+\cos\theta}$ in polar coordinates.

Solution. This equation does not match any of the forms in Theorem 11.6.2 because they all require a constant term of 1 in the denominator. However, we can put the equation into one of these forms by dividing the numerator and denominator by 2 to obtain

$$r = \frac{3}{1+\frac{1}{2}\cos\theta}$$

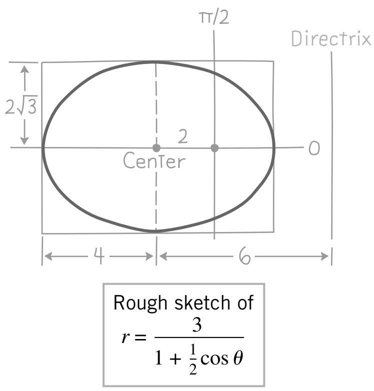


Figure 11.6.6

This is an exact match to (3) with $d = 6$ and $e = \frac{1}{2}$, so the graph is an ellipse with the directrix 6 units to the right of the pole. The distance r_0 from the focus to the closest vertex can be obtained by setting $\theta = 0$ in this equation, and the distance r_1 to the farthest vertex can be obtained by setting $\theta = \pi$. This yields

$$r_0 = \frac{3}{1+\frac{1}{2}\cos 0} = \frac{3}{\frac{3}{2}} = 2, \quad r_1 = \frac{3}{1+\frac{1}{2}\cos\pi} = \frac{3}{\frac{1}{2}} = 6$$

Thus, from Formulas (8), (10), and (9), respectively, we obtain

$$a = \tfrac{1}{2}(r_1 + r_0) = 4, \quad b = \sqrt{r_0 r_1} = 2\sqrt{3}, \quad c = \tfrac{1}{2}(r_1 - r_0) = 2$$

Thus, the ellipse looks roughly like that sketched in Figure 11.6.6. ◀

All of the important information about a hyperbola can be obtained from the values of a, b, and c in Figure 11.6.7. As with the ellipse, one way to find these values from the polar equation of a hyperbola is based on finding the distances from the focus to the vertices. As shown in the figure, let r_0 be the distance from the focus to the closest vertex and r_1 the distance to the farthest vertex. Thus,

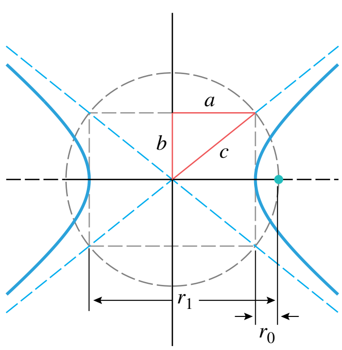


Figure 11.6.7

$$r_0 = c - a \quad \text{and} \quad r_1 = c + a \tag{11}$$

from which it follows that

$$\boxed{a = \tfrac{1}{2}(r_1 - r_0)} \tag{12}$$

and

$$\boxed{c = \tfrac{1}{2}(r_1 + r_0)} \tag{13}$$

Moreover, it also follows from (11) that

$$r_0 r_1 = c^2 - a^2 = b^2$$

from which it follows that

$$\boxed{b = \sqrt{r_0 r_1}} \tag{14}$$

Example 3 Sketch the graph of $r = \dfrac{2}{1+2\sin\theta}$ in polar coordinates.

Solution. This equation is an exact match to (5) with $d = 1$ and $e = 2$. Thus, the graph is a hyperbola with its directrix 1 unit above the pole. However, it is not so straightforward to compute the values of r_0 and r_1, since hyperbolas in polar coordinates are generated in a strange way as θ varies from 0 to 2π. This can be seen from Figure 11.6.8*a*, which is the graph of the given equation in rectangular coordinates. It follows from this graph that the corresponding polar graph is generated in pieces (see Figure 11.6.8*b*):

- As θ varies over the interval $0 \le \theta < 7\pi/6$, the value of r is positive and varies down to $2/3$ and then to $+\infty$, which generates part of the lower branch.

- As θ varies over the interval $7\pi/6 < \theta \le 3\pi/2$, the value of r is negative and varies from $-\infty$ to -2, which generates the right part of the upper branch.
- As θ varies over the interval $3\pi/2 \le \theta < 11\pi/6$, the value of r is negative and varies from -2 to $-\infty$, which generates the left part of the upper branch.
- As θ varies over the interval $11\pi/6 < \theta \le 2\pi$, the value of r is positive and varies from $+\infty$ to 2, which fills in the missing piece of the lower right branch.

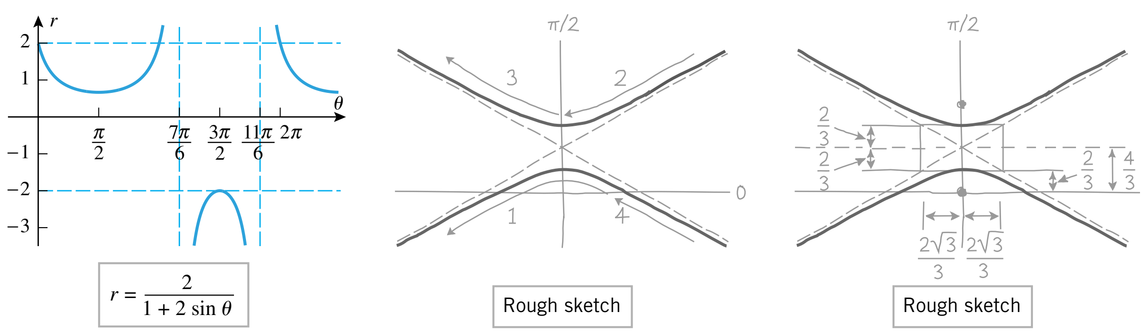
It is now clear that we can obtain r_0 by setting $\theta = \pi/2$ and r_1 by setting $\theta = 3\pi/2$. Keeping in mind that r_0 and r_1 are positive, this yields

$$r_0 = \frac{2}{1+2\sin(\pi/2)} = \frac{2}{3}, \quad r_1 = \left|\frac{2}{1+2\sin(3\pi/2)}\right| = \left|\frac{2}{-1}\right| = 2$$

Thus, from Formulas (12), (14), and (13), respectively, we obtain

$$a = \frac{1}{2}(r_1 - r_0) = \frac{2}{3}, \quad b = \sqrt{r_0 r_1} = \frac{2\sqrt{3}}{3}, \quad c = \frac{1}{2}(r_1 + r_0) = \frac{4}{3}$$

Thus, the hyperbola looks roughly like that sketched in Figure 11.6.8*c*. ◀



(*a*) (*b*) (*c*)

Figure 11.6.8

APPLICATIONS IN ASTRONOMY

In 1609 Johannes Kepler* published a book known as *Astronomia Nova* (or sometimes *Commentaries on the Motions of Mars*) in which he succeeded in distilling thousands of years of observational astronomy into three beautiful laws of planetary motion (Figure 11.6.9).

