Johann Bernoulli
Jacques Bernoulli

12

THREE-DIMENSIONAL SPACE; VECTORS

In this chapter we will discuss rectangular coordinate systems in three dimensions, and we will study the analytic geometry of lines, planes, and other basic surfaces. The second theme of this chapter is the study of vectors. These are the mathematical objects that physicists and engineers use to study forces, displacements, and velocities of objects moving on curved paths. More generally, vectors are used to represent all physical entities that involve both a magnitude and a direction for their complete description. We will introduce various algebraic operations on vectors, and we will apply these operations to problems involving force, work, and rotational tendencies in two and three dimensions. Finally, we will discuss cylindrical and spherical coordinate systems, which are appropriate in problems that involve various kinds of symmetries and also have specific applications in navigation and celestial mechanics.

March 6, 2001 14:27 g65-ch12 Sheet number 2 Page number 792 cyan magenta yellow black

12.1 RECTANGULAR COORDINATES IN 3-SPACE; SPHERES; CYLINDRICAL SURFACES

In this section we will discuss coordinate systems in three-dimensional space and some basic facts about surfaces in three dimensions.

RECTANGULAR COORDINATE SYSTEMS

In the remainder of this text we will call three-dimensional space ***3-space***, two-dimensional space (a plane) ***2-space***, and one-dimensional space (a line) ***1-space***. Just as points in 2-space can be placed in one-to-one correspondence with pairs of real numbers using two perpendicular coordinate lines, so points in 3-space can be placed in one-to-one correspondence with triples of real numbers by using three mutually perpendicular coordinate lines, called the ***x-axis***, the ***y-axis***, and the ***z-axis***, positioned so that their origins coincide (Figure 12.1.1). The three coordinate axes form a three-dimensional ***rectangular coordinate system*** (or ***Cartesian coordinate system***). The point of intersection of the coordinate axes is called the ***origin*** of the coordinate system.

Rectangular coordinate systems in 3-space fall into two categories: ***left-handed*** and ***right-handed***. A right-handed system has the property that when the fingers of the right hand are cupped so that they curve from the positive x-axis toward the positive y-axis, the thumb points (roughly) in the direction of the positive z-axis (Figure 12.1.2a). Similarly for a left-handed coordinate system (Figure 12.1.2b). We will use only right-handed coordinate systems in this text.

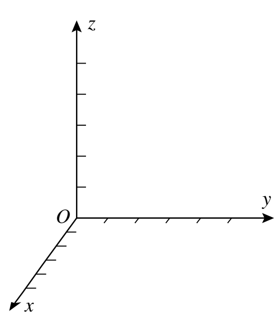


Figure 12.1.1

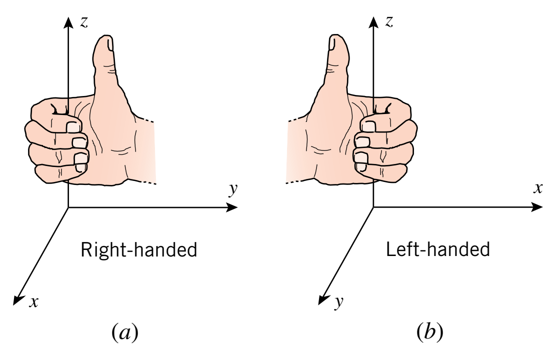


Figure 12.1.2

The coordinate axes, taken in pairs, determine three ***coordinate planes***: the ***xy-plane***, the ***xz-plane***, and the ***yz-plane***. To each point P in 3-space we can assign a triple of real numbers by passing three planes through P parallel to the coordinate planes and letting a, b, and c be the coordinates of the intersections of those planes with the x-axis, y-axis, and z-axis, respectively (Figure 12.1.3). We call a, b, and c the ***x-coordinate***, ***y-coordinate***, and ***z-coordinate*** of P, respectively, and we denote the point P by (a, b, c) or by $P(a, b, c)$. Figure 12.1.4 shows the points (4, 5, 6) and (−3, 2, −4).

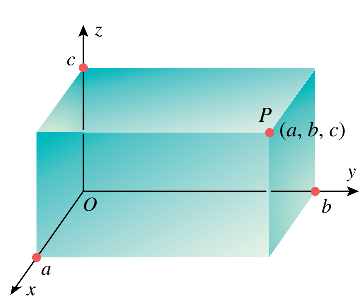


Figure 12.1.3

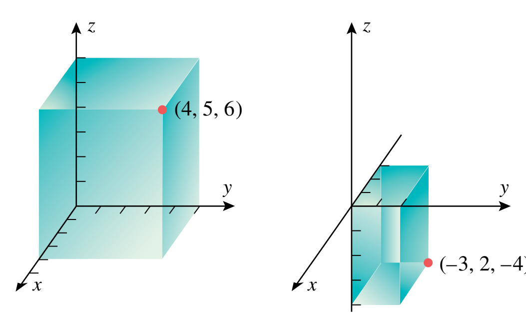


Figure 12.1.4

Just as the coordinate axes in a two-dimensional coordinate system divide 2-space into four quadrants, so the coordinate planes of a three-dimensional coordinate system divide 3-space into eight parts, called ***octants***. The set of points with three positive coordinates forms the ***first octant***; the remaining octants have no standard numbering.

You should be able to visualize the following facts about three-dimensional rectangular coordinate systems:

REGION	DESCRIPTION
xy-plane	Consists of all points of the form $(x, y, 0)$
xz-plane	Consists of all points of the form $(x, 0, z)$
yz-plane	Consists of all points of the form $(0, y, z)$
x-axis	Consists of all points of the form $(x, 0, 0)$
y-axis	Consists of all points of the form $(0, y, 0)$
z-axis	Consists of all points of the form $(0, 0, z)$

DISTANCE IN 3-SPACE

To derive a formula for the distance between two points in 3-space, we start by considering a box whose sides have lengths a, b, and c (Figure 12.1.5). The length d of a diagonal of the box can be obtained by applying the Theorem of Pythagoras twice: first to show that a diagonal of the base has length $\sqrt{a^2+b^2}$, then again to show that a diagonal of the box has length

$$d = \sqrt{(\sqrt{a^2+b^2}\,)^2 + c^2} = \sqrt{a^2+b^2+c^2} \tag{1}$$

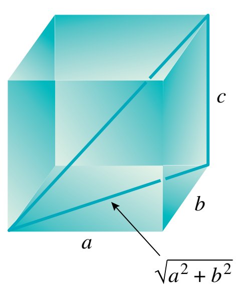


Figure 12.1.5

We can now obtain a formula for the distance d between two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ in 3-space by finding the length of the diagonal of a box that has these points as diagonal corners (Figure 12.1.6). The sides of such a box have lengths

$$|x_2 - x_1|, \quad |y_2 - y_1|, \quad \text{and} \quad |z_2 - z_1|$$

and hence from (1) the distance d between the points P_1 and P_2 is

$$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2} \tag{2}$$

(where we have omitted the unnecessary absolute value signs).

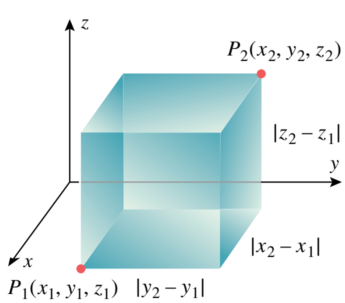


Figure 12.1.6

REMARK. Recall that in 2-space the distance d between points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

$$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$

Thus, the distance formula in 3-space has the same form as the formula in 2-space, but it has a third term to account for the additional dimension. We will see that this is a common occurrence in extending formulas from 2-space to 3-space.

Example 1 Find the distance d between the points $(2, 3, -1)$ and $(4, -1, 3)$.

Solution. From Formula (2)

$$d = \sqrt{(4-2)^2 + (-1-3)^2 + (3+1)^2} = \sqrt{36} = 6$$ ◀

GRAPHS IN 3-SPACE

Recall that in an xy-coordinate system, the set of points (x, y) whose coordinates satisfy an equation in x and y is called the *graph* of the equation. Analogously, in an xyz-coordinate system, the set of points (x, y, z) whose coordinates satisfy an equation in x, y, and z is called the ***graph*** of the equation. For example, consider the equation

$$x^2 + y^2 + z^2 = 25$$

The coordinates of a point (x, y, z) satisfy this equation if and only if the distance from the origin to the point is 5 (why?). Thus, the graph of this equation is a sphere of radius 5 centered at the origin (Figure 12.1.7).

SPHERES

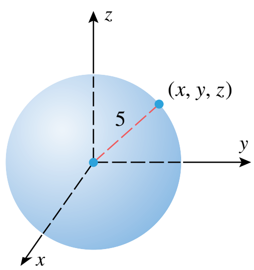


Figure 12.1.7

The sphere with center (x_0, y_0, z_0) and radius r consists of those points (x, y, z) whose coordinates satisfy

$$\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}=r$$

or, equivalently,

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=r^2 \qquad (3)$$

This is called the ***standard equation of the sphere*** with center (x_0, y_0, z_0) and radius r. Some examples are given in the following table.

EQUATION	GRAPH
$(x-3)^2+(y-2)^2+(z-1)^2=9$	Sphere with center (3, 2, 1) and radius 3
$(x+1)^2+y^2+(z+4)^2=5$	Sphere with center (–1, 0, –4) and radius $\sqrt{5}$
$x^2+y^2+z^2=1$	Sphere with center (0, 0, 0) and radius 1

Recall that in 2-space the standard equation of the circle with center (x_0, y_0) and radius r is

$$(x-x_0)^2+(y-y_0)^2=r^2$$

Thus, the standard equation of a sphere in 3-space has the same form as the standard equation of a circle in 2-space, but with an additional term to account for the third coordinate.

Example 2 A sphere S has center in the first octant and is tangent to each of the three coordinate planes. The distance from the origin to the sphere is $3-\sqrt{3}$ units. What is the equation of the sphere?

Solution. Let $P(x_0, y_0, z_0)$ and r denote the center and radius of S, respectively. In order for S to be tangent to the xy-plane, the distance $|z_0|$ from $P(x_0, y_0, z_0)$ to the xy-plane must equal r. Since $P(x_0, y_0, z_0)$ is in the first octant, we conclude that $z_0=|z_0|=r$. Similarly, $x_0=y_0=r$ and the center of S is $P(r, r, r)$. The distance from the origin to the center of S is then $\sqrt{r^2+r^2+r^2}=\sqrt{3r^2}=\sqrt{3}r$, from which it follows that the distance $3-\sqrt{3}=\sqrt{3}(\sqrt{3}-1)$ from the origin to S is given by $\sqrt{3}r-r=(\sqrt{3}-1)r$. Solving the equation $(\sqrt{3}-1)r=\sqrt{3}(\sqrt{3}-1)$ yields the solution $r=\sqrt{3}$. Therefore, the equation of the sphere is

$$(x-\sqrt{3})^2+(y-\sqrt{3})^2+(z-\sqrt{3})^2=3$$ ◀

If the terms in (3) are expanded and like terms are then collected, then the resulting equation has the form

$$x^2+y^2+z^2+Gx+Hy+Iz+J=0 \qquad (4)$$

The following example shows how the center and radius of a sphere that is expressed in this form can be obtained by completing the squares.

Example 3 Find the center and radius of the sphere

$$x^2+y^2+z^2-2x-4y+8z+17=0$$

Solution. We can put the equation in the form of (3) by completing the squares:

$$(x^2 - 2x) + (y^2 - 4y) + (z^2 + 8z) = -17$$
$$(x^2 - 2x + 1) + (y^2 - 4y + 4) + (z^2 + 8z + 16) = -17 + 21$$
$$(x - 1)^2 + (y - 2)^2 + (z + 4)^2 = 4$$

which is the equation of the sphere with center $(1, 2, -4)$ and radius 2. ◀

In general, completing the squares in (4) produces an equation of the form

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = k$$

If $k > 0$, then the graph of this equation is a sphere with center (x_0, y_0, z_0) and radius $\sqrt{k}$. If $k = 0$, then the sphere has radius zero, so the graph is the single point (x_0, y_0, z_0). If $k < 0$, the equation is not satisfied by any values of x, y, and z (why?), so it has no graph.

12.1.1 THEOREM. *An equation of the form*

$$x^2 + y^2 + z^2 + Gx + Hy + Iz + J = 0$$

represents a sphere, a point, or has no graph.

CYLINDRICAL SURFACES

Although it is natural to graph equations in two variables in 2-space and equations in three variables in 3-space, it is also possible to graph equations in two variables in 3-space. For example, the graph of the equation $y = x^2$ in an xy-coordinate system is a parabola; however, there is nothing to prevent us from writing this equation as $y = x^2 + 0z$ and inquiring about its graph in an xyz-coordinate system. To obtain this graph we need only observe that the equation $y = x^2$ does not impose any restrictions on z. Thus, if we find values of x and y that satisfy this equation, then the coordinates of the point (x, y, z) will also satisfy the equation for *arbitrary* values of z. Geometrically, the point (x, y, z) lies on the vertical line through the point $(x, y, 0)$ in the xy-plane, which means that we can obtain the graph of $y = x^2$ in an xyz-coordinate system by first graphing the equation in the xy-plane and then translating that graph parallel to the z-axis to generate the entire graph (Figure 12.1.8).

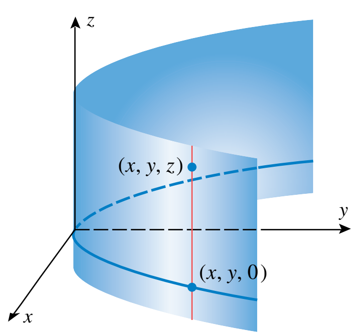


Figure 12.1.8

The process of generating a surface by translating a plane curve parallel to some line is called ***extrusion***, and surfaces that are generated by extrusion are called ***cylindrical surfaces***. A familiar example is the surface of a right circular cylinder, which can be generated by translating a circle parallel to the axis of the cylinder. The following theorem provides basic information about graphing equations in two variables in 3-space:

12.1.2 THEOREM. *An equation that contains only two of the variables x, y, and z represents a cylindrical surface in an xyz-coordinate system. The surface can be obtained by graphing the equation in the coordinate plane of the two variables that appear in the equation and then translating that graph parallel to the axis of the missing variable.*

Example 4 Sketch the graph of $x^2 + z^2 = 1$ in 3-space.

Solution. Since y does not appear in this equation, the graph is a cylindrical surface generated by extrusion parallel to the y-axis. In the xz-plane the graph of the equation $x^2 + z^2 = 1$ is a circle (Figure 12.1.9). Thus, in 3-space the graph is a right circular cylinder along the y-axis. ◀

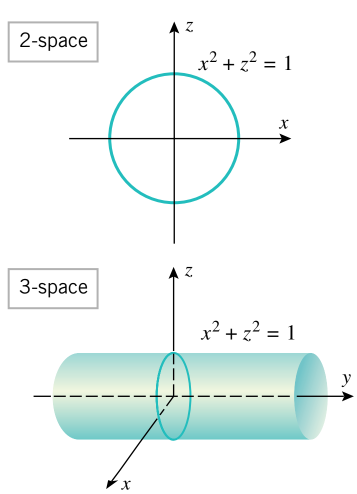


Figure 12.1.9

Example 5 Sketch the graph of $z = \sin y$ in 3-space.

Solution. (See Figure 12.1.10.) ◀

• FOR THE READER. Describe the graph of $x = 1$ in an xyz-coordinate system.

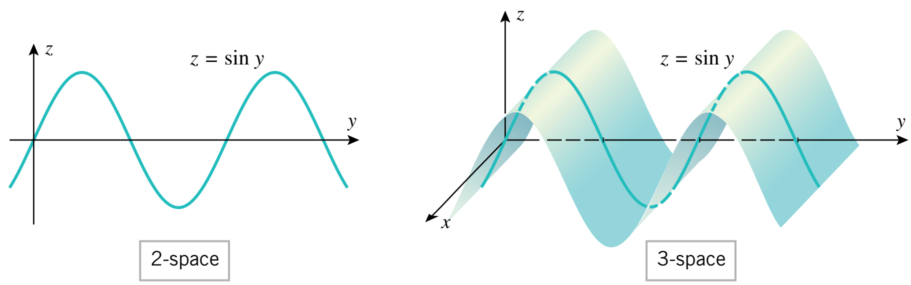
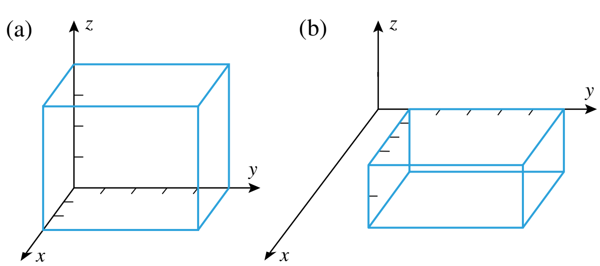


Figure 12.1.10

EXERCISE SET 12.1 Graphing Utility

1. In each part, find the coordinates of the eight corners of the box.



2. A cube of side 4 has its geometric center at the origin and its faces parallel to the coordinate planes. Sketch the cube and give the coordinates of the corners.

3. Suppose that a box has its faces parallel to the coordinate planes and the points $(4, 2, -2)$ and $(-6, 1, 1)$ are endpoints of a diagonal. Sketch the box and give the coordinates of the remaining six corners.

4. Suppose that a box has its faces parallel to the coordinate planes and the points (x_1, y_1, z_1) and (x_2, y_2, z_2) are endpoints of a diagonal.
   (a) Find the coordinates of the remaining six corners.
   (b) Show that the midpoint of the line segment joining (x_1, y_1, z_1) and (x_2, y_2, z_2) is
   $$\left(\tfrac{1}{2}(x_1 + x_2), \tfrac{1}{2}(y_1 + y_2), \tfrac{1}{2}(z_1 + z_2)\right)$$
   [*Suggestion:* Apply Theorem D.2 in Appendix D to three appropriate edges of the box.]

5. Find the center and radius of the sphere that has $(1, -2, 4)$ and $(3, 4, -12)$ as endpoints of a diameter. [See Exercise 4.]

6. Show that $(4, 5, 2)$, $(1, 7, 3)$, and $(2, 4, 5)$ are vertices of an equilateral triangle.

7. (a) Show that $(2, 1, 6)$, $(4, 7, 9)$, and $(8, 5, -6)$ are the vertices of a right triangle.
   (b) Which vertex is at the $90°$ angle?
   (c) Find the area of the triangle.

8. Find the distance from the point $(-5, 2, -3)$ to the
   (a) xy-plane (b) xz-plane (c) yz-plane
   (d) x-axis (e) y-axis (f) z-axis.

9. In each part, find the standard equation of the sphere that satisfies the stated conditions.
   (a) Center $(1, 0, -1)$; diameter $= 8$.
   (b) Center $(-1, 3, 2)$ and passing through the origin.
   (c) A diameter has endpoints $(-1, 2, 1)$ and $(0, 2, 3)$.

10. Find equations of two spheres that are centered at the origin and are tangent to the sphere of radius 1 centered at $(3, -2, 4)$.

11. In each part, find an equation of the sphere with center $(2, -1, -3)$ and satisfying the given condition.
    (a) Tangent to the xy-plane
    (b) Tangent to the xz-plane
    (c) Tangent to the yz-plane

12. (a) Find an equation of the sphere that is inscribed in the cube that is centered at the point $(-2, 1, 3)$ and has sides of length 1 that are parallel to the coordinate planes.
    (b) Find an equation of the sphere that is circumscribed about the cube in part (a).

In Exercises 13–18, describe the surface whose equation is given.