*JOHANNES KEPLER (1571–1630). German astronomer and physicist, Kepler, whose work provided our contemporary view of planetary motion, led a fascinating but ill-starred life. His alcoholic father made him work in a family-owned tavern as a child, later withdrawing him from elementary school and hiring him out as a field laborer, where the boy contracted smallpox, permanently crippling his hands and impairing his eyesight. In later years, Kepler's first wife and several children died, his mother was accused of witchcraft, and being a Protestant he was often subjected to persecution by Catholic authorities. He was often impoverished, eking out a living as an astrologer and prognosticator. Looking back on his unhappy childhood, Kepler described his father as "criminally inclined" and "quarrelsome" and his mother as "garrulous" and "bad-tempered." However, it was his mother who left an indelible mark on the six-year-old Kepler by showing him the comet of 1577; and in later life he personally prepared her defense against the witchcraft charges. Kepler became acquainted with the work of Copernicus as a student at the University of Tübingen, where he received his master's degree in 1591. He continued on as a theological student, but at the urging of the university officials he abandoned his clerical studies and accepted a position as a mathematician and teacher in Graz, Austria. However, he was expelled from the city when it came under Catholic control, and in 1600 he finally moved on to Prague, where he became an assistant at the observatory of the famous Danish astronomer Tycho Brahe. Brahe was a brilliant and meticulous astronomical observer who amassed the most accurate astronomical data known at that time; and when Brahe died in 1601 Kepler inherited the treasure-trove of data. After eight years of intense labor, Kepler deciphered the underlying principles buried in the data and in 1609 published his monumental work, *Astronomia Nova*, in which he stated his first two laws of planetary motion. Commenting on his discovery of elliptical orbits, Kepler wrote, "I was almost driven to madness in considering and calculating this matter. I could not find out why the planet would rather go on an elliptical orbit (rather than a circle). Oh ridiculous me!" It ultimately remained for Isaac Newton to discover the laws of gravitation that explained the reason for elliptical orbits.

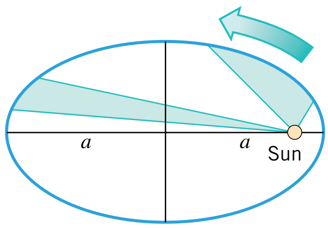


Figure 11.6.9

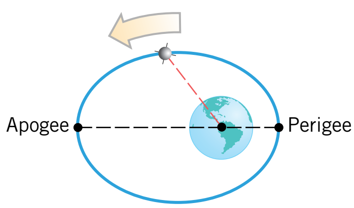


Figure 11.6.10

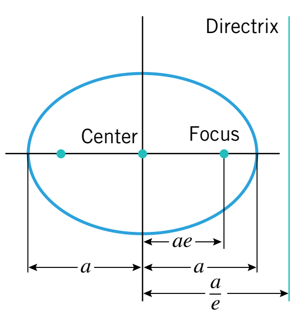


Figure 11.6.11

11.6.3 KEPLER'S LAWS.

- First law (***Law of Orbits***). Each planet moves in an elliptical orbit with the Sun at a focus.
- Second law (***Law of Areas***). The radial line from the center of the Sun to the center of a planet sweeps out equal areas in equal times.
- Third law (***Law of Periods***). The square of a planet's period (the time it takes the planet to complete one orbit about the Sun) is proportional to the cube of the semimajor axis of its orbit.

Kepler's laws, although stated for planetary motion around the Sun, apply to all orbiting celestial bodies that are subjected to a *single* central gravitational force—artificial satellites subjected only to the central force of Earth's gravity and moons subjected only to the central gravitational force of a planet, for example. Later in the text we will derive Kepler's laws from basic principles, but for now we will show how they can be used in basic astronomical computations.

In an elliptical orbit, the closest point to the focus is called the ***perigee*** and the farthest point the ***apogee*** (Figure 11.6.10). The distances from the focus to the perigee and apogee are called the ***perigee distance*** and ***apogee distance***, respectively. For orbits around the Sun, it is more common to use the terms ***perihelion*** and ***aphelion***, rather than perigee and apogee, and to measure time in Earth years and distances in astronomical units (AU), where 1 AU is the semimajor axis a of the Earth's orbit (approximately 150×10^6 km or 92.9×10^6 mi). With this choice of units, the constant of proportionality in Kepler's third law is 1, since $a = 1$ AU produces a period of $T = 1$ Earth year. In this case Kepler's third law can be expressed as

$$T = a^{3/2} \tag{15}$$

Shapes of elliptical orbits are often specified by giving the eccentricity e and the semimajor axis a, so it is useful to express the polar equations of an ellipse in terms of these constants. Figure 11.6.11, which can be obtained from the ellipse in Figure 11.6.1 and the relationship $c = ea$, implies that the distance d between the focus and the directrix is

$$d = \frac{a}{e} - c = \frac{a}{e} - ea = \frac{a(1-e^2)}{e} \tag{16}$$

from which it follows that $ed = a(1-e^2)$. Thus, depending on the orientation of the ellipse, the formulas in Theorem 11.6.2 can be expressed in terms of a and e as

$$r = \frac{a(1-e^2)}{1 \pm e\cos\theta} \qquad r = \frac{a(1-e^2)}{1 \pm e\sin\theta} \tag{17–18}$$

$+$: Directrix right of pole
$-$: Directrix left of pole

$+$: Directrix above pole
$-$: Directrix below pole

Moreover, it is evident from Figure 11.6.11 that the distances from the focus to the closest and farthest vertices can be expressed in terms of a and e as

$$r_0 = a - ea = a(1-e) \quad \text{and} \quad r_1 = a + ea = a(1+e) \tag{19–20}$$

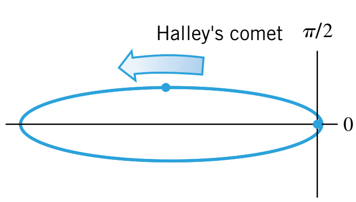


Figure 11.6.12

Example 4 Halley's comet (last seen in 1986) has an eccentricity of 0.97 and a semimajor axis of $a = 18.1$ AU.

(a) Find the equation of its orbit in the polar coordinate system shown in Figure 11.6.12.

(b) Find the period of its orbit.

(c) Find its perihelion and aphelion distances.

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Halley's comet photographed April 21, 1910 in Peru

Solution (a). From (17), the polar equation of the orbit has the form

$$r = \frac{a(1-e^2)}{1 + e\cos\theta}$$

But $a(1-e^2) = 18.1[1-(0.97)^2] \approx 1.07$. Thus, the equation of the orbit is

$$r = \frac{1.07}{1 + 0.97\cos\theta}$$

Solution (b). From (15), with $a = 18.1$, the period of the orbit is

$$T = (18.1)^{3/2} \approx 77 \text{ years}$$

Solution (c). Since the perihelion and aphelion distances are the distances to the closest and farthest vertices, respectively, it follows from (19) and (20) that

$$r_0 = a - ea = a(1-e) = 18.1(1-0.97) \approx 0.543 \text{ AU}$$
$$r_1 = a + ea = a(1+e) = 18.1(1+0.97) \approx 35.7 \text{ AU}$$

or since 1 AU $\approx 150 \times 10^6$ km, the perihelion and aphelion distances in kilometers are

$$r_0 = 18.1(1-0.97)(150 \times 10^6) \approx 81{,}500{,}000 \text{ km}$$
$$r_1 = 18.1(1+0.97)(150 \times 10^6) \approx 5{,}350{,}000{,}000 \text{ km}$$ ◀

FOR THE READER. Use the polar equation of the orbit of Halley's comet to check the values of r_0 and r_1.

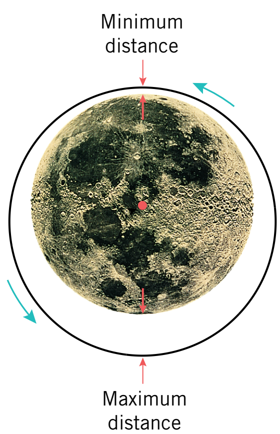

Figure 11.6.13

Example 5 An Apollo lunar lander orbits the Moon in an elliptic orbit with eccentricity $e = 0.12$ and semimajor axis $a = 2015$ km. Assuming the Moon to be a sphere of radius 1740 km, find the minimum and maximum heights of the lander above the lunar surface (Figure 11.6.13).

Solution. If we let r_0 and r_1 denote the minimum and maximum distances from the center of the Moon, then the minimum and maximum distances from the surface of the Moon will be

$$d_{\min} = r_0 - 1740$$
$$d_{\max} = r_1 - 1740$$

or from Formulas (19) and (20)

$$d_{\min} = r_0 - 1740 = a(1-e) - 1740 = 2015(0.88) - 1740 = 33.2 \text{ km}$$
$$d_{\max} = r_1 - 1740 = a(1+e) - 1740 = 2015(1.12) - 1740 = 516.8 \text{ km}$$ ◀

EXERCISE SET 11.6 Graphing Utility

For the conics in Exercises 1 and 2, find the eccentricity and the distance from the pole to the directrix, and sketch the graph in polar coordinates.

1. (a) $r = \dfrac{3}{2 - 2\cos\theta}$ (b) $r = \dfrac{3}{2 + \sin\theta}$

(c) $r = \dfrac{4}{2 + 3\cos\theta}$ (d) $r = \dfrac{5}{3 + 3\sin\theta}$

2. (a) $r = \dfrac{4}{3 - 2\cos\theta}$ (b) $r = \dfrac{3}{3 - 4\sin\theta}$

(c) $r = \dfrac{1}{3 + 3\sin\theta}$ (d) $r = \dfrac{1}{2 + 6\sin\theta}$

In Exercises 3 and 4, use Formulas (3)–(6) to name and describe the orientation of the conic, and then check your answer by generating the graph with a graphing utility.

3. (a) $r = \dfrac{8}{1 - \sin\theta}$ (b) $r = \dfrac{16}{4 + 3\sin\theta}$

(c) $r = \dfrac{4}{2 - 3\sin\theta}$ (d) $r = \dfrac{12}{4 + \cos\theta}$

4. (a) $r = \dfrac{15}{1 + \cos\theta}$ (b) $r = \dfrac{2}{3 + 3\cos\theta}$

(c) $r = \dfrac{64}{7 - 12\sin\theta}$ (d) $r = \dfrac{12}{3 - 2\cos\theta}$

In Exercises 5–8, find a polar equation for the conic that has its focus at the pole and satisfies the stated conditions. Points are in polar coordinates and directrices in rectangular coordinates for simplicity. (In some cases there may be more than one conic that satisfies the conditions.)

5. (a) Ellipse; $e = \frac{2}{3}$; directrix $x = 1$.
(b) Parabola; directrix $x = -1$.
(c) Hyperbola; $e = \frac{3}{2}$; directrix $y = 1$.

6. (a) Ellipse; $e = \frac{2}{3}$; directrix $y = -1$.
(b) Parabola; directrix $y = 1$.
(c) Hyperbola; $e = \frac{4}{3}$; directrix $x = -1$.

7. (a) Ellipse; vertices $(6, 0)$ and $(4, \pi)$.
(b) Parabola; vertex $(1, 3\pi/2)$.
(c) Hyperbola; vertices $(3, \pi/2)$ and $(-7, 3\pi/2)$.

8. (a) Ellipse; ends of major axis $(1, \pi/2)$ and $(4, 3\pi/2)$.
(b) Parabola; vertex $(3, \pi)$.
(c) Hyperbola; equilateral; vertex $(5, 0)$.

In Exercises 9 and 10, find the distances from the pole to the vertices, and then apply Formulas (8)–(10) to find the equation of the ellipse in rectangular coordinates.