13. $x^2 + y^2 + z^2 + 10x + 4y + 2z - 19 = 0$

14. $x^2 + y^2 + z^2 - y = 0$

15. $2x^2 + 2y^2 + 2z^2 - 2x - 3y + 5z - 2 = 0$

16. $x^2 + y^2 + z^2 + 2x - 2y + 2z + 3 = 0$

17. $x^2 + y^2 + z^2 - 3x + 4y - 8z + 25 = 0$

18. $x^2 + y^2 + z^2 - 2x - 6y - 8z + 1 = 0$

19. In each part, sketch the portion of the surface that lies in the first octant.
    (a) $y = x$ (b) $y = z$ (c) $x = z$

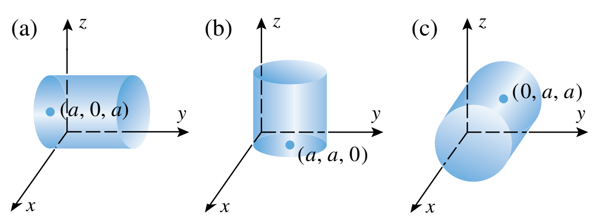
20. In each part, sketch the graph of the equation in 3-space.
    (a) $x = 1$ (b) $y = 1$ (c) $z = 1$

21. In each part, sketch the graph of the equation in 3-space.
    (a) $x^2 + y^2 = 25$ (b) $y^2 + z^2 = 25$ (c) $x^2 + z^2 = 25$

22. In each part, sketch the graph of the equation in 3-space.
    (a) $x = y^2$ (b) $z = x^2$ (c) $y = z^2$

23. In each part, write an equation for the surface.
(a) The plane that contains the x-axis and the point $(0, 1, 2)$.
(b) The plane that contains the y-axis and the point $(1, 0, 2)$.
(c) The right circular cylinder that has radius 1 and is centered on the line parallel to the z-axis that passes through the point $(1, 1, 0)$.
(d) The right circular cylinder that has radius 1 and is centered on the line parallel to the y-axis that passes through the point $(1, 0, 1)$.

24. Find equations for the following right circular cylinders. Each cylinder has radius a and is "tangent" to two coordinate planes.



In Exercises 25–34, sketch the surface in 3-space.

25. $y = \sin x$

26. $y = e^x$

27. $z = 1 - y^2$

28. $z = \cos x$

29. $2x + z = 3$

30. $2x + 3y = 6$

31. $4x^2 + 9z^2 = 36$

32. $z = \sqrt{3 - x}$

33. $y^2 - 4z^2 = 4$

34. $yz = 1$

35. Use a graphing utility to generate the curve $y = x^3/(1+x^2)$ in the xy-plane, and then use the graph to help sketch the surface $z = y^3/(1 + y^2)$ in 3-space.

36. Use a graphing utility to generate the curve $y = x/(1+x^4)$ in the xy-plane, and then use the graph to help sketch the surface $z = y/(1 + y^4)$ in 3-space.

37. If a bug walks on the sphere

$$x^2 + y^2 + z^2 + 2x - 2y - 4z - 3 = 0$$

how close and how far can it get from the origin?

38. Describe the set of all points in 3-space whose coordinates satisfy the inequality $x^2 + y^2 + z^2 - 2x + 8z \leq 8$.

39. Describe the set of all points in 3-space whose coordinates satisfy the inequality $y^2 + z^2 + 6y - 4z > 3$.

40. The distance between a point $P(x, y, z)$ and the point $A(1, -2, 0)$ is twice the distance between P and the point $B(0, 1, 1)$. Show that the set of all such points is a sphere, and find the center and radius of the sphere.

41. As shown in the accompanying figure, a bowling ball of radius R is placed inside a box just large enough to hold it, and it is secured for shipping by packing a Styrofoam sphere into each corner of the box. Find the radius of the largest Styrofoam sphere that can be used. [*Hint:* Take the origin of a Cartesian coordinate system at a corner of the box with the coordinate axes along the edges.]

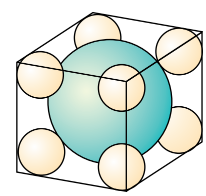


Figure Ex-41

42. Consider the equation

$$x^2 + y^2 + z^2 + Gx + Hy + Iz + J = 0$$

and let $K = G^2 + H^2 + I^2 - 4J$.
(a) Prove that the equation represents a sphere if $K > 0$, a point if $K = 0$, and has no graph if $K < 0$.
(b) In the case where $K > 0$, find the center and radius of the sphere.

43. Show that for all values of θ and ϕ, the point

$$(a \sin\phi \cos\theta, a \sin\phi \sin\theta, a \cos\phi)$$

lies on the sphere $x^2 + y^2 + z^2 = a^2$.

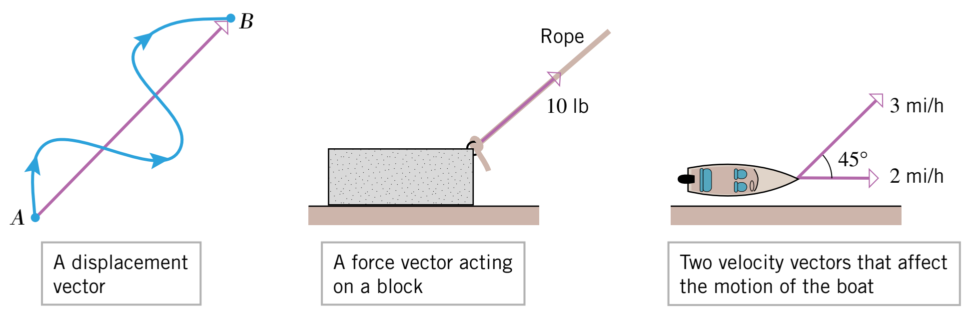
12.2 VECTORS

Many physical quantities such as area, length, mass, and temperature are completely described once the magnitude of the quantity is given. Such quantities are called "scalars." Other physical quantities, called "vectors," are not completely determined until both a magnitude and a direction are specified. For example, winds are usually described by giving their speed and direction, say 20 mi/h northeast. The wind speed and wind direction together form a vector quantity called the wind velocity. Other examples of vectors are force and displacement. In this section we will develop the basic mathematical properties of vectors.

VECTORS IN PHYSICS AND ENGINEERING

A particle that moves along a line can move in only two directions, so its direction of motion can be described by taking one direction to be positive and the other negative. Thus, the *displacement* or *change in position* of the point can be described by a signed real number. For example, a displacement of 3 (= +3) describes a position change of 3 units in the positive direction, and a displacement of −3 describes a position change of 3 units in the negative direction. However, for a particle that moves in two dimensions or three dimensions, a plus or minus sign is no longer sufficient to specify the direction of motion—other methods are required. One method is to use an arrow, called a ***vector***, that points in the direction of motion and whose length represents the distance from the starting point to the ending point; this is called the ***displacement vector*** for the motion. For example, Figure 12.2.1*a* shows the displacement vector of a particle that moves from point A to point B along a circuitous path. Note that the length of the arrow describes the distance between the starting and ending points and not the actual distance traveled by the particle.

Arrows are not limited to describing displacements—they can be used to describe any physical quantity that involves both a magnitude and direction. Two important examples are forces and velocities. For example, the arrow in Figure 12.2.1*b* shows a force vector of 10 lb acting in a specific direction on a block, and the arrows in Figure 12.2.1*c* show the velocity vector of a boat whose motor propels it parallel to the shore at 2 mi/h and the velocity vector of a 3 mi/h wind acting at an angle of 45° with the shoreline. Intuition suggests that the two velocity vectors will combine to produce some net velocity for the boat at an angle to the shoreline. Thus, our first objective in this section is to define mathematical operations on vectors that can be used to determine the combined effect of vectors.

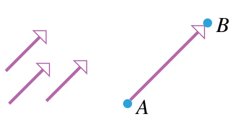


(*a*) (*b*) (*c*)

Figure 12.2.1

VECTORS VIEWED GEOMETRICALLY

Vectors can be represented geometrically by arrows in 2-space or 3-space; the direction of the arrow specifies the direction of the vector and the length of the arrow describes its magnitude. The tail of the arrow is called the ***initial point*** of the vector, and the tip of the arrow the ***terminal point***. We will denote vectors with lowercase boldface type such as **a**, **k**, **v**, **w**, and **x**. When discussing vectors, we will refer to real numbers as ***scalars***. Scalars will be denoted by lowercase italic type such as a, k, v, w, and x. Two vectors, **v** and **w**, are considered to be ***equal*** (also called ***equivalent***) if they have the same length and same direction, in which case we write $\mathbf{v} = \mathbf{w}$. Geometrically, two vectors are equal if they are translations of one another; thus, the three vectors in Figure 12.2.2*a* are equal, even though they are in different positions.



(*a*) (*b*)

Figure 12.2.2

Because vectors are not affected by translation, the initial point of a vector **v** can be moved to any convenient point A by making an appropriate translation. If the initial point of **v** is A and the terminal point is B, then we write $\mathbf{v} = \overrightarrow{AB}$ when we want to emphasize the initial and terminal points (Figure 12.2.2*b*). If the initial and terminal points of a vector coincide, then the vector has length zero; we call this the ***zero vector*** and denote it by **0**. The zero vector does not have a specific direction, so we will agree that it can be assigned any convenient direction in a specific problem.

There are various algebraic operations that are performed on vectors, all of whose definitions originated in physics. We begin with vector addition.

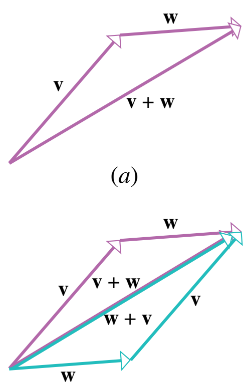


Figure 12.2.3

> **12.2.1** DEFINITION. If **v** and **w** are vectors, then the ***sum*** $\mathbf{v}+\mathbf{w}$ is the vector from the initial point of **v** to the terminal point of **w** when the vectors are positioned so the initial point of **w** is at the terminal point of **v** (Figure 12.2.3*a*).

In Figure 12.2.3*b* we have constructed two sums, $\mathbf{v}+\mathbf{w}$ (purple arrows) and $\mathbf{w}+\mathbf{v}$ (green arrows). It is evident that

$$\mathbf{v}+\mathbf{w}=\mathbf{w}+\mathbf{v}$$

and that the sum coincides with the diagonal of the parallelogram determined by **v** and **w** when these vectors are positioned so they have the same initial point.

Since the initial and terminal points of **0** coincide, it follows that

$$\mathbf{0}+\mathbf{v}=\mathbf{v}+\mathbf{0}=\mathbf{v}$$

> **12.2.2** DEFINITION. If **v** is a nonzero vector and k is a nonzero real number (a scalar), then the ***scalar multiple*** $k\mathbf{v}$ is defined to be the vector whose length is $|k|$ times the length of **v** and whose direction is the same as that of **v** if $k>0$ and opposite to that of **v** if $k<0$. We define $k\mathbf{v}=\mathbf{0}$ if $k=0$ or $\mathbf{v}=\mathbf{0}$.

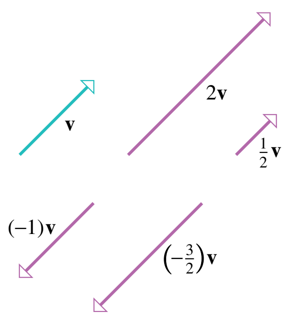


Figure 12.2.4

Figure 12.2.4 shows the geometric relationship between a vector **v** and various scalar multiples of it. Observe that if k and **v** are nonzero, then the vectors **v** and $k\mathbf{v}$ lie on the same line if their initial points coincide and lie on parallel or coincident lines if they do not. Thus, we say that **v** and $k\mathbf{v}$ are ***parallel vectors***. Observe also that the vector $(-1)\mathbf{v}$ has the same length as **v** but is oppositely directed. We call $(-1)\mathbf{v}$ the ***negative*** of **v** and denote it by $-\mathbf{v}$ (Figure 12.2.5). In particular, $-\mathbf{0}=(-1)\mathbf{0}=\mathbf{0}$.

Vector subtraction is defined in terms of addition and scalar multiplication by

$$\mathbf{v}-\mathbf{w}=\mathbf{v}+(-\mathbf{w})$$

The difference $\mathbf{v}-\mathbf{w}$ can be obtained geometrically by first constructing the vector $-\mathbf{w}$ and then adding **v** and $-\mathbf{w}$, say by the parallelogram method (Figure 12.2.6*a*). However, if **v** and **w** are positioned so their initial points coincide, then $\mathbf{v}-\mathbf{w}$ can be formed more directly, as shown in Figure 12.2.6*b*, by drawing the vector from the terminal point of **w** (the second term) to the terminal point of **v** (the first term). In the special case where $\mathbf{v}=\mathbf{w}$ the terminal points of the vectors coincide, so their difference is **0**; that is,

$$\mathbf{v}+(-\mathbf{v})=\mathbf{v}-\mathbf{v}=\mathbf{0}$$

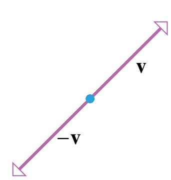


Figure 12.2.5

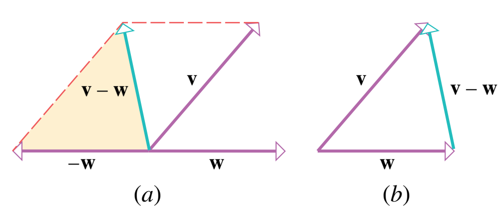


Figure 12.2.6

VECTORS IN COORDINATE SYSTEMS

Problems involving vectors are often best solved by introducing a rectangular coordinate system. If a vector **v** is positioned with its initial point at the origin of a rectangular coordinate system, then its terminal point will have coordinates of the form (v_1, v_2) or (v_1, v_2, v_3), depending on whether the vector is in 2-space or 3-space (Figure 12.2.7). We call these coordinates the ***components*** of **v**, and we write

$$\underbrace{\mathbf{v}=\langle v_1, v_2\rangle}_{\text{2-space}} \quad \text{or} \quad \underbrace{\mathbf{v}=\langle v_1, v_2, v_3\rangle}_{\text{3-space}}$$

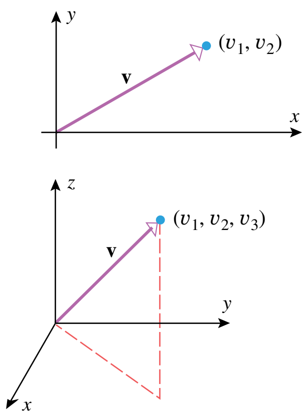


Figure 12.2.7

In particular, the zero vector is

$$\mathbf{0} = \langle 0, 0 \rangle \quad \text{and} \quad \mathbf{0} = \langle 0, 0, 0 \rangle$$

2-space 3-space

Components provide a simple way of identifying equivalent vectors. For example, consider the vectors $\mathbf{v} = \langle v_1, v_2 \rangle$ and $\mathbf{w} = \langle w_1, w_2 \rangle$ in 2-space. If $\mathbf{v} = \mathbf{w}$, then the vectors have the same length and same direction, and this means that their terminal points coincide when their initial points are placed at the origin. It follows that $v_1 = w_1$ and $v_2 = w_2$, so we have shown that equivalent vectors have the same components. Conversely, if $v_1 = w_1$ and $v_2 = w_2$, then the terminal points of the vectors coincide when their initial points are placed at the origin. It follows that the vectors have the same length and same direction, so we have shown that vectors with the same components are equivalent. A similar argument holds for vectors in 3-space, so we have the following result:

12.2.3 THEOREM. *Two vectors are equivalent if and only if their corresponding components are equal.*

For example,

$$\langle a, b, c \rangle = \langle 1, -4, 2 \rangle$$

if and only if $a = 1$, $b = -4$, and $c = 2$.

ARITHMETIC OPERATIONS ON VECTORS

The next theorem shows how to perform arithmetic operations on vectors using components.

12.2.4 THEOREM. *If $\mathbf{v} = \langle v_1, v_2 \rangle$ and $\mathbf{w} = \langle w_1, w_2 \rangle$ are vectors in 2-space and k is any scalar, then*

$$\mathbf{v} + \mathbf{w} = \langle v_1 + w_1, v_2 + w_2 \rangle \qquad (1)$$
$$\mathbf{v} - \mathbf{w} = \langle v_1 - w_1, v_2 - w_2 \rangle \qquad (2)$$
$$k\mathbf{v} = \langle kv_1, kv_2 \rangle \qquad (3)$$

Similarly, if $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ are vectors in 3-space and k is any scalar, then

$$\mathbf{v} + \mathbf{w} = \langle v_1 + w_1, v_2 + w_2, v_3 + w_3 \rangle \qquad (4)$$
$$\mathbf{v} - \mathbf{w} = \langle v_1 - w_1, v_2 - w_2, v_3 - w_3 \rangle \qquad (5)$$
$$k\mathbf{v} = \langle kv_1, kv_2, kv_3 \rangle \qquad (6)$$

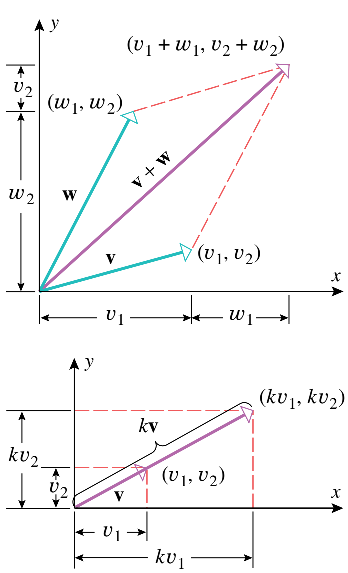


Figure 12.2.8

We will not prove this theorem. However, results (1) and (3) should be evident from Figure 12.2.8. Similar figures in 3-space can be used to motivate (4) and (6). Formulas (2) and (5) can be obtained by writing $\mathbf{v} + \mathbf{w} = \mathbf{v} + (-1)\mathbf{w}$.

Example 1 If $\mathbf{v} = \langle -2, 0, 1 \rangle$ and $\mathbf{w} = \langle 3, 5, -4 \rangle$, then

$$\mathbf{v} + \mathbf{w} = \langle -2, 0, 1 \rangle + \langle 3, 5, -4 \rangle = \langle 1, 5, -3 \rangle$$
$$3\mathbf{v} = \langle -6, 0, 3 \rangle$$
$$-\mathbf{w} = \langle -3, -5, 4 \rangle$$
$$\mathbf{w} - 2\mathbf{v} = \langle 3, 5, -4 \rangle - \langle -4, 0, 2 \rangle = \langle 7, 5, -6 \rangle \qquad ◀$$

VECTORS WITH INITIAL POINT NOT AT THE ORIGIN

Recall that we defined the components of a vector to be the coordinates of its terminal point when its initial point is at the origin. We will now consider the problem of finding

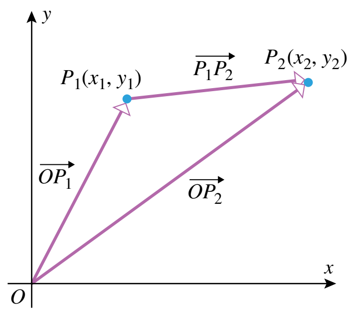


Figure 12.2.9

the components of a vector whose initial point is not at the origin. To be specific, suppose that $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are points in 2-space and we are interested in finding the components of the vector $\overrightarrow{P_1P_2}$. As illustrated in Figure 12.2.9, we can write this vector as

$$\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1} = \langle x_2, y_2 \rangle - \langle x_1, y_1 \rangle = \langle x_2 - x_1, y_2 - y_1 \rangle$$

Thus, we have shown that the components of the vector $\overrightarrow{P_1P_2}$ can be obtained by subtracting the coordinates of its initial point from the coordinates of its terminal point. Similar computations hold in 3-space, so we have established the following result:

12.2.5 THEOREM. *If $\overrightarrow{P_1P_2}$ is a vector in 2-space with initial point $P_1(x_1, y_1)$ and terminal point $P_2(x_2, y_2)$, then*

$$\overrightarrow{P_1P_2} = \langle x_2 - x_1, y_2 - y_1 \rangle \tag{7}$$

Similarly, if $\overrightarrow{P_1P_2}$ is a vector in 3-space with initial point $P_1(x_1, y_1, z_1)$ and terminal point $P_2(x_2, y_2, z_2)$, then

$$\overrightarrow{P_1P_2} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle \tag{8}$$

Example 2 In 2-space the vector from $P_1(1, 3)$ to $P_2(4, -2)$ is

$$\overrightarrow{P_1P_2} = \langle 4 - 1, -2 - 3 \rangle = \langle 3, -5 \rangle$$

and in 3-space the vector from $A(0, -2, 5)$ to $B(3, 4, -1)$ is

$$\overrightarrow{AB} = \langle 3 - 0, 4 - (-2), -1 - 5 \rangle = \langle 3, 6, -6 \rangle$$ ◀

RULES OF VECTOR ARITHMETIC

The following theorem shows that many of the familiar rules of ordinary arithmetic also hold for vector arithmetic.