9. (a) $r = \dfrac{6}{2 + \sin\theta}$ (b) $r = \dfrac{1}{2 - \cos\theta}$

10. (a) $r = \dfrac{6}{5 + 2\cos\theta}$ (b) $r = \dfrac{8}{4 - 3\sin\theta}$

In Exercises 11 and 12, find the distances from the pole to the vertices, and then apply Formulas (12)–(14) to find the equation of the hyperbola in rectangular coordinates.

11. (a) $r = \dfrac{2}{1 + 3\sin\theta}$ (b) $r = \dfrac{10}{6 - 9\cos\theta}$

12. (a) $r = \dfrac{4}{1 - 2\sin\theta}$ (b) $r = \dfrac{15}{2 + 8\cos\theta}$

In Exercises 13 and 14, find a polar equation for the ellipse that has its focus at the pole and satisfies the stated conditions.

13. (a) Directrix to the right of the pole; $a = 8$; $e = \frac{1}{2}$.
(b) Directrix below the pole; $a = 4$; $e = \frac{3}{5}$.
(c) Directrix to the left of the pole; $b = 4$; $e = \frac{3}{5}$.
(d) Directrix above the pole; $c = 5$; $e = \frac{1}{5}$.

14. (a) Directrix above the pole; $a = 10$; $e = \frac{1}{2}$.
(b) Directrix to the left of the pole; $a = 6$; $e = \frac{1}{5}$.
(c) Directrix below the pole; $b = 4$; $e = \frac{3}{4}$.
(d) Directrix to the right of the pole; $c = 10$; $e = \frac{4}{5}$.

15. (a) Show that the eccentricity of an ellipse can be expressed in terms of r_0 and r_1 as

$$e = \frac{r_1 - r_0}{r_1 + r_0}$$

(b) Show that

$$\frac{r_1}{r_0} = \frac{1 + e}{1 - e}$$

16. (a) Show that the eccentricity of a hyperbola can be expressed in terms of r_0 and r_1 as

$$e = \frac{r_1 + r_0}{r_1 - r_0}$$

(b) Show that

$$\frac{r_1}{r_0} = \frac{e + 1}{e - 1}$$

In Exercises 17–22, use the following values, where needed:

radius of the Earth = 4000 mi = 6440 km
1 year (Earth year) = 365 days (Earth days)
1 AU = 92.9×10^6 mi = 150×10^6 km

17. The planet Pluto has eccentricity $e = 0.249$ and semimajor axis $a = 39.5$ AU.
(a) Find the period T in years.
(b) Find the perihelion and aphelion distances.
(c) Choose a polar coordinate system with the center of the Sun at the pole, and find a polar equation of Pluto's orbit in that coordinate system.
(d) Make a sketch of the orbit with reasonably accurate proportions.

18. (a) Let a be the semimajor axis of a planet's orbit around the Sun, and let T be its period. Show that if T is measured in days and a in kilometers, then $T = (365 \times 10^{-9})(a/150)^{3/2}$.
(b) Use the result in part (a) to find the period of the planet Mercury in days, given that its semimajor axis is $a = 57.95 \times 10^6$ km.
(c) Choose a polar coordinate system with the Sun at the pole, and find an equation for the orbit of Mercury in that coordinate system given that the eccentricity of the orbit is $e = 0.206$.
(d) Use a graphing utility to generate the orbit of Mercury from the equation obtained in part (c).

19. The Hale–Bopp comet, discovered independently on July 23, 1995 by Alan Hale and Thomas Bopp, has an orbital eccentricity of $e = 0.9951$ and a period of 2380 years.
(a) Find its semimajor axis in astronomical units (AU).
(b) Find its perihelion and aphelion distances.
(c) Choose a polar coordinate system with the center of the Sun at the pole, and find an equation for the Hale–Bopp orbit in that coordinate system.
(d) Make a sketch of the Hale–Bopp orbit with reasonably accurate proportions.

20. Mars has a perihelion distance of 204,520,000 km and an aphelion distance of 246,280,000 km.
(a) Use these data to calculate the eccentricity, and compare your answer to the value given in Table 11.6.1.

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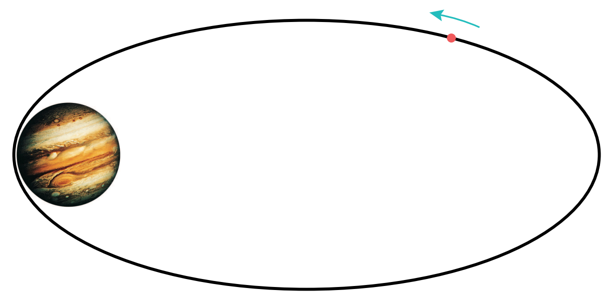
(b) Find the period of Mars.
(c) Choose a polar coordinate system with the center of the Sun at the pole, and find an equation for the orbit of Mars in that coordinate system.
(d) Use a graphing utility to generate the orbit of Mars from the equation obtained in part (c).

21. *Vanguard 1* was launched in March 1958 into an orbit around the Earth with eccentricity $e = 0.21$ and semimajor axis 8864.5 km. Find the minimum and maximum heights of *Vanguard 1* above the surface of the Earth.

22. The planet Jupiter is believed to have a rocky core of radius 10,000 km surrounded by two layers of hydrogen—a 40,000-km-thick layer of compressed metallic-like hydrogen and a 20,000-km-thick layer of ordinary molecular hydrogen. The visible features, such as the Great Red Spot, are at the outer surface of the molecular hydrogen layer. On November 6, 1997 the spacecraft *Galileo* was placed in a Jovian orbit to study the moon Europa. The orbit had eccentricity 0.814580 and semimajor axis 3,514,918.9 km. Find *Galileo*'s minimum and maximum heights above the molecular hydrogen layer (see the accompanying figure).



Not to scale

Figure Ex-22

23. What happens to the distance between the directrix and the center of an ellipse if the foci remain fixed and $e \to 0$?

24. (a) Show that the coordinates of the point P on the hyperbola in Figure 11.6.1 satisfy the equation

$$\sqrt{(x-c)^2 + y^2} = \frac{c}{a}x - a$$

(b) Use the result in part (a) to show that $PF/PD = c/a$.