12.2.6 THEOREM. *For any vectors* **u**, **v**, *and* **w** *and any scalars k and ℓ, the following relationships hold*:

(*a*) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	(*e*) $k(\ell\mathbf{u}) = (k\ell)\mathbf{u}$
(*b*) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$	(*f*) $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
(*c*) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$	(*g*) $(k + \ell)\mathbf{u} = k\mathbf{u} + \ell\mathbf{u}$
(*d*) $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$	(*h*) $1\mathbf{u} = \mathbf{u}$

The results in this theorem can be proved either algebraically by using components or geometrically by treating the vectors as arrows. We will prove part (*b*) both ways and leave some of the remaining proofs as exercises.

Proof (b) (Algebraic in 2-space). Let $\mathbf{u} = \langle u_1, u_2 \rangle$, $\mathbf{v} = \langle v_1, v_2 \rangle$, and $\mathbf{w} = \langle w_1, w_2 \rangle$. Then

$$\begin{aligned}
(\mathbf{u} + \mathbf{v}) + \mathbf{w} &= (\langle u_1, u_2 \rangle + \langle v_1, v_2 \rangle) + \langle w_1, w_2 \rangle \\
&= \langle u_1 + v_1, u_2 + v_2 \rangle + \langle w_1, w_2 \rangle \\
&= \langle (u_1 + v_1) + w_1, (u_2 + v_2) + w_2 \rangle \\
&= \langle u_1 + (v_1 + w_1), u_2 + (v_2 + w_2) \rangle \\
&= \langle u_1, u_2 \rangle + \langle v_1 + w_1, v_2 + w_2 \rangle \\
&= \mathbf{u} + (\mathbf{v} + \mathbf{w})
\end{aligned}$$

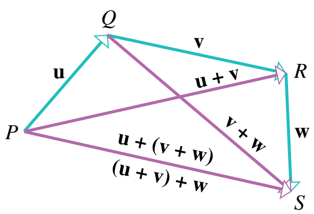


Figure 12.2.10

Proof (b) (Geometric). Let **u**, **v**, and **w** be represented by $\overrightarrow{PQ}$, $\overrightarrow{QR}$, and $\overrightarrow{RS}$ as shown in Figure 12.2.10. Then

$$\mathbf{v}+\mathbf{w}=\overrightarrow{QS} \quad \text{and} \quad \mathbf{u}+(\mathbf{v}+\mathbf{w})=\overrightarrow{PS}$$
$$\mathbf{u}+\mathbf{v}=\overrightarrow{PR} \quad \text{and} \quad (\mathbf{u}+\mathbf{v})+\mathbf{w}=\overrightarrow{PS}$$

Therefore,

$$(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$$

REMARK. It follows from part (*b*) of this theorem that the symbol $\mathbf{u}+\mathbf{v}+\mathbf{w}$ is unambiguous since the same vector results no matter how the terms are grouped. Moreover, Figure 12.2.10 shows that if the vectors **u**, **v**, and **w** are placed "tip to tail," then the sum $\mathbf{u}+\mathbf{v}+\mathbf{w}$ is the vector from the initial point of **u** to the terminal point of **w**. This also works for four or more vectors (Figure 12.2.11).

NORM OF A VECTOR

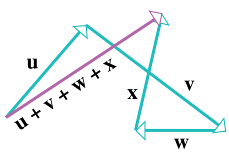


Figure 12.2.11

The distance between the initial and terminal points of a vector **v** is called the ***length***, the ***norm***, or the ***magnitude*** of **v** and is denoted by $\|\mathbf{v}\|$. This distance does not change if the vector is translated, so for purposes of calculating the norm we can assume that the vector is positioned with its initial point at the origin (Figure 12.2.12). This makes it evident that the norm of a vector $\mathbf{v}=\langle v_1, v_2\rangle$ in 2-space is given by

$$\|\mathbf{v}\|=\sqrt{v_1^2+v_2^2} \tag{9}$$

and the norm of a vector $\mathbf{v}=\langle v_1, v_2, v_3\rangle$ in 3-space is given by

$$\|\mathbf{v}\|=\sqrt{v_1^2+v_2^2+v_3^2} \tag{10}$$

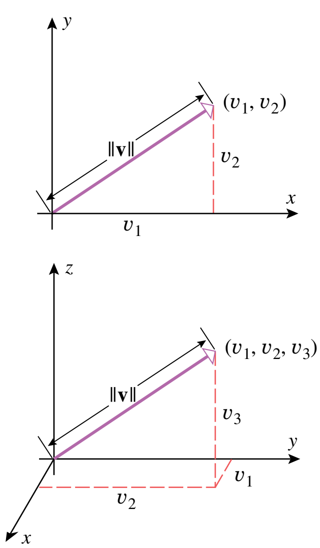


Figure 12.2.12

Example 3 Find the norms of $\mathbf{v}=\langle -2, 3\rangle$ and $\mathbf{w}=\langle 2, 3, 6\rangle$.

Solution. From (9) and (10)

$$\|\mathbf{v}\|=\sqrt{(-2)^2+3^2}=\sqrt{13}$$
$$\|\mathbf{w}\|=\sqrt{2^2+3^2+6^2}=\sqrt{49}=7$$

Recall from Definition 12.2.2 that the length of $k\mathbf{v}$ is $|k|$ times the length of **v**; that is,

$$\|k\mathbf{v}\|=|k|\|\mathbf{v}\| \tag{11}$$

Thus, for example,

$$\|3\mathbf{v}\|=|3|\|\mathbf{v}\|=3\|\mathbf{v}\|$$
$$\|-2\mathbf{v}\|=|-2|\|\mathbf{v}\|=2\|\mathbf{v}\|$$
$$\|-\mathbf{v}\|=|-1|\|\mathbf{v}\|=\|\mathbf{v}\|$$

This applies to vectors in 2-space and 3-space.

UNIT VECTORS

A vector of length 1 is called a ***unit vector***. In an *xy*-coordinate system the unit vectors along the *x*- and *y*-axes are denoted by **i** and **j**, respectively; and in an *xyz*-coordinate system the unit vectors along the *x*-, *y*-, and *z*-axes are denoted by **i**, **j**, and **k**, respectively (Figure 12.2.13). Thus,

$$\mathbf{i}=\langle 1, 0\rangle, \qquad \mathbf{j}=\langle 0, 1\rangle \qquad \text{In 2-space}$$
$$\mathbf{i}=\langle 1, 0, 0\rangle, \qquad \mathbf{j}=\langle 0, 1, 0\rangle, \qquad \mathbf{k}=\langle 0, 0, 1\rangle \qquad \text{In 3-space}$$

Every vector in 2-space is expressible uniquely in terms of **i** and **j**, and every vector in

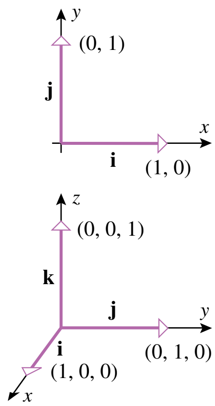


Figure 12.2.13

3-space is expressible uniquely in terms of **i**, **j**, and **k** as follows:

$$\mathbf{v} = \langle v_1, v_2 \rangle = \langle v_1, 0 \rangle + \langle 0, v_2 \rangle = v_1\langle 1, 0 \rangle + v_2\langle 0, 1 \rangle = v_1\mathbf{i} + v_2\mathbf{j}$$

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle = v_1\langle 1, 0, 0 \rangle + v_2\langle 0, 1, 0 \rangle + v_3\langle 0, 0, 1 \rangle = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

REMARK. The bracket and unit vector notations for vectors are completely interchangeable, the choice being a matter of convenience or personal preference.

Example 4

2-SPACE	3-SPACE
$\langle 2, 3 \rangle = 2\mathbf{i} + 3\mathbf{j}$	$\langle 2, -3, 4 \rangle = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$
$\langle -4, 0 \rangle = -4\mathbf{i} + 0\mathbf{j} = -4\mathbf{i}$	$\langle 0, 3, 0 \rangle = 3\mathbf{j}$
$\langle 0, 0 \rangle = 0\mathbf{i} + 0\mathbf{j} = \mathbf{0}$	$\langle 0, 0, 0 \rangle = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$
$(3\mathbf{i} + 2\mathbf{j}) + (4\mathbf{i} + \mathbf{j}) = 7\mathbf{i} + 3\mathbf{j}$	$(3\mathbf{i} + 2\mathbf{j} - \mathbf{k}) - (4\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = -\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$
$5(6\mathbf{i} - 2\mathbf{j}) = 30\mathbf{i} - 10\mathbf{j}$	$2(\mathbf{i} + \mathbf{j} - \mathbf{k}) + 4(\mathbf{i} - \mathbf{j}) = 6\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$
$\\|2\mathbf{i} - 3\mathbf{j}\\| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$	$\\|\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}\\| = \sqrt{1^2 + 2^2 + (-3)^2} = \sqrt{14}$
$\\|v_1\mathbf{i} + v_2\mathbf{j}\\| = \sqrt{v_1^2 + v_2^2}$	$\\|\langle v_1, v_2, v_3 \rangle\\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$

◀

NORMALIZING A VECTOR

A common problem in applications is to find a unit vector **u** that has the same direction as some given nonzero vector **v**. This can be done by multiplying **v** by the reciprocal of its length; that is,

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

is a unit vector with the same direction as **v**—the direction is the same because $k = 1/\|\mathbf{v}\|$ is a positive scalar, and the length is 1 because

$$\|\mathbf{u}\| = \|k\mathbf{v}\| = |k|\|\mathbf{v}\| = k\|\mathbf{v}\| = \frac{1}{\|\mathbf{v}\|}\|\mathbf{v}\| = 1$$

The process of multiplying a vector **v** by the reciprocal of its length to obtain a unit vector with the same direction is called ***normalizing*** **v**.

Example 5 Find the unit vector that has the same direction as $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

Solution. The vector **v** has length

$$\|\mathbf{v}\| = \sqrt{2^2 + 2^2 + (-1)^2} = 3$$

so the unit vector **u** in the same direction as **v** is

$$\mathbf{u} = \tfrac{1}{3}\mathbf{v} = \tfrac{2}{3}\mathbf{i} + \tfrac{2}{3}\mathbf{j} - \tfrac{1}{3}\mathbf{k}$$

◀

FOR THE READER. Many calculating utilities can perform vector operations, and some have built-in norm and normalization operations. If your calculating utility has these capabilities, use it to check the computations in Examples 1, 3, and 5.

VECTORS DETERMINED BY LENGTH AND ANGLE

If **v** is a nonzero vector with its initial point at the origin of an xy-coordinate system, and if ϕ is the angle from the positive x-axis to the radial line through **v**, then the x-component of **v** can be written as $\|\mathbf{v}\|\cos\phi$ and the y-component as $\|\mathbf{v}\|\sin\phi$ (Figure 12.2.14); and hence **v** can be expressed in trigonometric form as

$$\mathbf{v} = \|\mathbf{v}\|\langle \cos\phi, \sin\phi \rangle \quad \text{or} \quad \mathbf{v} = \|\mathbf{v}\|\cos\phi\,\mathbf{i} + \|\mathbf{v}\|\sin\phi\,\mathbf{j} \qquad (12)$$

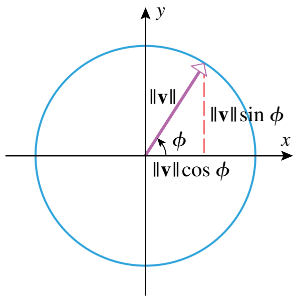


Figure 12.2.14

In the special case of a unit vector **u** this simplifies to

$$\mathbf{u} = \langle \cos\phi, \sin\phi \rangle \quad \text{or} \quad \mathbf{u} = \cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j} \tag{13}$$

Example 6

(a) Find the vector of length 2 that makes an angle of $\pi/4$ with the positive x-axis.

(b) Find the angle that the vector $\mathbf{v} = -\sqrt{3}\,\mathbf{i} + \mathbf{j}$ makes with the positive x-axis.

Solution (a). From (12)

$$\mathbf{v} = 2\cos\frac{\pi}{4}\mathbf{i} + 2\sin\frac{\pi}{4}\mathbf{j} = \sqrt{2}\,\mathbf{i} + \sqrt{2}\,\mathbf{j}$$

Solution (b). We will normalize **v**, then use (13) to find $\sin\phi$ and $\cos\phi$, and then use these values to find ϕ. Normalizing **v** yields

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{-\sqrt{3}\,\mathbf{i} + \mathbf{j}}{\sqrt{(-\sqrt{3}\,)^2 + 1^2}} = -\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$$

Thus, $\cos\phi = -\sqrt{3}/2$ and $\sin\phi = \frac{1}{2}$, from which we conclude that $\phi = 5\pi/6$. ◀

VECTORS DETERMINED BY LENGTH AND A VECTOR IN THE SAME DIRECTION

It is a common problem in many applications that a direction in 2-space or 3-space is determined by some known unit vector **u**, and it is of interest to find the components of a vector **v** that has the same direction as **u** and some specified length $\|\mathbf{v}\|$. This can be done by expressing **v** as

$$\mathbf{v} = \|\mathbf{v}\|\mathbf{u}$$ v is equal to its length times a unit vector in the same direction.

and then reading off the components of $\|\mathbf{v}\|\mathbf{u}$.

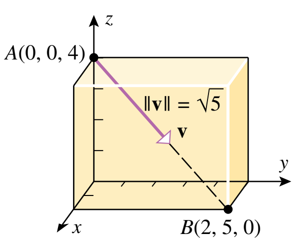


Figure 12.2.15

Example 7

Figure 12.2.15 shows a vector **v** of length $\sqrt{5}$ that extends along the line through A and B. Find the components of **v**.

Solution. First we will find the components of the vector $\overrightarrow{AB}$, then we will normalize this vector to obtain a unit vector in the direction of **v**, and then we will multiply this unit vector by $\|\mathbf{v}\|$ to obtain the vector **v**. The computations are as follows:

$$\overrightarrow{AB} = \langle 2, 5, 0 \rangle - \langle 0, 0, 4 \rangle = \langle 2, 5, -4 \rangle$$

$$\|\overrightarrow{AB}\| = \sqrt{2^2 + 5^2 + (-4)^2} = \sqrt{45} = 3\sqrt{5}$$

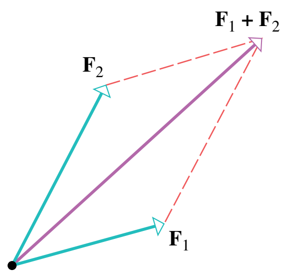
$$\frac{\overrightarrow{AB}}{\|\overrightarrow{AB}\|} = \left\langle \frac{2}{3\sqrt{5}}, \frac{5}{3\sqrt{5}}, -\frac{4}{3\sqrt{5}} \right\rangle$$

$$\mathbf{v} = \|\mathbf{v}\|\left(\frac{\overrightarrow{AB}}{\|\overrightarrow{AB}\|}\right) = \sqrt{5}\left\langle \frac{2}{3\sqrt{5}}, \frac{5}{3\sqrt{5}}, -\frac{4}{3\sqrt{5}} \right\rangle = \left\langle \frac{2}{3}, \frac{5}{3}, -\frac{4}{3} \right\rangle$$ ◀

RESULTANT OF TWO CONCURRENT FORCES

The effect that a force has on an object depends on the magnitude and direction of the force and the point at which it is applied. Thus, forces are regarded to be vector quantities and, indeed, the algebraic operations on vectors that we have defined in this section have their origin in the study of forces. For example, it is a fact of physics that if two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are applied at the same point on an object, then the two forces have the same effect on the object as the single force $\mathbf{F}_1 + \mathbf{F}_2$ applied at the point (Figure 12.2.16). Physicists and engineers call $\mathbf{F}_1 + \mathbf{F}_2$ the ***resultant*** of $\mathbf{F}_1$ and $\mathbf{F}_2$, and they say that the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are ***concurrent*** to indicate that they are applied at the same point.

In many applications, the magnitudes of two concurrent forces and the angle between them are known, and the problem is to find the magnitude and direction of the resultant. For example, referring to Figure 12.2.17, suppose that we know the magnitudes of the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ and the angle ϕ between them, and we are interested in finding the magnitude of



The single force $\mathbf{F}_1 + \mathbf{F}_2$ has the same effect as the two forces $\mathbf{F}_1$ and $\mathbf{F}_2$.

Figure 12.2.16

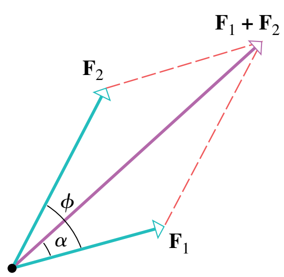


Figure 12.2.17

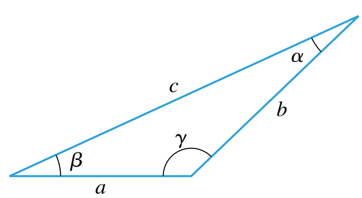


Figure 12.2.18

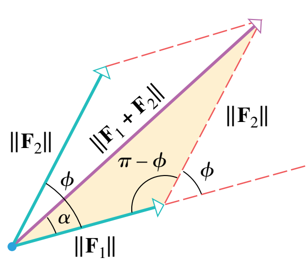


Figure 12.2.19

the resultant $\mathbf{F}_1 + \mathbf{F}_2$ and the angle α that the resultant makes with the force $\mathbf{F}_1$. This can be done by trigonometric methods based on the laws of sines and cosines. For this purpose, recall that the law of sines applied to the triangle in Figure 12.2.18 states that

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

and the law of cosines implies that

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

Referring to Figure 12.2.19, and using the fact that $\cos(\pi - \phi) = -\cos\phi$, it follows from the law of cosines that

$$\|\mathbf{F}_1 + \mathbf{F}_2\|^2 = \|\mathbf{F}_1\|^2 + \|\mathbf{F}_2\|^2 + 2\|\mathbf{F}_1\|\|\mathbf{F}_2\|\cos\phi \tag{14}$$

Moreover, it follows from the law of sines that

$$\frac{\|\mathbf{F}_2\|}{\sin\alpha} = \frac{\|\mathbf{F}_1 + \mathbf{F}_2\|}{\sin(\pi - \phi)}$$

which, with the help of the identity $\sin(\pi - \phi) = \sin\phi$, can be expressed as

$$\sin\alpha = \frac{\|\mathbf{F}_2\|}{\|\mathbf{F}_1 + \mathbf{F}_2\|}\sin\phi \tag{15}$$

Example 8 Suppose that two forces are applied to an eye bracket, as shown in Figure 12.2.20. Find the magnitude of the resultant and the angle θ that it makes with the positive x-axis.

Solution. We are given that $\|\mathbf{F}_1\| = 200$ N and $\|\mathbf{F}_2\| = 300$ N and that the angle between the vectors $\mathbf{F}_1$ and $\mathbf{F}_2$ is $\phi = 40^\circ$. Thus, it follows from (14) that the magnitude of the resultant is

$$\begin{aligned}\|\mathbf{F}_1 + \mathbf{F}_2\| &= \sqrt{\|\mathbf{F}_1\|^2 + \|\mathbf{F}_2\|^2 + 2\|\mathbf{F}_1\|\|\mathbf{F}_2\|\cos\phi} \\ &= \sqrt{(200)^2 + (300)^2 + 2(200)(300)\cos 40^\circ} \\ &\approx 471 \text{ N}\end{aligned}$$

Moreover, it follows from (15) that the angle α between $\mathbf{F}_1$ and the resultant is

$$\begin{aligned}\alpha &= \sin^{-1}\left(\frac{\|\mathbf{F}_2\|}{\|\mathbf{F}_1 + \mathbf{F}_2\|}\sin\phi\right) \\ &\approx \sin^{-1}\left(\frac{300}{471}\sin 40^\circ\right) \\ &\approx 24.2^\circ\end{aligned}$$

Thus, the angle θ that the resultant makes with the positive x-axis is

$$\theta = \alpha + 30^\circ \approx 24.2^\circ + 30^\circ = 54.2^\circ$$

(Figure 12.2.21). ◀

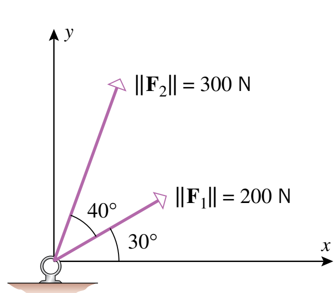


Figure 12.2.20

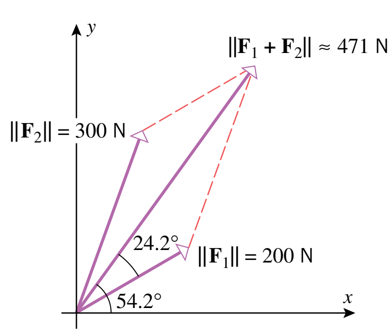


Figure 12.2.21

REMARK. The resultant of three or more concurrent forces can be found by working in pairs. For example, the resultant of three concurrent forces can be found by finding the resultant of any two of the three forces and then finding the resultant of that resultant with the third force.

EXERCISE SET 12.2

In Exercises 1–4, sketch the vectors with their initial points at the origin.

1. (a) $\langle 2, 5\rangle$ (b) $\langle -5, -4\rangle$ (c) $\langle 2, 0\rangle$
(d) $-5\mathbf{i} + 3\mathbf{j}$ (e) $3\mathbf{i} - 2\mathbf{j}$ (f) $-6\mathbf{j}$

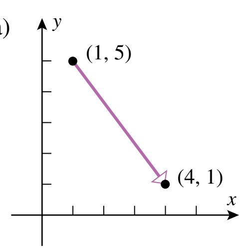
2. (a) $\langle -3, 7\rangle$ (b) $\langle 6, -2\rangle$ (c) $\langle 0, -8\rangle$
(d) $4\mathbf{i} + 2\mathbf{j}$ (e) $-2\mathbf{i} - \mathbf{j}$ (f) $4\mathbf{i}$

3. (a) $\langle 1, -2, 2\rangle$ (b) $\langle 2, 2, -1\rangle$
(c) $-\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ (d) $2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

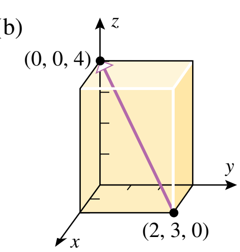
4. (a) $\langle -1, 3, 2\rangle$ (b) $\langle 3, 4, 2\rangle$
(c) $2\mathbf{j} - \mathbf{k}$ (d) $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$

In Exercises 5 and 6, find the components of the vector, and sketch an equivalent vector with its initial point at the origin.

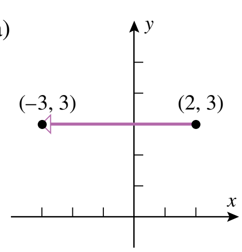
5. (a)



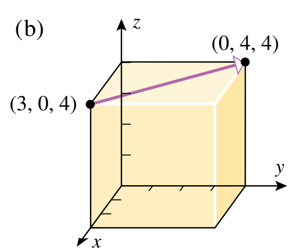
(b)



6. (a)



(b)



In Exercises 7 and 8, find the components of the vector $\overrightarrow{P_1P_2}$.

7. (a) $P_1(3, 5)$, $P_2(2, 8)$ (b) $P_1(7, -2)$, $P_2(0, 0)$
(c) $P_1(5, -2, 1)$, $P_2(2, 4, 2)$

8. (a) $P_1(-6, -2)$, $P_2(-4, -1)$
(b) $P_1(0, 0, 0)$, $P_2(-1, 6, 1)$
(c) $P_1(4, 1, -3)$, $P_2(9, 1, -3)$

9. (a) Find the terminal point of $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j}$ if the initial point is $(1, -2)$.
(b) Find the initial point of $\mathbf{v} = \langle -3, 1, 2\rangle$ if the terminal point is $(5, 0, -1)$.

10. (a) Find the terminal point of $\mathbf{v} = \langle 7, 6\rangle$ if the initial point is $(2, -1)$.
(b) Find the terminal point of $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ if the initial point is $(-2, 1, 4)$.

In Exercises 11 and 12, perform the stated operations on the vectors **u**, **v**, and **w**.

11. $\mathbf{u} = 3\mathbf{i} - \mathbf{k}$, $\mathbf{v} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $\mathbf{w} = 3\mathbf{j}$
(a) $\mathbf{w} - \mathbf{v}$ (b) $6\mathbf{u} + 4\mathbf{w}$
(c) $-\mathbf{v} - 2\mathbf{w}$ (d) $4(3\mathbf{u} + \mathbf{v})$
(e) $-8(\mathbf{v} + \mathbf{w}) + 2\mathbf{u}$ (f) $3\mathbf{w} - (\mathbf{v} - \mathbf{w})$

12. $\mathbf{u} = \langle 2, -1, 3\rangle$, $\mathbf{v} = \langle 4, 0, -2\rangle$, $\mathbf{w} = \langle 1, 1, 3\rangle$
(a) $\mathbf{u} - \mathbf{w}$ (b) $7\mathbf{v} + 3\mathbf{w}$ (c) $-\mathbf{w} + \mathbf{v}$
(d) $3(\mathbf{u} - 7\mathbf{v})$ (e) $-3\mathbf{v} - 8\mathbf{w}$ (f) $2\mathbf{v} - (\mathbf{u} + \mathbf{w})$

In Exercises 13 and 14, find the norm of **v**.

13. (a) $\mathbf{v} = \langle 1, -1\rangle$ (b) $\mathbf{v} = -\mathbf{i} + 7\mathbf{j}$
(c) $\mathbf{v} = \langle -1, 2, 4\rangle$ (d) $\mathbf{v} = -3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

14. (a) $\mathbf{v} = \langle 3, 4\rangle$ (b) $\mathbf{v} = \sqrt{2}\mathbf{i} - \sqrt{7}\mathbf{j}$
(c) $\mathbf{v} = \langle 0, -3, 0\rangle$ (d) $\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

15. Let $\mathbf{u} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$, $\mathbf{v} = \mathbf{i} + \mathbf{j}$, and $\mathbf{w} = 2\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$. Find
(a) $\|\mathbf{u} + \mathbf{v}\|$ (b) $\|\mathbf{u}\| + \|\mathbf{v}\|$
(c) $\|-2\mathbf{u}\| + 2\|\mathbf{v}\|$ (d) $\|3\mathbf{u} - 5\mathbf{v} + \mathbf{w}\|$
(e) $\dfrac{1}{\|\mathbf{w}\|}\mathbf{w}$ (f) $\left\|\dfrac{1}{\|\mathbf{w}\|}\mathbf{w}\right\|$.

16. Is it possible to have $\|\mathbf{u}\| + \|\mathbf{v}\| = \|\mathbf{u} + \mathbf{v}\|$ if **u** and **v** are nonzero vectors? Justify your conclusion geometrically.

In Exercises 17 and 18, find unit vectors that satisfy the stated conditions.

17. (a) Same direction as $-\mathbf{i} + 4\mathbf{j}$.
(b) Oppositely directed to $6\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$.
(c) Same direction as the vector from the point $A(-1, 0, 2)$ to the point $B(3, 1, 1)$.

18. (a) Oppositely directed to $3\mathbf{i} - 4\mathbf{j}$.
(b) Same direction as $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$.
(c) Same direction as the vector from the point $A(-3, 2)$ to the point $B(1, -1)$.

In Exercises 19 and 20, find vectors that satisfy the stated conditions.

19. (a) Oppositely directed to $\mathbf{v} = \langle 3, -4 \rangle$ and half the length of $\mathbf{v}$.
(b) Length $\sqrt{17}$ and same direction as $\mathbf{v} = \langle 7, 0, -6 \rangle$.

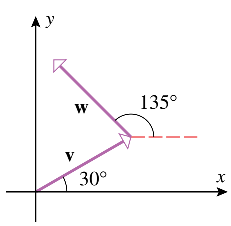
20. (a) Same direction as $\mathbf{v} = -2\mathbf{i} + 3\mathbf{j}$ and three times the length of $\mathbf{v}$.
(b) Length 2 and oppositely directed to $\mathbf{v} = -3\mathbf{i} + 4\mathbf{j} + \mathbf{k}$.

21. In each part, find the component form of the vector $\mathbf{v}$ in 2-space that has the stated length and makes the stated angle ϕ with the positive x-axis.
(a) $\|\mathbf{v}\| = 3;\ \phi = \pi/4$ (b) $\|\mathbf{v}\| = 2;\ \phi = 90^\circ$
(c) $\|\mathbf{v}\| = 5;\ \phi = 120^\circ$ (d) $\|\mathbf{v}\| = 1;\ \phi = \pi$

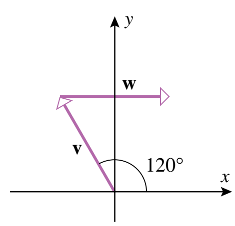
22. Find the component forms of $\mathbf{v} + \mathbf{w}$ and $\mathbf{v} - \mathbf{w}$ in 2-space, given that $\|\mathbf{v}\| = 1$, $\|\mathbf{w}\| = 1$, $\mathbf{v}$ makes an angle of $\pi/6$ with the positive x-axis, and $\mathbf{w}$ makes an angle of $3\pi/4$ with the positive x-axis.

In Exercises 23 and 24, find the component form of $\mathbf{v} + \mathbf{w}$, given that $\mathbf{v}$ and $\mathbf{w}$ are unit vectors.

23.

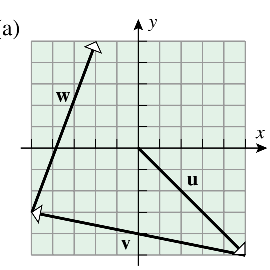


24.

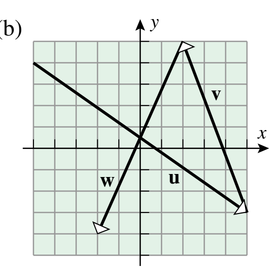


25. In each part, sketch the vector $\mathbf{u} + \mathbf{v} + \mathbf{w}$ and express it in component form.

(a)



(b)



26. In each part of Exercise 25, sketch the vector $\mathbf{u} - \mathbf{v} + \mathbf{w}$ and express it in component form.

27. Let $\mathbf{u} = \langle 1, 3 \rangle$, $\mathbf{v} = \langle 2, 1 \rangle$, $\mathbf{w} = \langle 4, -1 \rangle$. Find the vector $\mathbf{x}$ that satisfies $2\mathbf{u} - \mathbf{v} + \mathbf{x} = 7\mathbf{x} + \mathbf{w}$.

28. Let $\mathbf{u} = \langle -1, 1 \rangle$, $\mathbf{v} = \langle 0, 1 \rangle$, and $\mathbf{w} = \langle 3, 4 \rangle$. Find the vector $\mathbf{x}$ that satisfies $\mathbf{u} - 2\mathbf{x} = \mathbf{x} - \mathbf{w} + 3\mathbf{v}$.

29. Find $\mathbf{u}$ and $\mathbf{v}$ if $\mathbf{u} + 2\mathbf{v} = 3\mathbf{i} - \mathbf{k}$ and $3\mathbf{u} - \mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

30. Find $\mathbf{u}$ and $\mathbf{v}$ if $\mathbf{u} + \mathbf{v} = \langle 2, -3 \rangle$ and $3\mathbf{u} + 2\mathbf{v} = \langle -1, 2 \rangle$.

31. Use vectors to find the lengths of the diagonals of the parallelogram that has $\mathbf{i} + \mathbf{j}$ and $\mathbf{i} - 2\mathbf{j}$ as adjacent sides.

32. Use vectors to find the fourth vertex of a parallelogram, three of whose vertices are (0, 0), (1, 3), and (2, 4). [*Note:* There is more than one answer.]

33. (a) Given that $\|\mathbf{v}\| = 3$, find all values of k such that $\|k\mathbf{v}\| = 5$.
(b) Given that $k = -2$ and $\|k\mathbf{v}\| = 6$, find $\|\mathbf{v}\|$.

34. What do you know about k and $\mathbf{v}$ if $\|k\mathbf{v}\| = 0$?

35. In each part, find two unit vectors in 2-space that satisfy the stated condition.
(a) Parallel to the line $y = 3x + 2$
(b) Parallel to the line $x + y = 4$
(c) Perpendicular to the line $y = -5x + 1$

36. In each part, find two unit vectors in 3-space that satisfy the stated condition.
(a) Perpendicular to the xy-plane
(b) Perpendicular to the xz-plane
(c) Perpendicular to the yz-plane

37. Let $\mathbf{r} = \langle x, y \rangle$ be an arbitrary vector. In each part, describe the set of all points (x, y) in 2-space that satisfy the stated condition.
(a) $\|\mathbf{r}\| = 1$ (b) $\|\mathbf{r}\| \le 1$ (c) $\|\mathbf{r}\| > 1$

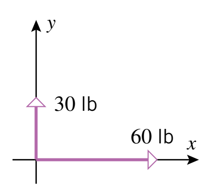
38. Let $\mathbf{r} = \langle x, y \rangle$ and $\mathbf{r}_0 = \langle x_0, y_0 \rangle$. In each part, describe the set of all points (x, y) in 2-space that satisfy the stated condition.
(a) $\|\mathbf{r} - \mathbf{r}_0\| = 1$ (b) $\|\mathbf{r} - \mathbf{r}_0\| \le 1$ (c) $\|\mathbf{r} - \mathbf{r}_0\| > 1$

39. Let $\mathbf{r} = \langle x, y, z \rangle$ be an arbitrary vector. In each part, describe the set of all points (x, y, z) in 3-space that satisfy the stated condition.
(a) $\|\mathbf{r}\| = 1$ (b) $\|\mathbf{r}\| \le 1$ (c) $\|\mathbf{r}\| > 1$

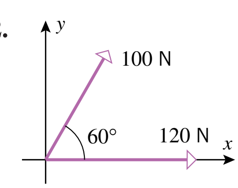
40. Let $\mathbf{r}_1 = \langle x_1, y_1 \rangle$, $\mathbf{r}_2 = \langle x_2, y_2 \rangle$, and $\mathbf{r} = \langle x, y \rangle$. Describe the set of all points (x, y) for which $\|\mathbf{r} - \mathbf{r}_1\| + \|\mathbf{r} - \mathbf{r}_2\| = k$, assuming that $k > \|\mathbf{r}_2 - \mathbf{r}_1\|$.

In Exercises 41–46, find the magnitude of the resultant force and the angle that it makes with the positive x-axis.

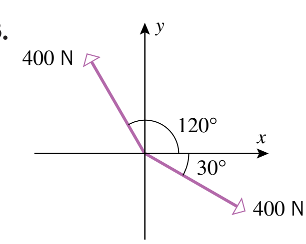
41.



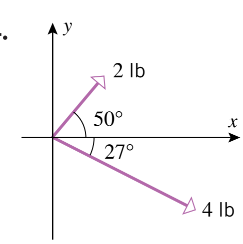
42.



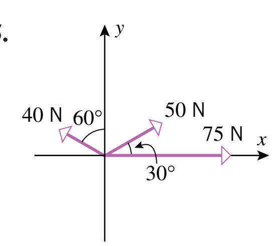
43.



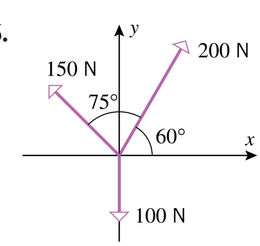
44.



45.

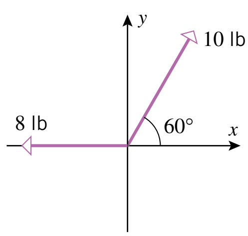


46.

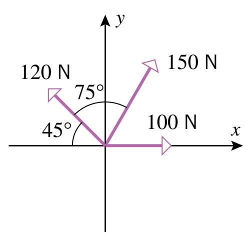


A particle is said to be in ***static equilibrium*** if the resultant of all forces applied to it is zero. In Exercises 47 and 48, find the force **F** that must be applied to the point to produce static equilibrium. Describe **F** by specifying its magnitude and the angle that it makes with the positive x-axis.

47.



48.



49. The accompanying figure shows a 250-lb traffic light supported by two flexible cables. The magnitudes of the forces that the cables apply to the eye ring are called the cable ***tensions***. Find the tensions in the cables if the traffic light is in static equilibrium (defined above Exercise 47).

50. Find the tensions in the cables shown in the accompanying figure if the block is in static equilibrium (see Exercise 49).

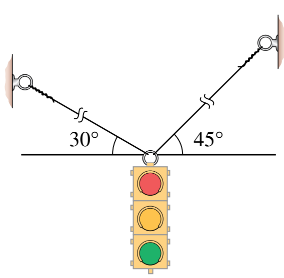


Figure Ex-49

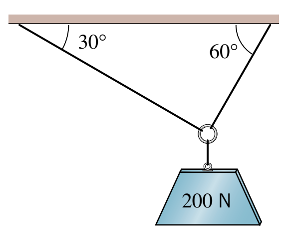


Figure Ex-50

51. A vector **w** is said to be a ***linear combination*** of the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ if **w** can be expressed as $\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, where c_1 and c_2 are scalars.
(a) Find scalars c_1 and c_2 to express the vector $4\mathbf{j}$ as a linear combination of the vectors $\mathbf{v}_1 = 2\mathbf{i} - \mathbf{j}$ and $\mathbf{v}_2 = 4\mathbf{i} + 2\mathbf{j}$.
(b) Show that the vector $\langle 3, 5\rangle$ cannot be expressed as a linear combination of the vectors $\mathbf{v}_1 = \langle 1, -3\rangle$ and $\mathbf{v}_2 = \langle -2, 6\rangle$.

52. A vector **w** is said to be a ***linear combination*** of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ if **w** can be expressed as $\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$, where c_1, c_2, and c_3 are scalars.
(a) Find scalars c_1, c_2, and c_3 to express $\langle -1, 1, 5\rangle$ as a linear combination of the vectors $\mathbf{v}_1 = \langle 1, 0, 1\rangle$, $\mathbf{v}_2 = \langle 3, 2, 0\rangle$, and $\mathbf{v}_3 = \langle 0, 1, 1\rangle$.
(b) Show that the vector $2\mathbf{i}+\mathbf{j}-\mathbf{k}$ cannot be expressed as a linear combination of the vectors $\mathbf{v}_1 = \mathbf{i}-\mathbf{j}$, $\mathbf{v}_2 = 3\mathbf{i}+\mathbf{k}$, and $\mathbf{v}_3 = 4\mathbf{i} - \mathbf{j} + \mathbf{k}$.

53. Use a theorem from plane geometry to show that if **u** and **v** are vectors in 2-space or 3-space, then

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$$

which is called the ***triangle inequality for vectors***. Give some examples to illustrate this inequality.