SUPPLEMENTARY EXERCISES

Graphing Utility C CAS

1. Under what conditions does a parametric curve $x = f(t)$, $y = g(t)$ have a horizontal tangent line? A vertical tangent line? A singular point?

2. Express the point whose xy-coordinates are $(-1, 1)$ in polar coordinates with

(a) $r > 0,\ 0 \le \theta < 2\pi$ (b) $r < 0,\ 0 \le \theta < 2\pi$
(c) $r > 0,\ -\pi < \theta \le \pi$ (d) $r < 0,\ -\pi < \theta \le \pi$.

3. In each part, state the name that describes the polar curve most precisely: a rose, a line, a circle, a limaçon, a cardioid, a spiral, a lemniscate, or none of these.

(a) $r = 3\cos\theta$ (b) $r = \cos 3\theta$
(c) $r = \dfrac{3}{\cos\theta}$ (d) $r = 3 - \cos\theta$
(e) $r = 1 - 3\cos\theta$ (f) $r^2 = 3\cos\theta$
(g) $r = (3\cos\theta)^2$ (h) $r = 1 + 3\theta$

4. In each part: (i) Identify the polar graph as a parabola, an ellipse, or a hyperbola; (ii) state whether the directrix is above, below, to the left, or to the right of the pole; and (iii) find the distance from the pole to the directrix.

(a) $r = \dfrac{1}{3 + \cos\theta}$ (b) $r = \dfrac{1}{1 - 3\cos\theta}$
(c) $r = \dfrac{1}{3(1 + \sin\theta)}$ (d) $r = \dfrac{3}{1 - \sin\theta}$

5. The accompanying figure shows the polar graph of the equation $r = f(\theta)$. Sketch the graph of

(a) $r = f(-\theta)$ (b) $r = f\left(\theta - \dfrac{\pi}{2}\right)$
(c) $r = f\left(\theta + \dfrac{\pi}{2}\right)$ (d) $r = -f(\theta)$
(e) $r = f(\theta) + 1$.

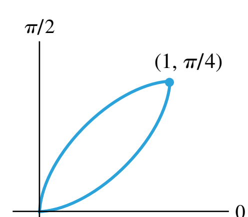


Figure Ex-5

6. Find equations for the two families of circles in the accompanying figure.

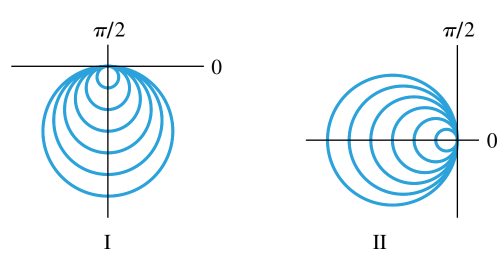


Figure Ex-6

7. In each part, identify the curve by converting the polar equation to rectangular coordinates. Assume that $a > 0$.
(a) $r = a\sec^2\dfrac{\theta}{2}$ (b) $r^2\cos 2\theta = a^2$
(c) $r = 4\csc\left(\theta - \dfrac{\pi}{4}\right)$ (d) $r = 4\cos\theta + 8\sin\theta$

8. Use a graphing utility to investigate how the family of polar curves $r = 1 + a\cos n\theta$ is affected by changing the values of a and n, where a is a positive real number and n is a positive integer. Write a brief paragraph to explain your conclusions.

In Exercises 9 and 10, find an equation in xy-coordinates for the conic section that satisfies the given conditions.

9. (a) Ellipse with eccentricity $e = \frac{2}{7}$ and ends of the minor axis at the points $(0, \pm 3)$.
(b) Parabola with vertex at the origin, focus on the y-axis, and directrix passing through the point $(7, 4)$.
(c) Hyperbola that has the same foci as the ellipse $3x^2 + 16y^2 = 48$ and asymptotes $y = \pm 2x/3$.

10. (a) Ellipse with center $(-3, 2)$, vertex $(2, 2)$, and eccentricity $e = \frac{4}{5}$.
(b) Parabola with focus $(-2, -2)$ and vertex $(-2, 0)$.
(c) Hyperbola with vertex $(-1, 7)$ and asymptotes $y - 5 = \pm 8(x + 1)$.

11. In each part, sketch the graph of the conic section with reasonably accurate proportions.
(a) $x^2 - 4x + 8y + 36 = 0$
(b) $3x^2 + 4y^2 - 30x - 8y + 67 = 0$
(c) $4x^2 - 5y^2 - 8x - 30y - 21 = 0$
(d) $x^2 + y^2 - 3xy - 3 = 0$

C **12.** If you have a CAS that can graph implicit equations, use it to check your work in Exercise 11.

13. It can be shown that hanging cables form parabolic arcs rather than catenaries if they are subjected to uniformly distributed downward forces along their length. For example, if the weight of the roadway in a suspension bridge is assumed to be uniformly distributed along the supporting cables, then the cables can be modeled by parabolas.
(a) Assuming a parabolic model, find an equation for the cable in the accompanying figure, taking the y-axis to be vertical and the origin at the low point of the cable.
(b) Find the length of the cable between the supports.

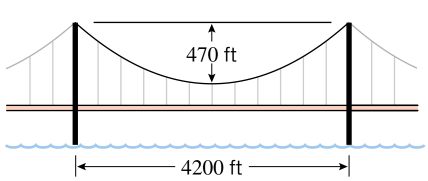


Figure Ex-13

14. A parametric curve of the form

$$x = a\cot t + b\cos t, \quad y = a + b\sin t \qquad (0 < t < 2\pi)$$

is called a ***conchoid of Nicomedes*** (see the accompanying figure for the case $0 < a < b$).
(a) Describe how the conchoid

$$x = \cot t + 4\cos t, \quad y = 1 + 4\sin t$$

is generated as t varies over the interval $0 < t < 2\pi$.
(b) Find the horizontal asymptote of the conchoid given in part (a).
(c) For what values of t does the conchoid in part (a) have a horizontal tangent line? A vertical tangent line?
(d) Find a polar equation $r = f(\theta)$ for the conchoid in part (a), and then find polar equations for the tangent lines to the conchoid at the pole.