54. Prove parts (a), (c), and (e) of Theorem 12.2.6 algebraically in 2-space.

55. Prove parts (d), (g), and (h) of Theorem 12.2.6 algebraically in 2-space.

56. Prove part (f) of Theorem 12.2.6 geometrically.

57. Use vectors to prove that the line segment joining the midpoints of two sides of a triangle is parallel to the third side and half as long.

58. Use vectors to prove that the midpoints of the sides of a quadrilateral are the vertices of a parallelogram.

12.3 DOT PRODUCT; PROJECTIONS

In the last section we defined three operations on vectors—addition, subtraction, and scalar multiplication. In scalar multiplication a vector is multiplied by a scalar and the result is a vector. In this section we will define a new kind of multiplication in which two vectors are multiplied to produce a scalar. This multiplication operation has many uses, some of which we will also discuss in this section.

DEFINITION OF THE DOT PRODUCT

12.3.1 DEFINITION. If $\mathbf{u} = \langle u_1, u_2\rangle$ and $\mathbf{v} = \langle v_1, v_2\rangle$ are vectors in 2-space, then the ***dot product*** of **u** and **v** is written as $\mathbf{u}\cdot\mathbf{v}$ and is defined as

$$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2$$

Similarly, if $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ are vectors in 3-space, then their dot product is defined as

$$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$$

In words, the dot product of two vectors is formed by multiplying their corresponding components and adding the products. Note that the dot product of two vectors is a scalar.

Example 1

$$\langle 3, 5\rangle \cdot \langle -1, 2\rangle = 3(-1) + 5(2) = 7$$
$$\langle 2, 3\rangle \cdot \langle -3, 2\rangle = 2(-3) + 3(2) = 0$$
$$\langle 1, -3, 4\rangle \cdot \langle 1, 5, 2\rangle = 1(1) + (-3)(5) + 4(2) = -6$$

Here are the same computations expressed another way:

$$(3\mathbf{i} + 5\mathbf{j}) \cdot (-\mathbf{i} + 2\mathbf{j}) = 3(-1) + 5(2) = 7$$
$$(2\mathbf{i} + 3\mathbf{j}) \cdot (-3\mathbf{i} + 2\mathbf{j}) = 2(-3) + 3(2) = 0$$
$$(\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}) \cdot (\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}) = 1(1) + (-3)(5) + 4(2) = -6$$ ◀

FOR THE READER. Many calculating utilities have a built-in dot product operation. If your calculating utility has this capability, use it to check the computations in Example 1.

ALGEBRAIC PROPERTIES OF THE DOT PRODUCT

The following theorem provides some of the basic algebraic properties of the dot product:

12.3.2 THEOREM. *If* **u**, **v**, *and* **w** *are vectors in 2- or 3-space and k is a scalar, then*

(*a*) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

(*b*) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

(*c*) $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (k\mathbf{v})$

(*d*) $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$

(*e*) $\mathbf{0} \cdot \mathbf{v} = 0$

We will prove parts (*c*) and (*d*) for vectors in 3-space and leave some of the others as exercises.

Proof (c). Let $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$. Then

$$k(\mathbf{u} \cdot \mathbf{v}) = k(u_1v_1 + u_2v_2 + u_3v_3) = (ku_1)v_1 + (ku_2)v_2 + (ku_3)v_3 = (k\mathbf{u}) \cdot \mathbf{v}$$

Similarly, $k(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (k\mathbf{v})$.

Proof (d). $\mathbf{v} \cdot \mathbf{v} = v_1v_1 + v_2v_2 + v_3v_3 = v_1^2 + v_2^2 + v_3^2 = \|\mathbf{v}\|^2$. ■

REMARK. Pay particular attention to the two zeros that appear in part (*e*) of the last theorem—the zero on the left side is the zero vector (boldface), and the zero on the right side is the zero scalar (lightface). It is also worth noting that the result in part (*d*) can be written as

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} \tag{1}$$

which provides a way of expressing the norm of a vector in terms of a dot product.

ANGLE BETWEEN VECTORS

Suppose that **u** and **v** are nonzero vectors in 2-space or 3-space that are positioned so their initial points coincide. We define the ***angle between* u *and* v** to be the angle θ determined by the vectors that satisfies the condition $0 \le \theta \le \pi$ (Figure 12.3.1). In 2-space, θ is the smallest counterclockwise angle through which one of the vectors can be rotated until it aligns with the other.

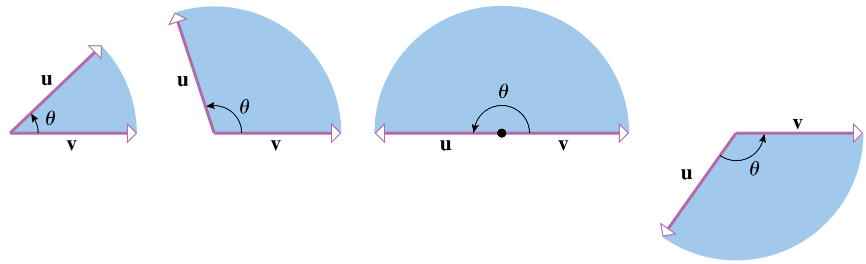


Figure 12.3.1

The next theorem provides a way of calculating the angle between two vectors from their components.

12.3.3 THEOREM. *If* **u** *and* **v** *are nonzero vectors in 2-space or 3-space, and if* θ *is the angle between them, then*

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} \tag{2}$$

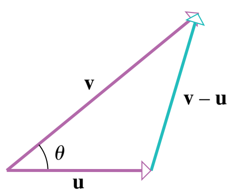


Figure 12.3.2

Proof. Suppose that the vectors **u**, **v**, and $\mathbf{v}-\mathbf{u}$ are positioned to form three sides of a triangle, as shown in Figure 12.3.2. It follows from the law of cosines that

$$\|\mathbf{v}-\mathbf{u}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta \tag{3}$$

Using the properties of the dot product in Theorem 12.3.2, we can rewrite the left side of this equation as

$$\begin{aligned}\|\mathbf{v}-\mathbf{u}\|^2 &= (\mathbf{v}-\mathbf{u})\cdot(\mathbf{v}-\mathbf{u})\\ &= (\mathbf{v}-\mathbf{u})\cdot\mathbf{v} - (\mathbf{v}-\mathbf{u})\cdot\mathbf{u}\\ &= \mathbf{v}\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{u}\\ &= \|\mathbf{v}\|^2 - 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{u}\|^2\end{aligned}$$

Substituting this back into (3) yields

$$\|\mathbf{v}\|^2 - 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{u}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta$$

which we can simplify and rewrite as

$$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta$$

Finally, dividing both sides of this equation by $\|\mathbf{u}\|\,\|\mathbf{v}\|$ yields (2). ∎

Example 2 Find the angle between the vector $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ and

(a) $\mathbf{v} = -3\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}$ (b) $\mathbf{w} = 2\mathbf{i} + 7\mathbf{j} + 6\mathbf{k}$ (c) $\mathbf{z} = -3\mathbf{i} + 6\mathbf{j} - 6\mathbf{k}$

Solution (a).

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{-11}{(3)(7)} = -\frac{11}{21}$$

Thus,

$$\theta = \cos^{-1}\left(-\tfrac{11}{21}\right) \approx 2.12 \text{ radians} \approx 121.6^\circ$$

Solution (b).

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{w}}{\|\mathbf{u}\|\,\|\mathbf{w}\|} = \frac{0}{\|\mathbf{u}\|\,\|\mathbf{w}\|} = 0$$

Thus, $\theta = \pi/2$, which means that the vectors are perpendicular.

Solution (c).

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{z}}{\|\mathbf{u}\|\|\mathbf{z}\|} = \frac{-27}{(3)(9)} = -1$$

Thus, $\theta = \pi$, which means that the vectors are oppositely directed. In retrospect, we could have seen this without computing θ, since $\mathbf{z} = -3\mathbf{u}$. ◀

INTERPRETING THE SIGN OF THE DOT PRODUCT

It will often be convenient to express Formula (2) as

$$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta \tag{4}$$

which expresses the dot product of $\mathbf{u}$ and $\mathbf{v}$ in terms of the lengths of these vectors and the angle between them. Since $\mathbf{u}$ and $\mathbf{v}$ are assumed to be nonzero vectors, this version of the formula makes it clear that the sign of $\mathbf{u}\cdot\mathbf{v}$ is the same as the sign of $\cos\theta$. Thus, we can tell from the dot product whether the angle between two vectors is acute or obtuse or whether the vectors are perpendicular (Figure 12.3.3).

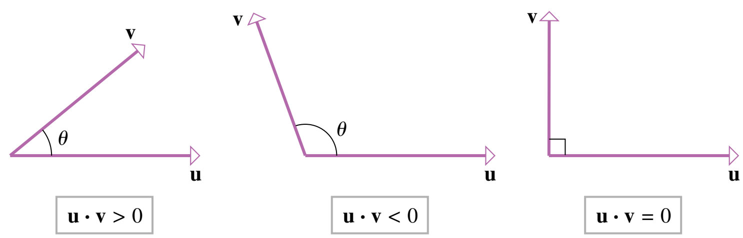


Figure 12.3.3

REMARK. The terms "perpendicular," "orthogonal," and "normal" are all commonly used to describe geometric objects that meet at right angles. For consistency, we will say that two vectors are *orthogonal*, a vector is *normal* to a plane, and two planes are *perpendicular*. Moreover, although the zero vector does not make a well-defined angle with other vectors, we will consider $\mathbf{0}$ to be orthogonal to *all* vectors. This convention allows us to say that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vectors if and only if $\mathbf{u}\cdot\mathbf{v} = 0$, and it makes Formula (4) valid if $\mathbf{u}$ or $\mathbf{v}$ (or both) is zero.

DIRECTION ANGLES

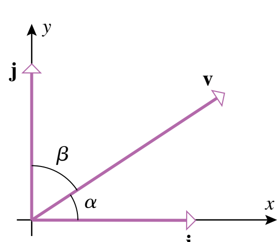


Figure 12.3.4

In an xy-coordinate system, the direction of a nonzero vector $\mathbf{v}$ is completely determined by the angles α and β between $\mathbf{v}$ and the unit vectors $\mathbf{i}$ and $\mathbf{j}$ (Figure 12.3.4), and in an xyz-coordinate system the direction is completely determined by the angles α, β, and γ between $\mathbf{v}$ and the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ (Figure 12.3.5). In both 2-space and 3-space the angles between a nonzero vector $\mathbf{v}$ and the vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are called the ***direction angles*** of $\mathbf{v}$, and the cosines of those angles are called the ***direction cosines*** of $\mathbf{v}$. Formulas for the direction cosines of a vector can be obtained from Formula (2). For example, if $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, then

$$\cos\alpha = \frac{\mathbf{v}\cdot\mathbf{i}}{\|\mathbf{v}\|\|\mathbf{i}\|} = \frac{v_1}{\|\mathbf{v}\|}, \quad \cos\beta = \frac{\mathbf{v}\cdot\mathbf{j}}{\|\mathbf{v}\|\|\mathbf{j}\|} = \frac{v_2}{\|\mathbf{v}\|}, \quad \cos\gamma = \frac{\mathbf{v}\cdot\mathbf{k}}{\|\mathbf{v}\|\|\mathbf{k}\|} = \frac{v_3}{\|\mathbf{v}\|}$$

Thus, we have the following result:

12.3.4 THEOREM. *The direction cosines of a nonzero vector* $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ *are*

$$\cos\alpha = \frac{v_1}{\|\mathbf{v}\|}, \quad \cos\beta = \frac{v_2}{\|\mathbf{v}\|}, \quad \cos\gamma = \frac{v_3}{\|\mathbf{v}\|}$$

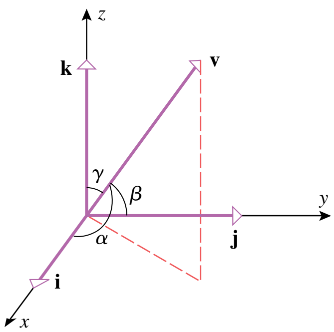


Figure 12.3.5

The direction cosines of a vector $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ can be computed by normalizing $\mathbf{v}$ and reading off the components of $\mathbf{v}/\|\mathbf{v}\|$, since

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{v_1}{\|\mathbf{v}\|}\mathbf{i} + \frac{v_2}{\|\mathbf{v}\|}\mathbf{j} + \frac{v_3}{\|\mathbf{v}\|}\mathbf{k} = (\cos\alpha)\mathbf{i} + (\cos\beta)\mathbf{j} + (\cos\gamma)\mathbf{k}$$

We leave it as an exercise for you to show that the direction cosines of a vector satisfy the equation

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \tag{5}$$

Example 3 Find the direction cosines of the vector $\mathbf{v} = 2\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}$, and approximate the direction angles to the nearest degree.

Solution. First we will normalize the vector $\mathbf{v}$ and then read off the components. We have $\|\mathbf{v}\| = \sqrt{4+16+16} = 6$, so that $\mathbf{v}/\|\mathbf{v}\| = \frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$. Thus,

$$\cos\alpha = \tfrac{1}{3}, \quad \cos\beta = -\tfrac{2}{3}, \quad \cos\gamma = \tfrac{2}{3}$$

With the help of a calculating utility we obtain

$$\alpha = \cos^{-1}\left(\tfrac{1}{3}\right) \approx 71^\circ, \quad \beta = \cos^{-1}\left(-\tfrac{2}{3}\right) \approx 132^\circ, \quad \gamma = \cos^{-1}\left(\tfrac{2}{3}\right) \approx 48^\circ$$ ◀

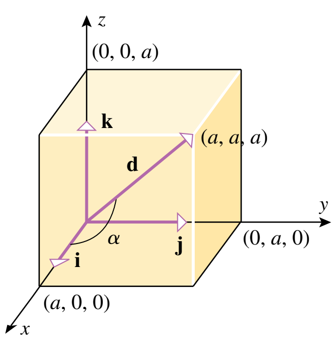


Figure 12.3.6

Example 4 Find the angle between a diagonal of a cube and one of its edges.

Solution. Assume that the cube has side a, and introduce a coordinate system as shown in Figure 12.3.6. In this coordinate system the vector

$$\mathbf{d} = a\mathbf{i} + a\mathbf{j} + a\mathbf{k}$$

is a diagonal of the cube and the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ run along the edges. By symmetry, the diagonal makes the same angle with each edge, so it is sufficient to find the angle between $\mathbf{d}$ and $\mathbf{i}$ (the direction angle α). Thus,

$$\cos\alpha = \frac{\mathbf{d}\cdot\mathbf{i}}{\|\mathbf{d}\|\,\|\mathbf{i}\|} = \frac{a}{\|\mathbf{d}\|} = \frac{a}{\sqrt{3a^2}} = \frac{1}{\sqrt{3}}$$

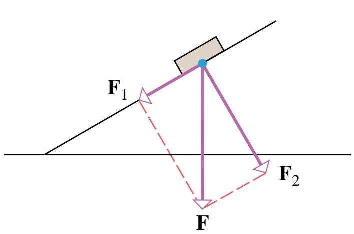
and hence

$$\alpha = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 0.955 \text{ radian} \approx 54.7^\circ$$ ◀

DECOMPOSING VECTORS INTO ORTHOGONAL COMPONENTS

In many applications it is desirable to "decompose" a vector into a sum of two orthogonal vectors with convenient specified directions. For example, Figure 12.3.7 shows a block on an inclined plane. The downward force $\mathbf{F}$ that gravity exerts on the block can be decomposed into the sum

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$$



The force of gravity pulls the block against the ramp and down the ramp.

Figure 12.3.7

where the force $\mathbf{F}_1$ is parallel to the ramp and the force $\mathbf{F}_2$ is perpendicular to the ramp. The forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are useful because $\mathbf{F}_1$ is the force that pulls the block *along* the ramp, and $\mathbf{F}_2$ is the force that the block exerts *against* the ramp.

Thus, our next objective is to develop a computational procedure for decomposing a vector into a sum of orthogonal vectors. For this purpose, suppose that $\mathbf{e}_1$ and $\mathbf{e}_2$ are two orthogonal *unit* vectors in 2-space, and suppose that we want to express a given vector $\mathbf{v}$ as a sum

$$\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2$$

where $\mathbf{w}_1$ is a scalar multiple of $\mathbf{e}_1$ and $\mathbf{w}_2$ is a scalar multiple of $\mathbf{e}_2$ (Figure 12.3.8*a*); that is, we want to find scalars k_1 and k_2 such that

$$\mathbf{v} = k_1\mathbf{e}_1 + k_2\mathbf{e}_2 \tag{6}$$

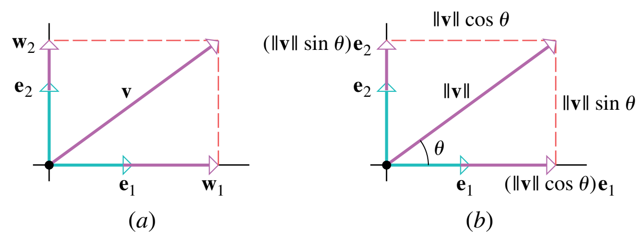


Figure 12.3.8

We can find k_1 by taking the dot product of $\mathbf{v}$ with $\mathbf{e}_1$. This yields

$$\mathbf{v} \cdot \mathbf{e}_1 = (k_1\mathbf{e}_1 + k_2\mathbf{e}_2) \cdot \mathbf{e}_1 = k_1(\mathbf{e}_1 \cdot \mathbf{e}_1) + k_2(\mathbf{e}_2 \cdot \mathbf{e}_1) = k_1\|\mathbf{e}_1\|^2 + 0 = k_1$$

and, similarly,

$$\mathbf{v} \cdot \mathbf{e}_2 = (k_1\mathbf{e}_1 + k_2\mathbf{e}_2) \cdot \mathbf{e}_2 = k_1(\mathbf{e}_1 \cdot \mathbf{e}_2) + k_2(\mathbf{e}_2 \cdot \mathbf{e}_2) = 0 + k_2\|\mathbf{e}_2\|^2 = k_2$$

Substituting these expressions for k_1 and k_2 in (6) yields

$$\mathbf{v} = (\mathbf{v} \cdot \mathbf{e}_1)\mathbf{e}_1 + (\mathbf{v} \cdot \mathbf{e}_2)\mathbf{e}_2 \tag{7}$$

In this formula we call $(\mathbf{v} \cdot \mathbf{e}_1)\mathbf{e}_1$ and $(\mathbf{v} \cdot \mathbf{e}_2)\mathbf{e}_2$ the ***vector components*** of $\mathbf{v}$ along $\mathbf{e}_1$ and $\mathbf{e}_2$, respectively; and we call $\mathbf{v} \cdot \mathbf{e}_1$ and $\mathbf{v} \cdot \mathbf{e}_2$ the ***scalar components*** of $\mathbf{v}$ along $\mathbf{e}_1$ and $\mathbf{e}_2$, respectively. If θ denotes the angle between $\mathbf{v}$ and $\mathbf{e}_1$, then the scalar components of $\mathbf{v}$ can be written in trigonometric form as

$$\mathbf{v} \cdot \mathbf{e}_1 = \|\mathbf{v}\|\cos\theta \quad \text{and} \quad \mathbf{v} \cdot \mathbf{e}_2 = \|\mathbf{v}\|\sin\theta \tag{8}$$

(Figure 12.3.8*b*). Moreover, the vector components of $\mathbf{v}$ can be expressed as

$$(\mathbf{v} \cdot \mathbf{e}_1)\mathbf{e}_1 = (\|\mathbf{v}\|\cos\theta)\mathbf{e}_1 \quad \text{and} \quad (\mathbf{v} \cdot \mathbf{e}_2)\mathbf{e}_2 = (\|\mathbf{v}\|\sin\theta)\mathbf{e}_2 \tag{9}$$

and the decomposition (6) can be expressed as

$$\mathbf{v} = (\|\mathbf{v}\|\cos\theta)\mathbf{e}_1 + (\|\mathbf{v}\|\sin\theta)\mathbf{e}_2 \tag{10}$$

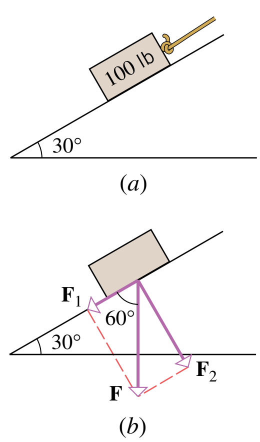


Figure 12.3.9

Example 5 A rope is attached to a 100-lb block on a ramp that is inclined at an angle of 30° with the ground (Figure 12.3.9*a*). How much force does the block exert against the ramp, and how much force must be applied to the rope in a direction parallel to the ramp to prevent the block from sliding down the ramp? (Assume that the ramp is smooth, that is, exerts no frictional forces.)

Solution. Let $\mathbf{F}$ denote the downward force of gravity on the block (so $\|\mathbf{F}\| = 100$ lb), and let $\mathbf{F}_1$ and $\mathbf{F}_2$ be the vector components of $\mathbf{F}$ parallel and perpendicular to the ramp (as shown in Figure 12.3.9*b*). The lengths of $\mathbf{F}_1$ and $\mathbf{F}_2$ are

$$\|\mathbf{F}_1\| = \|\mathbf{F}\|\cos 60^\circ = 100\left(\frac{1}{2}\right) = 50 \text{ lb}$$

$$\|\mathbf{F}_2\| = \|\mathbf{F}\|\sin 60^\circ = 100\left(\frac{\sqrt{3}}{2}\right) \approx 86.6 \text{ lb}$$

Thus, the block exerts a force of approximately 86.6 lb against the ramp, and it requires a force of 50 lb to prevent the block from sliding down the ramp. ◀

ORTHOGONAL PROJECTIONS

The vector components of $\mathbf{v}$ along $\mathbf{e}_1$ and $\mathbf{e}_2$ in (7) are also called the *orthogonal projections* of $\mathbf{v}$ on $\mathbf{e}_1$ and $\mathbf{e}_2$ and are commonly denoted by

$$\text{proj}_{\mathbf{e}_1}\mathbf{v} = (\mathbf{v} \cdot \mathbf{e}_1)\mathbf{e}_1 \quad \text{and} \quad \text{proj}_{\mathbf{e}_2}\mathbf{v} = (\mathbf{v} \cdot \mathbf{e}_2)\mathbf{e}_2$$

In general, if $\mathbf{e}$ is a unit vector, then we define the ***orthogonal projection of* v *on* e** to be

$$\text{proj}_{\mathbf{e}}\mathbf{v} = (\mathbf{v} \cdot \mathbf{e})\mathbf{e} \tag{11}$$

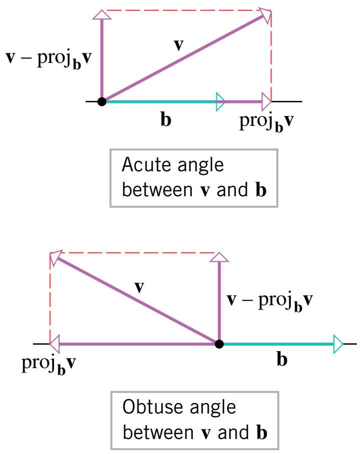


Figure 12.3.10

The orthogonal projection of **v** on an arbitrary nonzero vector **b** can be obtained by normalizing **b** and then applying Formula (11); that is,

$$\text{proj}_{\mathbf{b}}\mathbf{v} = \left(\mathbf{v} \cdot \frac{\mathbf{b}}{\|\mathbf{b}\|}\right)\left(\frac{\mathbf{b}}{\|\mathbf{b}\|}\right)$$

which can be rewritten as

$$\text{proj}_{\mathbf{b}}\mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{b}}{\|\mathbf{b}\|^2}\mathbf{b} \tag{12}$$

Geometrically, if **b** and **v** have a common initial point, then $\text{proj}_{\mathbf{b}}\mathbf{v}$ is the vector that is determined when a perpendicular is dropped from the terminal point of **v** to the line through **b** (illustrated in Figure 12.3.10 in two cases). Moreover, it is evident from Figure 12.3.10 that if we subtract $\text{proj}_{\mathbf{b}}\mathbf{v}$ from **v**, then the resulting vector

$$\mathbf{v} - \text{proj}_{\mathbf{b}}\mathbf{v}$$

will be orthogonal to **b**; we call this the ***vector component of* v *orthogonal to* b**.

Example 6 Find the orthogonal projection of $\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ on $\mathbf{b} = 2\mathbf{i} + 2\mathbf{j}$, and then find the vector component of **v** orthogonal to **b**.

Solution. We have

$$\mathbf{v} \cdot \mathbf{b} = (\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot (2\mathbf{i} + 2\mathbf{j}) = 2 + 2 + 0 = 4$$

$$\|\mathbf{b}\|^2 = 2^2 + 2^2 = 8$$

Thus, the orthogonal projection of **v** on **b** is

$$\text{proj}_{\mathbf{b}}\mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{b}}{\|\mathbf{b}\|^2}\mathbf{b} = \frac{4}{8}(2\mathbf{i} + 2\mathbf{j}) = \mathbf{i} + \mathbf{j}$$

and the vector component of **v** orthogonal to **b** is

$$\mathbf{v} - \text{proj}_{\mathbf{b}}\mathbf{v} = (\mathbf{i} + \mathbf{j} + \mathbf{k}) - (\mathbf{i} + \mathbf{j}) = \mathbf{k}$$

These results are consistent with Figure 12.3.11. ◀

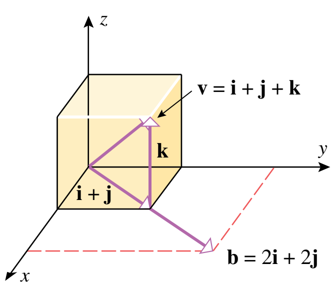


Figure 12.3.11

WORK

In Section 6.6 we discussed the work done by a constant force acting on an object that moves along a line. We defined the work W done on the object by a constant force of magnitude F acting in the direction of motion over a distance d to be

$$W = Fd = \text{force} \times \text{distance} \tag{13}$$

If we let **F** denote a force vector of magnitude $\|\mathbf{F}\| = F$ acting in the direction of motion, then we can write (13) as

$$W = \|\mathbf{F}\|d$$

Furthermore, if we assume that the object moves along a line from point P to point Q, then $d = \|\overrightarrow{PQ}\|$, so that the work can be expressed entirely in vector form as

$$W = \|\mathbf{F}\|\|\overrightarrow{PQ}\|$$

(Figure 12.3.12*a*). The vector $\overrightarrow{PQ}$ is called the ***displacement vector*** for the object. In the case where a constant force **F** is not in the direction of motion, but rather makes an angle θ

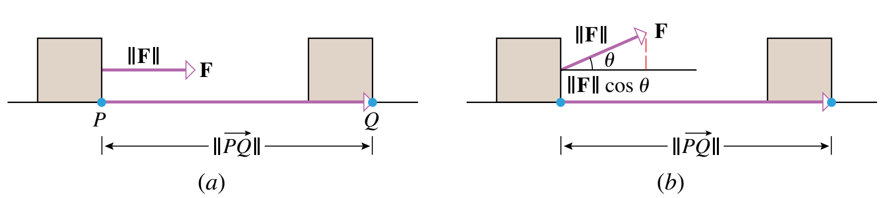


Figure 12.3.12

with the displacement vector, then we *define* the work W done by $\mathbf{F}$ to be

$$W = (\|\mathbf{F}\|\cos\theta)\|\overrightarrow{PQ}\| = \mathbf{F}\cdot\overrightarrow{PQ} \tag{14}$$

(Figure 12.3.12*b*).

REMARK. Note that in Formula (14) the quantity $\|\mathbf{F}\|\cos\theta$ is the scalar component of force along the displacement vector. Thus, in the case where $\cos\theta > 0$, a force of magnitude $\|\mathbf{F}\|$ acting at an angle θ does the same work as a force of magnitude $\|\mathbf{F}\|\cos\theta$ acting in the direction of motion.

Example 7 A wagon is pulled horizontally by exerting a constant force of 10 lb on the handle at an angle of 60° with the horizontal. How much work is done in moving the wagon 50 ft?

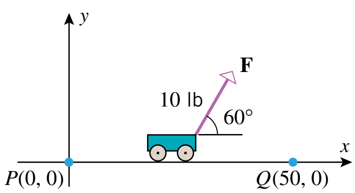


Figure 12.3.13

Solution. Introduce an xy-coordinate system so that the wagon moves from $P(0,0)$ to $Q(50,0)$ along the x-axis (Figure 12.3.13). In this coordinate system

$$\overrightarrow{PQ} = 50\mathbf{i}$$

and

$$\mathbf{F} = (10\cos 60^\circ)\mathbf{i} + (10\sin 60^\circ)\mathbf{j} = 5\mathbf{i} + 5\sqrt{3}\mathbf{j}$$

so the work done is

$$W = \mathbf{F}\cdot\overrightarrow{PQ} = (5\mathbf{i} + 5\sqrt{3}\mathbf{j})\cdot(50\mathbf{i}) = 250 \text{ (foot-pounds)}$$ ◀

EXERCISE SET 12.3 Graphing Utility C CAS

1. In each part, find the dot product of the vectors and the cosine of the angle between them.
(a) $\mathbf{u} = \mathbf{i} + 2\mathbf{j},\ \mathbf{v} = 6\mathbf{i} - 8\mathbf{j}$
(b) $\mathbf{u} = \langle -7, -3\rangle,\ \mathbf{v} = \langle 0, 1\rangle$
(c) $\mathbf{u} = \mathbf{i} - 3\mathbf{j} + 7\mathbf{k},\ \mathbf{v} = 8\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$
(d) $\mathbf{u} = \langle -3, 1, 2\rangle,\ \mathbf{v} = \langle 4, 2, -5\rangle$

2. In each part use the given information to find $\mathbf{u}\cdot\mathbf{v}$.
(a) $\|\mathbf{u}\| = 1$, $\|\mathbf{v}\| = 2$, the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\pi/6$.
(b) $\|\mathbf{u}\| = 2$, $\|\mathbf{v}\| = 3$, the angle between $\mathbf{u}$ and $\mathbf{v}$ is 135°.

3. In each part, determine whether $\mathbf{u}$ and $\mathbf{v}$ make an acute angle, an obtuse angle, or are orthogonal.
(a) $\mathbf{u} = 7\mathbf{i} + 3\mathbf{j} + 5\mathbf{k},\ \mathbf{v} = -8\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$
(b) $\mathbf{u} = 6\mathbf{i} + \mathbf{j} + 3\mathbf{k},\ \mathbf{v} = 4\mathbf{i} - 6\mathbf{k}$
(c) $\mathbf{u} = \langle 1, 1, 1\rangle,\ \mathbf{v} = \langle -1, 0, 0\rangle$
(d) $\mathbf{u} = \langle 4, 1, 6\rangle,\ \mathbf{v} = \langle -3, 0, 2\rangle$

4. Does the triangle in 3-space with vertices $(-1, 2, 3)$, $(2, -2, 0)$, and $(3, 1, -4)$ have an obtuse angle? Justify your answer.

5. The accompanying figure shows eight vectors that are equally spaced around a circle of radius 1. Find the dot product of $\mathbf{v}_0$ with each of the other seven vectors.

6. The accompanying figure shows six vectors that are equally spaced around a circle of radius 5. Find the dot product of $\mathbf{v}_0$ with each of the other five vectors.

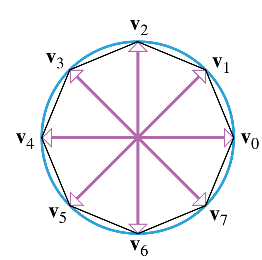


Figure Ex-5

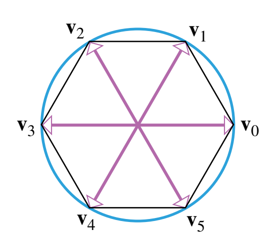


Figure Ex-6

7. (a) Use vectors to show that $A(2, -1, 1)$, $B(3, 2, -1)$, and $C(7, 0, -2)$ are vertices of a right triangle. At which vertex is the right angle?
(b) Use vectors to find the interior angles of the triangle with vertices $(-1, 0)$, $(2, -1)$, and $(1, 4)$. Express your answers to the nearest degree.

8. Find k so that the vector from the point $A(1, -1, 3)$ to the point $B(3, 0, 5)$ is orthogonal to the vector from A to the point $P(k, k, k)$.

9. (a) Show that if $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ is a vector in 2-space, then the vectors

$$\mathbf{v}_1 = -b\mathbf{i} + a\mathbf{j} \quad \text{and} \quad \mathbf{v}_2 = b\mathbf{i} - a\mathbf{j}$$

are both orthogonal to $\mathbf{v}$.

(b) Use the result in part (a) to find two unit vectors that are orthogonal to the vector $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j}$. Sketch the vectors $\mathbf{v}$, $\mathbf{v}_1$, and $\mathbf{v}_2$.

10. Find two unit vectors in 2-space that make an angle of 45° with $4\mathbf{i} + 3\mathbf{j}$.

11. Explain why each of the following expressions makes no sense.

(a) $\mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w})$ (b) $(\mathbf{u} \cdot \mathbf{v}) + \mathbf{w}$

(c) $\|\mathbf{u} \cdot \mathbf{v}\|$ (d) $k \cdot (\mathbf{u} + \mathbf{v})$

12. Verify parts (*b*) and (*c*) of Theorem 12.3.2 for the vectors $\mathbf{u} = 6\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $\mathbf{v} = 2\mathbf{i} + 7\mathbf{j} + 4\mathbf{k}$, $\mathbf{w} = \mathbf{i} + \mathbf{j} - 3\mathbf{k}$ and $k = -5$.

13. Let $\mathbf{u} = \langle 1, 2\rangle$, $\mathbf{v} = \langle 4, -2\rangle$, and $\mathbf{w} = \langle 6, 0\rangle$. Find

(a) $\mathbf{u} \cdot (7\mathbf{v} + \mathbf{w})$ (b) $\|(\mathbf{u} \cdot \mathbf{w})\mathbf{w}\|$

(c) $\|\mathbf{u}\|(\mathbf{v} \cdot \mathbf{w})$ (d) $(\|\mathbf{u}\|\mathbf{v}) \cdot \mathbf{w}$.

14. True or False? If $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$ and if $\mathbf{a} \neq \mathbf{0}$, then $\mathbf{b} = \mathbf{c}$. Justify your conclusion.

In Exercises 15 and 16, find the direction cosines of $\mathbf{v}$, and confirm that they satisfy Equation (5). Then use the direction cosines to approximate the direction angles to the nearest degree.

15. (a) $\mathbf{v} = \mathbf{i} + \mathbf{j} - \mathbf{k}$ (b) $\mathbf{v} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

16. (a) $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j} - 6\mathbf{k}$ (b) $\mathbf{v} = 3\mathbf{i} - 4\mathbf{k}$

17. Show that the direction cosines of a vector satisfy

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

18. Let θ and λ be the angles shown in the accompanying figure. Show that the direction cosines of $\mathbf{v}$ can be expressed as

$$\cos\alpha = \cos\lambda\cos\theta$$
$$\cos\beta = \cos\lambda\sin\theta$$
$$\cos\gamma = \sin\lambda$$

[*Hint:* Express $\mathbf{v}$ in component form and normalize.]

19. Use the result in Exercise 18 to find the direction angles of the vector shown in the accompanying figure to the nearest degree.

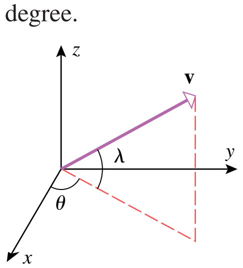


Figure Ex-18

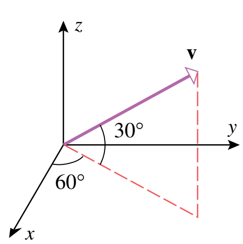


Figure Ex-19

20. Show that two nonzero vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal if and only if their direction cosines satisfy

$$\cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2 = 0$$

21. In each part, find the vector component of $\mathbf{v}$ along $\mathbf{b}$ and the vector component of $\mathbf{v}$ orthogonal to $\mathbf{b}$. Then sketch the vectors $\mathbf{v}$, $\text{proj}_{\mathbf{b}}\mathbf{v}$, and $\mathbf{v} - \text{proj}_{\mathbf{b}}\mathbf{v}$.

(a) $\mathbf{v} = 2\mathbf{i} - \mathbf{j}$, $\mathbf{b} = 3\mathbf{i} + 4\mathbf{j}$

(b) $\mathbf{v} = \langle 4, 5\rangle$, $\mathbf{b} = \langle 1, -2\rangle$

(c) $\mathbf{v} = -3\mathbf{i} - 2\mathbf{j}$, $\mathbf{b} = 2\mathbf{i} + \mathbf{j}$

22. In each part, find the vector component of $\mathbf{v}$ along $\mathbf{b}$ and the vector component of $\mathbf{v}$ orthogonal to $\mathbf{b}$.

(a) $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

(b) $\mathbf{v} = \langle 4, -1, 7\rangle$, $\mathbf{b} = \langle 2, 3, -6\rangle$

In Exercises 23 and 24, express the vector $\mathbf{v}$ as the sum of a vector parallel to $\mathbf{b}$ and a vector orthogonal to $\mathbf{b}$.

23. (a) $\mathbf{v} = 2\mathbf{i} - 4\mathbf{j}$, $\mathbf{b} = \mathbf{i} + \mathbf{j}$

(b) $\mathbf{v} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - \mathbf{k}$

24. (a) $\mathbf{v} = \langle -3, 5\rangle$, $\mathbf{b} = \langle 1, 1\rangle$

(b) $\mathbf{v} = \langle -2, 1, 6\rangle$, $\mathbf{b} = \langle 0, -2, 1\rangle$

25. If L is a line in 2-space or 3-space that passes through the points A and B, then the distance from a point P to the line L is equal to the length of the component of the vector $\overrightarrow{AP}$ that is orthogonal to the vector $\overrightarrow{AB}$ (see the accompanying figure). Use this result to find the distance from the point $P(1, 0)$ to the line through $A(2, -3)$ and $B(5, 1)$.

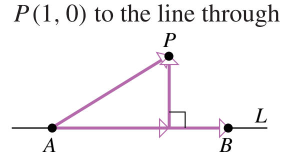


Figure Ex-25

26. Use the method of Exercise 25 to find the distance from the point $P(-3, 1, 2)$ to the line through $A(1, 1, 0)$ and $B(-2, 3, -4)$.

27. As shown in the accompanying figure, a block with a mass of 10 kg rests on a smooth (frictionless) ramp that is inclined at an angle of 45° with the ground. How much force does the block exert on the ramp, and how much force must be applied in the direction of $\mathbf{P}$ to prevent the block from sliding down the ramp? Take the acceleration due to gravity to be 9.8 m/s^2.

28. For the block in Exercise 27, how much force must be applied in the direction of $\mathbf{Q}$ (shown in the accompanying figure) to prevent the block from sliding down the ramp?

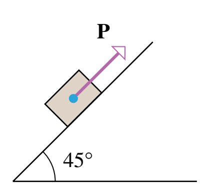


Figure Ex-27

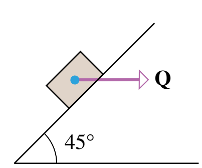


Figure Ex-28

29. A block weighing 300 lb is suspended by cables A and B, as shown in the accompanying figure. Determine the forces that the block exerts along the cables.