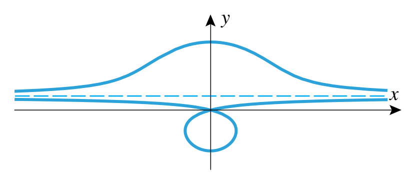


Figure Ex-14

15. Find the area of the region that is common to the circles $r = 1$, $r = 2\cos\theta$, and $r = 2\sin\theta$.

16. Find the area of the region that is inside the cardioid $r = a(1 + \sin\theta)$ and outside the circle $r = a\sin\theta$.

17. (a) Find the arc length of the polar curve $r = 1/\theta$ for $\pi/4 \le \theta \le \pi/2$.
(b) What can you say about the arc length of the portion of the curve that lies inside the circle $r = 1$?

18. (a) If a thread is unwound from a fixed circle while being held taut (i.e., tangent to the circle), then the end of the thread traces a curve called an ***involute of a circle***. Show that if the circle is centered at the origin, has radius a, and the end of the thread is initially at the point $(a, 0)$, then the involute can be expressed parametrically as

$$x = a(\cos\theta + \theta\sin\theta), \quad y = a(\sin\theta - \theta\cos\theta)$$

where θ is the angle shown in part (*a*) of Figure Ex-18 (next page).
(b) Assuming that the dog in part (*b*) of Figure Ex-18 (next page) unwinds its leash while keeping it taut, for what values of θ in the interval $0 \le \theta \le 2\pi$ will the dog be walking North? South? East? West?
(c) Use a graphing utility to generate the curve traced by the dog, and show that it is consistent with your answer in part (b).

19. Let R be the region that is above the x-axis and enclosed between the curve $b^2x^2 - a^2y^2 = a^2b^2$ and the line $x = \sqrt{a^2 + b^2}$.
(a) Sketch the solid generated by revolving R about the x-axis, and find its volume.

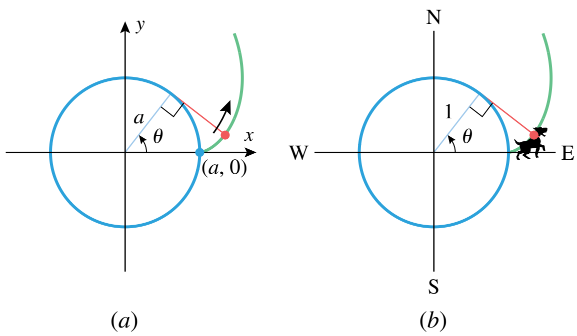


Figure Ex-18

(b) Sketch the solid generated by revolving R about the y-axis, and find its volume.

20. (a) Sketch the curves

$$r = \frac{1}{1+\cos\theta} \quad \text{and} \quad r = \frac{1}{1-\cos\theta}$$

(b) Find polar coordinates of the intersections of the curves in part (a).

(c) Show that the curves are *orthogonal*, that is, their tangent lines are perpendicular at the points of intersection.

21. How is the shape of a hyperbola affected as its eccentricity approaches 1? As it approaches $+\infty$? Draw some pictures to illustrate your conclusions.

22. Use the formula obtained in part (a) of Exercise 67 of Section 11.1 to find the distance between successive tips of the three-petal rose $r = \sin 3\theta$, and check your answer using trigonometry.

23. (a) Find the minimum and maximum x-coordinates of points on the cardioid $r = 1 + \cos\theta$.

(b) Find the minimum and maximum y-coordinates of points on the cardioid in part (a).

24. (a) Show that the maximum value of the y-coordinate of points on the curve $r = 1/\sqrt{\theta}$ for θ in the interval $(0, \pi]$ occurs when $\tan\theta = 2\theta$.

(b) Use Newton's Method to solve the equation in part (a) for θ to at least four decimal-place accuracy.

(c) Use the result of part (b) to approximate the maximum value of y for $0 < \theta \le \pi$.

25. Define the width of a petal of a rose curve to be the dimension shown in the accompanying figure. Show that the width w of a petal of the four-petal rose $r = \cos 2\theta$ is $w = 2\sqrt{6}/9$. [*Hint:* Express y in terms of θ, and investigate the maximum value of y.]

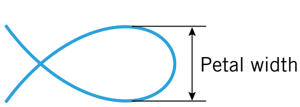


Figure Ex-25

26. A nuclear cooling tower is to have a height of h feet and the shape of the solid that is generated by revolving the region R enclosed by the right branch of the hyperbola $1521x^2 - 225y^2 = 342{,}225$ and the lines $x = 0$, $y = -h/2$, and $y = h/2$ about the y-axis.

(a) Find the volume of the tower.

(b) Find the lateral surface area of the tower.

27. The amusement park rides illustrated in the accompanying figure consist of two connected rotating arms of length 1—an inner arm that rotates counterclockwise at 1 radian per second and an outer arm that can be programmed to rotate either clockwise at 2 radians per second (the Scrambler ride) or counterclockwise at 2 radians per second (the Calypso ride). The center of the rider cage is at the end of the outer arm.

(a) Show that in the Scrambler ride the center of the cage has parametric equations

$$x = \cos t + \cos 2t, \quad y = \sin t - \sin 2t$$

(b) Find parametric equations for the center of the cage in the Calypso ride, and use a graphing utility to confirm that the center traces the curve shown in the accompanying figure.

(c) Do you think that a rider travels the same distance in one revolution of the Scrambler ride as in one revolution of the Calypso ride? Justify your conclusion.

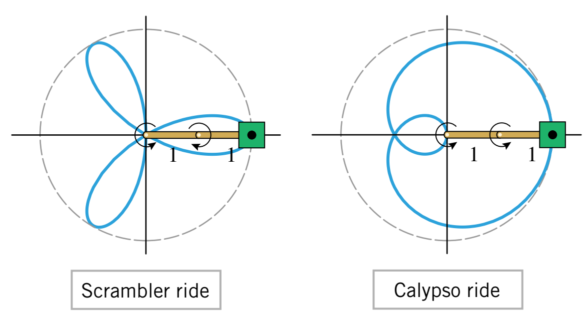


Figure Ex-27

28. Use a graphing utility to explore the effect of changing the rotation rates and the arm lengths in Exercise 27.

29. Use the parametric equations $x = a\cos t$, $y = b\sin t$ to show that the circumference C of an ellipse with semimajor axis a and eccentricity e is

$$C = 4a\int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 u}\, du$$

30. Use Simpson's rule or the numerical integration capability of a graphing utility to approximate the circumference of the ellipse $4x^2 + 9y^2 = 36$ from the integral obtained in Exercise 29.