30. A block weighing 100 N is suspended by cables A and B, as shown in the accompanying figure.
(a) Use a graphing utility to graph the forces that the block exerts along cables A and B as functions of the "sag" d.
(b) Does increasing the sag increase or decrease the forces on the cables?
(c) How much sag is required if the cables cannot tolerate forces in excess of 150 N?

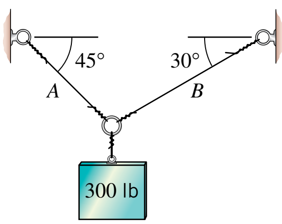


Figure Ex-29

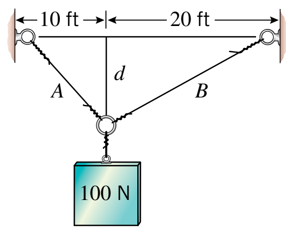


Figure Ex-30

31. Find the work done by a force $\mathbf{F} = -3\mathbf{j}$ (pounds) applied to a point that moves on a line from (1, 3) to (4, 7). Assume that distance is measured in feet.

32. A boat travels 100 meters due north while the wind exerts a force of 500 newtons toward the northeast. How much work does the wind do?

33. A box is dragged along the floor by a rope that applies a force of 50 lb at an angle of 60° with the floor. How much work is done in moving the box 15 ft?

34. A force of $\mathbf{F} = 4\mathbf{i} - 6\mathbf{j} + \mathbf{k}$ newtons is applied to a point that moves a distance of 15 meters in the direction of the vector $\mathbf{i} + \mathbf{j} + \mathbf{k}$. How much work is done?

35. Find, to the nearest degree, the acute angle formed by two diagonals of a cube.

36. Find, to the nearest degree, the angles that a diagonal of a box with dimensions 10 cm by 15 cm by 25 cm makes with the edges of the box.

37. Let $\mathbf{u}$ and $\mathbf{v}$ be adjacent sides of a parallelogram. Use vectors to prove that the diagonals of the parallelogram are perpendicular if the sides are equal in length.

38. Let $\mathbf{u}$ and $\mathbf{v}$ be adjacent sides of a parallelogram. Use vectors to prove that the parallelogram is a rectangle if the diagonals are equal in length.

39. Prove that

$$\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$$

and interpret the result geometrically by translating it into a theorem about parallelograms.

40. Prove: $\mathbf{u} \cdot \mathbf{v} = \frac{1}{4}\|\mathbf{u} + \mathbf{v}\|^2 - \frac{1}{4}\|\mathbf{u} - \mathbf{v}\|^2$.

41. Show that if $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are mutually orthogonal nonzero vectors in 3-space, and if a vector $\mathbf{v}$ in 3-space is expressed as

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

then the scalars c_1, c_2, and c_3 are given by the formulas

$$c_i = (\mathbf{v} \cdot \mathbf{v}_i)/\|\mathbf{v}_i\|^2, \quad i = 1, 2, 3$$

42. Show that the three vectors

$$\mathbf{v}_1 = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k},\ \mathbf{v}_2 = \mathbf{i} + \mathbf{j} - \mathbf{k},\ \mathbf{v}_3 = \mathbf{i} - 5\mathbf{j} - 4\mathbf{k}$$

are mutually orthogonal, and then use the result of Exercise 41 to find scalars c_1, c_2, and c_3 so that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{i} - \mathbf{j} + \mathbf{k}$$

43. For each x in $(-\infty, +\infty)$, let $\mathbf{u}(x)$ be the vector from the origin to the point $P(x, y)$ on the curve $y = x^2 + 1$, and $\mathbf{v}(x)$ the vector from the origin to the point $Q(x, y)$ on the line $y = -x - 1$.
(a) Use a CAS to find, to the nearest degree, the minimum angle between $\mathbf{u}(x)$ and $\mathbf{v}(x)$ for x in $(-\infty, +\infty)$.
(b) Determine whether there are any real values of x for which $\mathbf{u}(x)$ and $\mathbf{v}(x)$ are orthogonal.

44. Let $\mathbf{u}$ be a unit vector in the xy-plane of an xyz-coordinate system, and let $\mathbf{v}$ be a unit vector in the yz-plane. Let θ_1 be the angle between $\mathbf{u}$ and $\mathbf{i}$, let θ_2 be the angle between $\mathbf{v}$ and $\mathbf{k}$, and let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$.
(a) Show that $\cos\theta = \pm\sin\theta_1 \sin\theta_2$.
(b) Find θ if θ is acute and $\theta_1 = \theta_2 = 45°$.
(c) Use a CAS to find, to the nearest degree, the maximum and minimum values of θ if θ is acute and $\theta_2 = 2\theta_1$.

45. Prove parts (b) and (e) of Theorem 12.3.2 for vectors in 3-space.

12.4 CROSS PRODUCT

In many applications of vectors in mathematics, physics, and engineering, there is a need to find a vector that is orthogonal to two given vectors. In this section we will discuss a new type of vector multiplication that can be used for this purpose.

DETERMINANTS

Some of the concepts that we will develop in this section require basic ideas about ***determinants***, which are functions that assign numerical values to square arrays of numbers. For example, if a_1, a_2, b_1, and b_2 are real numbers, then we define a **2 × 2 *determinant*** by

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 \qquad (1)$$

The purpose of the arrows is to help you remember the formula—the determinant is the product of the entries on the rightward arrow minus the product of the entries on the leftward arrow. For example,

$$\begin{vmatrix} 3 & -2 \\ 4 & 5 \end{vmatrix} = (3)(5) - (-2)(4) = 15 + 8 = 23$$

A **3 × 3 *determinant*** is defined in terms of 2 × 2 determinants by

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \tag{2}$$

The right side of this formula is easily remembered by noting that a_1, a_2, and a_3 are the entries in the first "row" of the left side, and the 2 × 2 determinants on the right side arise by deleting the first row and an appropriate column from the left side. The pattern is as follows:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

For example,

$$\begin{vmatrix} 3 & -2 & -5 \\ 1 & 4 & -4 \\ 0 & 3 & 2 \end{vmatrix} = 3 \begin{vmatrix} 4 & -4 \\ 3 & 2 \end{vmatrix} - (-2) \begin{vmatrix} 1 & -4 \\ 0 & 2 \end{vmatrix} + (-5) \begin{vmatrix} 1 & 4 \\ 0 & 3 \end{vmatrix}$$

$$= 3(20) + 2(2) - 5(3) = 49$$

There are also definitions of 4 × 4 determinants, 5 × 5 determinants, and higher, but we will not need them in this text. Properties of determinants are studied in a branch of mathematics called ***linear algebra***, but we will only need the two properties stated in the following theorem:

> **12.4.1 THEOREM.**
> (*a*) *If two rows in the array of a determinant are the same, then the value of the determinant is* 0.
> (*b*) *Interchanging two rows in the array of a determinant multiplies its value by* −1.

We will give the proofs of parts (*a*) and (*b*) for 2 × 2 determinants and leave the proofs for 3 × 3 determinants as exercises.

Proof (a).

$$\begin{vmatrix} a_1 & a_2 \\ a_1 & a_2 \end{vmatrix} = a_1a_2 - a_2a_1 = 0$$

Proof (b).

$$\begin{vmatrix} b_1 & b_2 \\ a_1 & a_2 \end{vmatrix} = b_1a_2 - b_2a_1 = -(a_1b_2 - a_2b_1) = -\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$ ■

CROSS PRODUCT

We now turn to the main concept in this section.

> **12.4.2 DEFINITION.** If $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ are vectors in 3-space, then the ***cross product*** $\mathbf{u} \times \mathbf{v}$ is the vector defined by
>
> $$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k} \tag{3}$$
>
> or, equivalently,
>
> $$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k} \tag{4}$$

March 6, 2001 14:27 g65-ch12 Sheet number 29 Page number 819 cyan magenta yellow black

Observe that the right side of Formula (3) has the same form as the right side of Formula (2), the difference being notation and the order of the factors in the three terms. Thus, we can rewrite (3) as

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \tag{5}$$

However, this is just a mnemonic device and not a true determinant since the entries in a determinant are numbers, not vectors.

Example 1 Let $\mathbf{u} = \langle 1, 2, -2 \rangle$ and $\mathbf{v} = \langle 3, 0, 1 \rangle$. Find

(a) $\mathbf{u} \times \mathbf{v}$ (b) $\mathbf{v} \times \mathbf{u}$

Solution (a).

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -2 \\ 3 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} \mathbf{k} = 2\mathbf{i} - 7\mathbf{j} - 6\mathbf{k}$$

Solution (b). We could use the method of part (a), but it is really not necessary to perform any computations. We need only observe that reversing **u** and **v** interchanges the second and third rows in (5), which in turn interchanges the rows in the arrays for the 2×2 determinants in (3). But interchanging the rows in the array of a 2×2 determinant reverses its sign, so the net effect of reversing the factors in a cross product is to reverse the signs of the components. Thus, by inspection

$$\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = -2\mathbf{i} + 7\mathbf{j} + 6\mathbf{k}$$ ◀

Example 2 Show that $\mathbf{u} \times \mathbf{u} = \mathbf{0}$ for any vector **u** in 3-space.

Solution. We could let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and apply the method in part (a) of Example 1 to show that

$$\mathbf{u} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \end{vmatrix} = 0$$

However, the actual computations are unnecessary. We need only observe that if the two factors in a cross product are the same, then each 2×2 determinant in (3) is zero because its array has identical rows. Thus, $\mathbf{u} \times \mathbf{u} = \mathbf{0}$ by inspection. ◀

ALGEBRAIC PROPERTIES OF THE CROSS PRODUCT

Our next goal is to establish some of the basic algebraic properties of the cross product. As you read the discussion, keep in mind the essential differences between the cross product and the dot product:

- The cross product is defined only for vectors in 3-space, whereas the dot product is defined for vectors in 2-space and 3-space.
- The cross product of two vectors is a vector, whereas the dot product of two vectors is a scalar.

The main algebraic properties of the cross product are listed in the next theorem.

12.4.3 THEOREM. *If* **u**, **v**, *and* **w** *are any vectors in 3-space and k is any scalar, then*

(*a*) $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$

(*b*) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$

(*c*) $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$

(*d*) $k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v})$

(*e*) $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$

(*f*) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

Parts (*a*) and (*f*) were addressed in Examples 1 and 2. The other proofs are left as exercises.

WARNING. In ordinary multiplication and in dot products the order of the factors does not matter, but in cross products it does. Part (*a*) of the last theorem shows that reversing the order of the factors in a cross product reverses the direction of the resulting vector.

The following cross products occur so frequently that it is helpful to be familiar with them:

$$\begin{array}{lll} \mathbf{i} \times \mathbf{j} = \mathbf{k} & \mathbf{j} \times \mathbf{k} = \mathbf{i} & \mathbf{k} \times \mathbf{i} = \mathbf{j} \\ \mathbf{j} \times \mathbf{i} = -\mathbf{k} & \mathbf{k} \times \mathbf{j} = -\mathbf{i} & \mathbf{i} \times \mathbf{k} = -\mathbf{j} \end{array} \tag{6}$$

These results are easy to obtain; for example,

$$\mathbf{i} \times \mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\mathbf{k} = \mathbf{k}$$

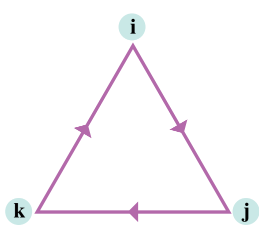


Figure 12.4.1

However, rather than computing these cross products each time you need them, you can use the diagram in Figure 12.4.1. In this diagram, the cross product of two consecutive vectors in the clockwise direction is the next vector around, and the cross product of two consecutive vectors in the counterclockwise direction is the negative of the next vector around.

WARNING. We can write a product of three real numbers as uvw because the associative law $u(vw) = (uv)w$ ensures that the same value for the product results no matter where the parentheses are inserted. However, the associative law *does not* hold for cross products. For example,

$$\mathbf{i} \times (\mathbf{j} \times \mathbf{j}) = \mathbf{i} \times \mathbf{0} = \mathbf{0} \quad \text{and} \quad (\mathbf{i} \times \mathbf{j}) \times \mathbf{j} = \mathbf{k} \times \mathbf{j} = -\mathbf{i}$$

so that $\mathbf{i} \times (\mathbf{j} \times \mathbf{j}) \neq (\mathbf{i} \times \mathbf{j}) \times \mathbf{j}$. Thus, we cannot write a cross product with three vectors as $\mathbf{u} \times \mathbf{v} \times \mathbf{w}$, since this expression is ambiguous without parentheses.

GEOMETRIC PROPERTIES OF THE CROSS PRODUCT

The following theorem shows that the cross product of two vectors is orthogonal to both factors.

12.4.4 THEOREM. *If* **u** *and* **v** *are vectors in 3-space, then*

(*a*) $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ ($\mathbf{u} \times \mathbf{v}$ *is orthogonal to* $\mathbf{u}$)

(*b*) $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ ($\mathbf{u} \times \mathbf{v}$ *is orthogonal to* $\mathbf{v}$)

We will prove part (*a*). The proof of part (*b*) is similar.

Proof (a). Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$. Then from (4)

$$\mathbf{u} \times \mathbf{v} = \langle u_2v_3 - u_3v_2, \quad u_3v_1 - u_1v_3, \quad u_1v_2 - u_2v_1 \rangle \tag{7}$$

so that

$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = u_1(u_2v_3 - u_3v_2) + u_2(u_3v_1 - u_1v_3) + u_3(u_1v_2 - u_2v_1) = 0$$ ■

Example 3 In Example 1 we showed that the cross product $\mathbf{u} \times \mathbf{v}$ of $\mathbf{u} = \langle 1, 2, -2 \rangle$ and $\mathbf{v} = \langle 3, 0, 1 \rangle$ is

$$\mathbf{u} \times \mathbf{v} = 2\mathbf{i} - 7\mathbf{j} - 6\mathbf{k} = \langle 2, -7, -6 \rangle$$

Theorem 12.4.4 guarantees that this vector is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$; this is confirmed by the computations

$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = \langle 1, 2, -2 \rangle \cdot \langle 2, -7, -6 \rangle = (1)(2) + (2)(-7) + (-2)(-6) = 0$$
$$\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = \langle 3, 0, 1 \rangle \cdot \langle 2, -7, -6 \rangle = (3)(2) + (0)(-7) + (1)(-6) = 0$$ ◀

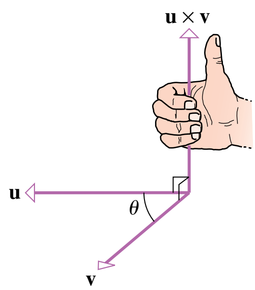


Figure 12.4.2

It can be proved that if $\mathbf{u}$ and $\mathbf{v}$ are nonzero and nonparallel vectors, then the direction of $\mathbf{u} \times \mathbf{v}$ relative to $\mathbf{u}$ and $\mathbf{v}$ is determined by a right-hand rule;* that is, if the fingers of the right hand are cupped so they curl from $\mathbf{u}$ toward $\mathbf{v}$ in the direction of rotation that takes $\mathbf{u}$ into $\mathbf{v}$ in less than 180°, then the thumb will point (roughly) in the direction of $\mathbf{u} \times \mathbf{v}$ (Figure 12.4.2). For example, we stated in (6) that

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}$$

all of which are consistent with the right-hand rule (verify).

The next theorem lists some more important geometric properties of the cross product.

12.4.5 THEOREM. *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be nonzero vectors in 3-space, and let* θ *be the angle between these vectors when they are positioned so their initial points coincide.*

(*a*) $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\| \sin\theta$

(*b*) *The area A of the parallelogram that has* $\mathbf{u}$ *and* $\mathbf{v}$ *as adjacent sides is*

$$A = \|\mathbf{u} \times \mathbf{v}\| \tag{8}$$

(*c*) $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ *if and only if* $\mathbf{u}$ *and* $\mathbf{v}$ *are parallel vectors, that is, if and only if they are scalar multiples of one another.*

Proof (a).

$$\begin{aligned}
\|\mathbf{u}\|\|\mathbf{v}\|\sin\theta &= \|\mathbf{u}\|\|\mathbf{v}\|\sqrt{1-\cos^2\theta} \\
&= \|\mathbf{u}\|\|\mathbf{v}\|\sqrt{1-\frac{(\mathbf{u}\cdot\mathbf{v})^2}{\|\mathbf{u}\|^2\|\mathbf{v}\|^2}} \qquad \text{Theorem 12.3.3} \\
&= \sqrt{\|\mathbf{u}\|^2\|\mathbf{v}\|^2-(\mathbf{u}\cdot\mathbf{v})^2} \\
&= \sqrt{(u_1^2+u_2^2+u_3^2)(v_1^2+v_2^2+v_3^2)-(u_1v_1+u_2v_2+u_3v_3)^2} \\
&= \sqrt{(u_2v_3-u_3v_2)^2+(u_1v_3-u_3v_1)^2+(u_1v_2-u_2v_1)^2} \\
&= \|\mathbf{u}\times\mathbf{v}\| \qquad \text{See Formula (4).}
\end{aligned}$$

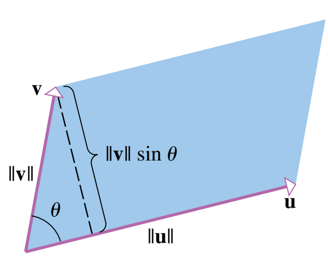


Figure 12.4.3

Proof (b). Referring to Figure 12.4.3, the parallelogram that has $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides can be viewed as having base $\|\mathbf{u}\|$ and altitude $\|\mathbf{v}\|\sin\theta$. Thus, its area A is

$$A = (\text{base})(\text{altitude}) = \|\mathbf{u}\|\|\mathbf{v}\|\sin\theta = \|\mathbf{u}\times\mathbf{v}\|$$

Proof (c). Since $\mathbf{u}$ and $\mathbf{v}$ are assumed to be nonzero vectors, it follows from part (*a*) that $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ if and only if $\sin\theta = 0$; this is true if and only if $\theta = 0$ or $\theta = \pi$ (since

*Recall that we agreed to consider only right-handed coordinate systems in this text. Had we used left-handed systems instead, a "left-hand rule" would apply here.

$0 \le \theta \le \pi$). Geometrically, this means that $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are parallel vectors.

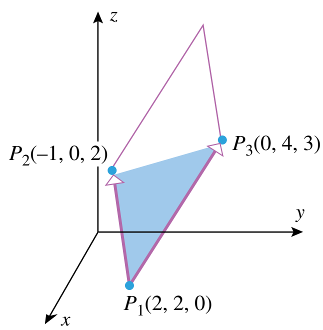


Figure 12.4.4

Example 4 Find the area of the triangle that is determined by the points $P_1(2, 2, 0)$, $P_2(-1, 0, 2)$, and $P_3(0, 4, 3)$.

Solution. The area A of the triangle is half the area of the parallelogram determined by the vectors $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$ (Figure 12.4.4). But $\overrightarrow{P_1P_2} = \langle -3, -2, 2 \rangle$ and $\overrightarrow{P_1P_3} = \langle -2, 2, 3 \rangle$, so

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \langle -10, 5, -10 \rangle$$

(verify), and consequently

$$A = \tfrac{1}{2}\|\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}\| = \tfrac{15}{2}$$ ◀

SCALAR TRIPLE PRODUCTS

If $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ are vectors in 3-space, then the number

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$$

is called the ***scalar triple product*** of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. It is not necessary to compute the dot product and cross product to evaluate a scalar triple product—the value can be obtained directly from the formula

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \tag{9}$$

the validity of which can be seen by writing

$$\begin{aligned} \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) &= \mathbf{u} \cdot \left(\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \mathbf{k} \right) \\ &= u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \\ &= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \end{aligned}$$

Example 5 Calculate the scalar triple product $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ of the vectors

$$\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} - 5\mathbf{k}, \quad \mathbf{v} = \mathbf{i} + 4\mathbf{j} - 4\mathbf{k}, \quad \mathbf{w} = 3\mathbf{j} + 2\mathbf{k}$$

Solution.