31. (a) Calculate the eccentricity of the Earth's orbit, given that the ratio of the distance between the center of the Earth and the center of the Sun at perihelion to the distance between the centers at aphelion is $\frac{59}{61}$.

(b) Find the distance between the center of the Earth and the center of the Sun at perihelion, given that the average value of the perihelion and aphelion distances between the centers is 93 million miles.

(c) Use the result in Exercise 29 and Simpson's rule or the numerical integration capability of a graphing utility to approximate the distance that the Earth travels in 1 year (one revolution around the Sun).

32. It will be shown later in this text that if a projectile is launched with speed v_0 at an angle α with the horizontal and at a height y_0 above ground level, then the resulting trajectory relative to the coordinate system in the accompanying figure will have parametric equations

$$x = (v_0 \cos\alpha)t, \quad y = y_0 + (v_0 \sin\alpha)t - \tfrac{1}{2}gt^2$$

where g is the acceleration due to gravity.

(a) Show that the trajectory is a parabola.

(b) Find the coordinates of the vertex.

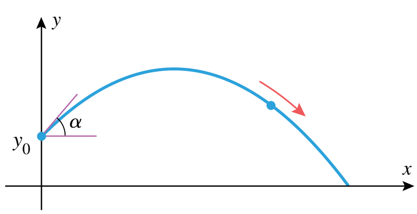


Figure Ex-32

33. Mickey Mantle is recognized as baseball's unofficial king of long home runs. On April 17, 1953 Mantle blasted a pitch by Chuck Stobbs of the hapless Washington Senators out of Griffith Stadium, just clearing the 50-ft wall at the 391-ft marker in left center. Assuming that the ball left the bat at a height of 3 ft above the ground and at an angle of 45°, use the parametric equations in Exercise 32 with $g = 32$ ft/s^2 to find

(a) the speed of the ball as it left the bat

(b) the maximum height of the ball

(c) the distance along the ground from home plate where the ball struck the ground.

C **34.** Recall from Section 7.5 that the Fresnel sine and cosine functions are defined as

$$S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt \quad \text{and} \quad C(x) = \int_0^x \cos\left(\frac{\pi t^2}{2}\right) dt$$

The following parametric curve, which is used to study amplitudes of light waves in optics, is called a ***clothoid*** or ***Cornu spiral*** in honor of the French scientist Marie Alfred Cornu (1841–1902):

$$\begin{aligned} x &= C(t) = \int_0^t \cos\left(\frac{\pi u^2}{2}\right) du \\ y &= S(t) = \int_0^t \sin\left(\frac{\pi u^2}{2}\right) du \end{aligned} \qquad (-\infty < t < +\infty)$$

(a) Use a CAS to graph the Cornu spiral.

(b) Describe the behavior of the spiral as $t \to +\infty$ and as $t \to -\infty$.

(c) Find the arc length of the spiral for $-1 \le t \le 1$.

35. As illustrated in the accompanying figure, let $P(r, \theta)$ be a point on the polar curve $r = f(\theta)$, let ψ be the smallest counterclockwise angle from the extended radius OP to the tangent line at P, and let ϕ be the angle of inclination of the tangent line. Derive the formula

$$\tan\psi = \frac{r}{dr/d\theta}$$

by substituting $\tan\phi$ for dy/dx in Formula (7) of Section 11.2 and applying the trigonometric identity

$$\tan(\phi - \theta) = \frac{\tan\phi - \tan\theta}{1 + \tan\phi\tan\theta}$$

In Exercises 36 and 37, use the formula for ψ obtained in Exercise 35.

36. (a) Use the trigonometric identity

$$\tan\frac{\theta}{2} = \frac{1 - \cos\theta}{\sin\theta}$$

to show that if (r, θ) is a point on the cardioid

$$r = 1 - \cos\theta \quad (0 \le \theta < 2\pi)$$

then $\psi = \theta/2$.

(b) Sketch the cardioid and show the angle ψ at the points where the cardioid crosses the y-axis.

(c) Find the angle ψ at the points where the cardioid crosses the y-axis.

37. Show that for a logarithmic spiral $r = ae^{b\theta}$, the angle from the radial line to the tangent line is constant along the spiral (see the accompanying figure). [*Note:* For this reason, logarithmic spirals are sometimes called ***equiangular spirals***.]

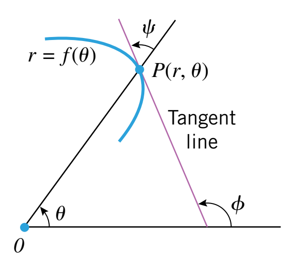


Figure Ex-35

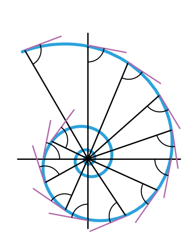


Figure Ex-37

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EXPANDING THE CALCULUS HORIZON

Comet Collision

The Earth lives in a cosmic shooting gallery of comets and asteroids. Although the probability that the Earth will be hit by a comet or asteroid in any given year is small, the consequences of such a collision are so catastrophic that the international community is now beginning to track **near Earth objects** *(NEOs). Your job, as part of the international NEO tracking team, is to compute the orbits of incoming comets and asteroids, determine how close they will come to colliding with the Earth, and issue a notification if there is danger of a collision or near miss.*

At the time when the Earth is at its *aphelion* (its farthest point from the Sun), your NEO tracking team receives a notification from the NASA/Caltech Jet Propulsion Laboratory that a previously unknown comet (designation Rogue 2000) is traveling in the plane of Earth's orbit and hurtling in the direction of the Earth. You immediately transmit a request to NASA for the orbital parameters and the current positions of the Earth and Rogue 2000 and receive the following report:

ORBITAL PARAMETERS

EARTH	ROGUE 2000
Eccentricity: $e_1 = 0.017$	Eccentricity: $e_2 = 0.98$
Semimajor axis: $a_1 = 1\text{ AU} = 1.496 \times 10^8$ km	Semimajor axis: $a_2 = 5\text{ AU} = 7.48 \times 10^8$ km
Period: $T_1 = 1$ year	Period: $T_2 = 5\sqrt{5}$ years

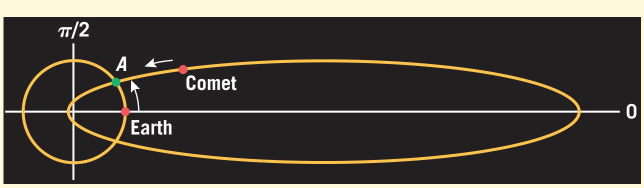
INITIAL POSITION INFORMATION

The major axes of Earth and Rogue 2000 lie on the same line.
The aphelions of Earth and Rogue 2000 are on the same side of the Sun.
Initial polar angle of Earth: $\theta = 0$ radians.
Initial polar angle of Rogue 2000: $\theta = 0.45$ radian.

The Calculation Strategy

Since the immediate concern is a possible collision at intersection A in Figure 1, your team works out the following plan:

Step 1. Find the polar equations for Earth and Rogue 2000.
Step 2. Find the polar coordinates of intersection A.
Step 3. Determine how long it will take the Earth to reach intersection A.
Step 4. Determine where Rogue 2000 will be when the Earth reaches intersection A.
Step 5. Determine how far Rogue 2000 will be from the Earth when the Earth is at intersection A.



Initial configuration of Earth and Rogue 2000

Figure 1

Polar Equations of the Orbits

Exercise 1 Write polar equations of the form

$$r = \frac{a(1-e^2)}{1 - e\cos\theta}$$

for the orbits of Earth and Rogue 2000 using AU units for r.

Exercise 2 Use a graphing utility to generate the two orbits on the same screen.

Intersection of the Orbits

The second step in your team's calculation plan is to find the polar coordinates of intersection A in Figure 1.

Exercise 3 For simplicity, let $k_1 = a_1(1-e_1^2)$ and $k_2 = a_2(1-e_2^2)$, and use the polar equations obtained in Exercise 1 to show that the angle θ at intersection A satisfies the equation

$$\cos\theta = \frac{k_1 - k_2}{k_1 e_2 - k_2 e_1}$$

Exercise 4 Use the result in Exercise 3 and the inverse cosine capability of a calculating utility to show that the angle θ at intersection A in Figure 1 is $\theta = 0.607$ radian.

Exercise 5 Use the result in Exercise 4 and either polar equation obtained in Exercise 1 to show that if r is in AU units, then the polar coordinates of intersection A are $(r, \theta) = (1.014, 0.607)$.

Time Required for Earth to Reach Intersection *A*

According to Kepler's second law (see 11.6.3), the radial line from the center of the Sun to the center of an object orbiting around it sweeps out equal areas in equal times. Thus, if t is the time that it takes for the radial line to sweep out an "elliptic sector" from some initial angle θ_{I} to some final angle θ_{F} (Figure 2), and if T is the period of the object (the time for one complete revolution), then

$$\frac{t}{T} = \frac{\text{area of the "elliptic sector"}}{\text{area of the entire ellipse}} \tag{1}$$

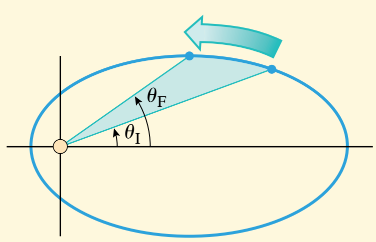


Figure 2

Exercise 6 Use Formula (1) to show that

$$t = \frac{T\displaystyle\int_{\theta_{\mathrm{I}}}^{\theta_{\mathrm{F}}} r^2\, d\theta}{2\pi a^2\sqrt{1-e^2}} \tag{2}$$

Exercise 7 Use a calculating utility with a numerical integration capability, Formula (2), and the polar equation for the orbit of the Earth obtained in Exercise 1 to find the time t (in years) required for the Earth to move from its initial position to intersection A.

Position of Rogue 2000 When the Earth Is at Intersection A

The fourth step in your team's calculation strategy is to determine the position of Rogue 2000 when the Earth reaches intersection A.

Exercise 8 During the time that it takes for the Earth to move from its initial position to intersection A, the polar angle of Rogue 2000 will change from its initial value $\theta_I = 0.45$ radian to some final value θ_F that remains to be determined. Apply Formula (2) using the orbital data for Rogue 2000 and the time t obtained in Exercise 7 to show that θ_F satisfies the equation

$$\int_{0.45}^{\theta_F} \left[\frac{a_2(1-e_2^2)}{1-e_2\cos\theta}\right]^2 d\theta = \frac{2t\pi a_2^2\sqrt{1-e_2^2}}{5\sqrt{5}} \tag{3}$$

Your team is now faced with the problem of solving Equation (3) for the unknown upper limit θ_F. Some members of the team plan to use a CAS to perform the integration, some plan to use integration tables, and others plan to use hand calculation by making the substitution $u = \tan(\theta/2)$ and applying the formulas in (5) of Section 8.6.

Exercise 9

(a) Evaluate the integral in (3) using a CAS or by hand calculation.

(b) Use the root-finding capability of a calculating utility to find the polar angle of Rogue 2000 when the Earth is at intersection A.

Calculating the Critical Distance

It is the policy of your NEO tracking team to issue a notification to various governmental agencies for any asteroid or comet that will be within 4 million kilometers of the Earth at an orbital intersection. (This distance is roughly 10 times that between the Earth and the Moon.) Accordingly, the final step in your team's plan is to calculate the distance between the Earth and Rogue 2000 when the Earth is at intersection A, and then determine whether a notification should be issued.

Exercise 10 Use the polar equation of Rogue 2000 obtained in Exercise 1 and the result in Exercise 9(b) to find polar coordinates of Rogue 2000 with r in AU units when the Earth is at intersection A.

Exercise 11 Use the distance formula in Exercise 67(a) of Section 11.1 to calculate the distance between the Earth and Rogue 2000 in AU units when the Earth is at intersection A, and then use the conversion factor 1 AU $= 1.496 \times 10^8$ km to determine whether a government notification should be issued.

Note: One of the closest near misses in recent history occurred on October 30, 1937 when the asteroid Hermes passed within 900,000 km of the Earth. More recently, on June 14, 1968 the asteroid Icarus passed within 23,000,000 km of the Earth.

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