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 3 & -2 & -5 \\ 1 & 4 & -4 \\ 0 & 3 & 2 \end{vmatrix} = 49$$ ◀

FOR THE READER. Many calculating utilities have built-in cross product and determinant operations. If your calculating utility has these capabilities, use it to check the computations in Examples 1 and 5.

GEOMETRIC PROPERTIES OF THE SCALAR TRIPLE PRODUCT

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are nonzero vectors in 3-space that are positioned so their initial points coincide, then these vectors form the adjacent sides of a parallelepiped (Figure 12.4.5). The following theorem establishes a relationship between the volume of this parallelepiped and the scalar triple product of the sides.

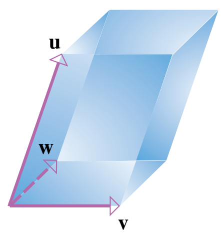


Figure 12.4.5

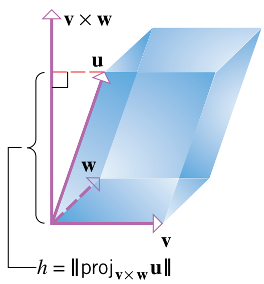


Figure 12.4.6

12.4.6 THEOREM. *Let* **u**, **v**, *and* **w** *be nonzero vectors in 3-space.*

(*a*) *The volume V of the parallelepiped that has* **u**, **v**, *and* **w** *as adjacent edges is*

$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| \tag{10}$$

(*b*) $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 0$ *if and only if* **u**, **v**, *and* **w** *lie in the same plane.*

Proof (a). Referring to Figure 12.4.6, let us regard the base of the parallelepiped with **u**, **v**, and **w** as adjacent sides to be the parallelogram determined by **v** and **w**. Thus, the area of the base is $\|\mathbf{v} \times \mathbf{w}\|$, and the altitude h of the parallelepiped (shown in the figure) is the length of the orthogonal projection of **u** on the vector $\mathbf{v} \times \mathbf{w}$. Therefore, from Formula (12) of Section 12.3 we have

$$h = \|\text{proj}_{\mathbf{v}\times\mathbf{w}}\mathbf{u}\| = \frac{|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|}{\|\mathbf{v} \times \mathbf{w}\|^2}\|\mathbf{v} \times \mathbf{w}\| = \frac{|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|}{\|\mathbf{v} \times \mathbf{w}\|}$$

It now follows that the volume of the parallelepiped is

$$V = (\text{area of base})(\text{height}) = \|\mathbf{v} \times \mathbf{w}\|h = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$$

Proof (b). The vectors **u**, **v**, and **w** lie in the same plane if and only if the parallelepiped with these vectors as adjacent sides has volume zero (why?). Thus, from part (*a*) the vectors lie in the same plane if and only if $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 0$. ∎

REMARK. It follows from Formula (10) that

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \pm V$$

The + occurs when **u** makes an acute angle with $\mathbf{v} \times \mathbf{w}$ and the − occurs when it makes an obtuse angle.

ALGEBRAIC PROPERTIES OF THE SCALAR TRIPLE PRODUCT

We observed earlier in this section that the expression $\mathbf{u} \times \mathbf{v} \times \mathbf{w}$ must be avoided because it is ambiguous without parentheses. However, the expression $\mathbf{u} \cdot \mathbf{v} \times \mathbf{w}$ is not ambiguous—it has to mean $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ and not $(\mathbf{u} \cdot \mathbf{v}) \times \mathbf{w}$ because we cannot form the cross product of a scalar and a vector. Similarly, the expression $\mathbf{u} \times \mathbf{v} \cdot \mathbf{w}$ must mean $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ and not $\mathbf{u} \times (\mathbf{v} \cdot \mathbf{w})$. Thus, when you see an expression of the form $\mathbf{u} \cdot \mathbf{v} \times \mathbf{w}$ or $\mathbf{u} \times \mathbf{v} \cdot \mathbf{w}$, the cross product is formed first and the dot product second.

Since interchanging two rows of a determinant multiplies its value by −1, making two row interchanges in a determinant has no effect on its value. This being the case, it follows that

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) \tag{11}$$

since the 3×3 determinants that are used to compute these scalar triple products can be obtained from one another by two row interchanges (verify).

REMARK. Observe that the second expression in (11) can be obtained from the first by leaving the dot, the cross, and the parentheses fixed, moving the first two vectors to the right, and bringing the third vector to the first position. The same procedure produces the third expression from the second and the first expression from the third (verify).

Another useful formula can be obtained by rewriting the first equality in (11) as

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$$

and then omitting the superfluous parentheses to obtain

$$\mathbf{u} \cdot \mathbf{v} \times \mathbf{w} = \mathbf{u} \times \mathbf{v} \cdot \mathbf{w} \tag{12}$$

In words, this formula states that the dot and cross in a scalar triple product can be interchanged (provided the factors are grouped appropriately).

March 6, 2001 14:27 g65-ch12 Sheet number 34 Page number 824 cyan magenta yellow black

DOT AND CROSS PRODUCTS ARE COORDINATE INDEPENDENT

In Definitions 12.3.1 and 12.4.2 we defined the dot product and the cross product of two vectors in terms of the components of those vectors in a coordinate system. Thus, it is theoretically possible that changing the coordinate system might change $\mathbf{u} \cdot \mathbf{v}$ or $\mathbf{u} \times \mathbf{v}$, since the components of a vector depend on the coordinate system that is chosen. However, the relationships

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta \tag{13}$$

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|\sin\theta \tag{14}$$

that were obtained in Theorems 12.3.3 and 12.4.5 show that this is not the case. Formula (13) shows that the value of $\mathbf{u} \cdot \mathbf{v}$ depends only on the lengths of the vectors and the angle between them—not on the coordinate system. Similarly, Formula (14), in combination with the right-hand rule and Theorem 12.4.4, shows that $\mathbf{u} \times \mathbf{v}$ does not depend on the coordinate system (as long as it is right handed). These facts are important in applications because they allow us to choose any convenient coordinate system for solving a problem with full confidence that the choice will not affect computations that involve dot products or cross products.

MOMENTS AND ROTATIONAL MOTION IN 3-SPACE

Astronauts use tools that are designed to limit forces that would impart unintended rotational motion to a satellite.

Cross products play an important role in describing rotational motion in 3-space. For example, suppose that an astronaut on a satellite repair mission in space applies a force $\mathbf{F}$ at a point Q on the surface of a spherical satellite. If the force is directed along a line that passes through the center P of the satellite, then Newton's Second Law of Motion implies that the force will accelerate the satellite in the direction of $\mathbf{F}$. However, if the astronaut applies the same force at an angle θ with the vector $\overrightarrow{PQ}$, then $\mathbf{F}$ will tend to cause a rotation, as well as an acceleration in the direction of $\mathbf{F}$. To see why this is so, let us resolve $\mathbf{F}$ into a sum of orthogonal components $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$, where $\mathbf{F}_1$ is the orthogonal projection of $\mathbf{F}$ on the vector $\overrightarrow{PQ}$ and $\mathbf{F}_2$ is the component of $\mathbf{F}$ orthogonal to $\overrightarrow{PQ}$ (Figure 12.4.7). Since the force $\mathbf{F}_1$ acts along the line through the center of the satellite, it contributes to the linear acceleration of the satellite but does not cause any rotation. However, the force $\mathbf{F}_2$ is tangent to the circle around the satellite in the plane of $\mathbf{F}$ and $\overrightarrow{PQ}$, so it causes the satellite to rotate about an axis that is perpendicular to that plane.

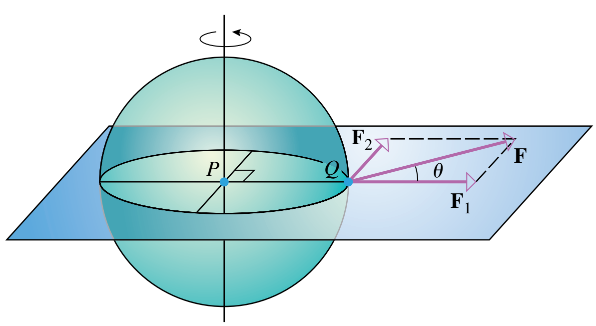


Figure 12.4.7

You know from your own experience that the "tendency" for rotation about an axis depends both on the amount of force and how far from the axis it is applied. For example, it is easier to close a door by pushing on its outer edge than applying the same force close to the hinges. In fact, the tendency of rotation of the satellite can be measured by

$$\|\overrightarrow{PQ}\|\|\mathbf{F}_2\| \quad \boxed{\text{distance from the center} \times \text{magnitude of the force}} \tag{15}$$

However, $\|\mathbf{F}_2\| = \|\mathbf{F}\|\sin\theta$, so we can rewrite (15) as

$$\|\overrightarrow{PQ}\|\|\mathbf{F}\|\sin\theta = \|\overrightarrow{PQ} \times \mathbf{F}\|$$

This is called the ***scalar moment*** or ***torque*** of $\mathbf{F}$ about the point P. Scalar moments have units of force times distance—pound-feet or newton-meters, for example. The vector $\overrightarrow{PQ} \times \mathbf{F}$ is called the ***vector moment*** or ***torque vector*** of $\mathbf{F}$ about P.

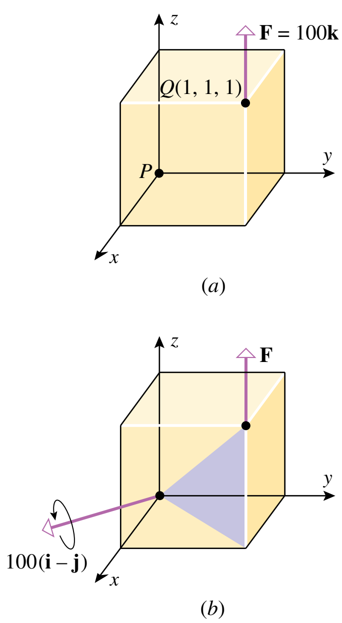


Figure 12.4.8

Recalling that the direction of $\overrightarrow{PQ} \times \mathbf{F}$ is determined by the right-hand rule, it follows that the direction of rotation about P that results by applying the force $\mathbf{F}$ at the point Q is counterclockwise looking down the axis of $\overrightarrow{PQ} \times \mathbf{F}$ (Figure 12.4.7). Thus, the vector moment $\overrightarrow{PQ} \times \mathbf{F}$ captures the essential information about the rotational effect of the force—the magnitude of the cross product provides the scalar moment of the force, and the cross product vector itself provides the axis and direction of rotation.

Example 6 Figure 12.4.8*a* shows a force $\mathbf{F}$ of 100 N applied in the positive z-direction at the point $Q(1, 1, 1)$ of a cube whose sides have a length of 1 m. Assuming that the cube is free to rotate about the point $P(0, 0, 0)$ (the origin), find the scalar moment of the force about P, and describe the direction of rotation.

Solution. The force vector is $\mathbf{F} = 100\mathbf{k}$, and the vector from P to Q is $\overrightarrow{PQ} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, so the vector moment of $\mathbf{F}$ about P is

$$\overrightarrow{PQ} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 0 & 0 & 100 \end{vmatrix} = 100\mathbf{i} - 100\mathbf{j}$$

Thus, the scalar moment of $\mathbf{F}$ about P is $\|100\mathbf{i} - 100\mathbf{j}\| = 100\sqrt{2} \approx 141$ N·m, and the direction of rotation is counterclockwise looking along the vector $100\mathbf{i} - 100\mathbf{j} = 100(\mathbf{i} - \mathbf{j})$ toward its initial point (Figure 12.4.8*b*). ◀

EXERCISE SET 12.4 [C] CAS

1. (a) Use a determinant to find the cross product

$$\mathbf{i} \times (\mathbf{i} + \mathbf{j} + \mathbf{k})$$

(b) Check your answer in part (a) by rewriting the cross product as

$$\mathbf{i} \times (\mathbf{i} + \mathbf{j} + \mathbf{k}) = (\mathbf{i} \times \mathbf{i}) + (\mathbf{i} \times \mathbf{j}) + (\mathbf{i} \times \mathbf{k})$$

and evaluating each term.

2. In each part, use the two methods in Exercise 1 to find (a) $\mathbf{j} \times (\mathbf{i} + \mathbf{j} + \mathbf{k})$ (b) $\mathbf{k} \times (\mathbf{i} + \mathbf{j} + \mathbf{k})$.

In Exercises 3–6, find $\mathbf{u} \times \mathbf{v}$, and check that it is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

3. $\mathbf{u} = \langle 1, 2, -3 \rangle$, $\mathbf{v} = \langle -4, 1, 2 \rangle$

4. $\mathbf{u} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{v} = -\mathbf{i} - 3\mathbf{j} + \mathbf{k}$

5. $\mathbf{u} = \langle 0, 1, -2 \rangle$, $\mathbf{v} = \langle 3, 0, -4 \rangle$

6. $\mathbf{u} = 4\mathbf{i} + \mathbf{k}$, $\mathbf{v} = 2\mathbf{i} - \mathbf{j}$

7. Let $\mathbf{u} = \langle 2, -1, 3 \rangle$, $\mathbf{v} = \langle 0, 1, 7 \rangle$, and $\mathbf{w} = \langle 1, 4, 5 \rangle$. Find
(a) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ (b) $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$
(c) $(\mathbf{u} \times \mathbf{v}) \times (\mathbf{v} \times \mathbf{w})$ (d) $(\mathbf{v} \times \mathbf{w}) \times (\mathbf{u} \times \mathbf{v})$.

[C] 8. Use a CAS or a calculating utility that can compute determinants or cross products to solve Exercise 7.

9. Find the direction cosines of $\mathbf{u} \times \mathbf{v}$ for the vectors $\mathbf{u}$ and $\mathbf{v}$ in the accompanying figure.

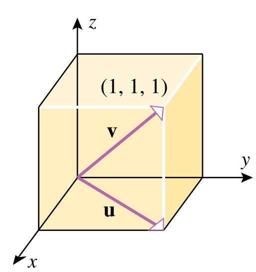


Figure Ex-9

10. Find two unit vectors that are normal to both

$$\mathbf{u} = -7\mathbf{i} + 3\mathbf{j} + \mathbf{k}, \quad \mathbf{v} = 2\mathbf{i} + 4\mathbf{k}$$

11. Find two unit vectors that are perpendicular to the plane determined by the points $A(0, -2, 1)$, $B(1, -1, -2)$, and $C(-1, 1, 0)$.

12. Find two unit vectors that are parallel to the yz-plane and are orthogonal to the vector $3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

In Exercises 13 and 14, find the area of the parallelogram that has $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides.

13. $\mathbf{u} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $\mathbf{v} = 3\mathbf{j} + \mathbf{k}$

14. $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j}$, $\mathbf{v} = -\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$

In Exercises 15 and 16, find the area of the triangle with vertices P, Q, and R.

15. $P(1, 5, -2)$, $Q(0, 0, 0)$, $R(3, 5, 1)$

16. $P(2, 0, -3)$, $Q(1, 4, 5)$, $R(7, 2, 9)$

In Exercises 17–20, find $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$.

17. $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$, $\mathbf{v} = 4\mathbf{i} + \mathbf{j} - 3\mathbf{k}$, $\mathbf{w} = \mathbf{j} + 5\mathbf{k}$

18. $\mathbf{u} = \langle 1, -2, 2\rangle$, $\mathbf{v} = \langle 0, 3, 2\rangle$, $\mathbf{w} = \langle -4, 1, -3\rangle$

19. $\mathbf{u} = \langle 2, 1, 0\rangle$, $\mathbf{v} = \langle 1, -3, 1\rangle$, $\mathbf{w} = \langle 4, 0, 1\rangle$

20. $\mathbf{u} = \mathbf{i}$, $\mathbf{v} = \mathbf{i} + \mathbf{j}$, $\mathbf{w} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

In Exercises 21 and 22, use a scalar triple product to find the volume of the parallelepiped that has **u**, **v**, and **w** as adjacent edges.

21. $\mathbf{u} = \langle 2, -6, 2\rangle$, $\mathbf{v} = \langle 0, 4, -2\rangle$, $\mathbf{w} = \langle 2, 2, -4\rangle$

22. $\mathbf{u} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, $\mathbf{v} = 4\mathbf{i} + 5\mathbf{j} + \mathbf{k}$, $\mathbf{w} = \mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$

23. In each part, use a scalar triple product to determine whether the vectors lie in the same plane.
(a) $\mathbf{u} = \langle 1, -2, 1\rangle$, $\mathbf{v} = \langle 3, 0, -2\rangle$, $\mathbf{w} = \langle 5, -4, 0\rangle$
(b) $\mathbf{u} = 5\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $\mathbf{v} = 4\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{w} = \mathbf{i} - \mathbf{j}$
(c) $\mathbf{u} = \langle 4, -8, 1\rangle$, $\mathbf{v} = \langle 2, 1, -2\rangle$, $\mathbf{w} = \langle 3, -4, 12\rangle$

24. Suppose that $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 3$. Find
(a) $\mathbf{u} \cdot (\mathbf{w} \times \mathbf{v})$ (b) $(\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u}$
(c) $\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$ (d) $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{w})$
(e) $(\mathbf{u} \times \mathbf{w}) \cdot \mathbf{v}$ (f) $\mathbf{v} \cdot (\mathbf{w} \times \mathbf{w})$.

25. Consider the parallelepiped with adjacent edges

$$\mathbf{u} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$$
$$\mathbf{v} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$$
$$\mathbf{w} = \mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$$

(a) Find the volume.
(b) Find the area of the face determined by **u** and **w**.
(c) Find the angle between **u** and the plane containing the face determined by **v** and **w**.

26. Show that in 3-space the distance d from a point P to the line L through points A and B can be expressed as

$$d = \frac{\|\overrightarrow{AP} \times \overrightarrow{AB}\|}{\|\overrightarrow{AB}\|}$$

27. Use the result in Exercise 26 to find the distance between the point P and the line through the points A and B.
(a) $P(-3, 1, 2)$, $A(1, 1, 0)$, $B(-2, 3, -4)$
(b) $P(4, 3)$, $A(2, 1)$, $B(0, 2)$

28. It is a theorem of solid geometry that the volume of a tetrahedron is $\frac{1}{3}$(area of base) · (height). Use this result to prove that the volume of a tetrahedron with adjacent edges given by the vectors **u**, **v**, and **w** is $\frac{1}{6}|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$.

29. Use the result of Exercise 28 to find the volume of the tetrahedron with vertices

$$P(-1, 2, 0),\quad Q(2, 1, -3),\quad R(1, 0, 1),\quad S(3, -2, 3)$$

30. Let θ be the angle between the vectors $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$.
(a) Use the dot product to find $\cos\theta$.
(b) Use the cross product to find $\sin\theta$.
(c) Confirm that $\sin^2\theta + \cos^2\theta = 1$.

31. What can you say about the angle between nonzero vectors **u** and **v** if $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u} \times \mathbf{v}\|$?

32. Show that if **u** and **v** are vectors in 3-space, then

$$\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2$$

[*Note:* This result is sometimes called ***Lagrange's identity***.]

33. The accompanying figure shows a force **F** of 10 lb applied in the positive y-direction to the point $Q(1, 1, 1)$ of a cube whose sides have a length of 1 ft. In each part, find the scalar moment of **F** about the point P, and describe the direction of rotation, if any, if the cube is free to rotate about P.
(a) P is the point $(0, 0, 0)$. (b) P is the point $(1, 0, 0)$.
(c) P is the point $(1, 0, 1)$.

34. The accompanying figure shows a force **F** of 1000 N applied to the corner of a box.
(a) Find the scalar moment of **F** about the point P.
(b) Find the direction angles of the vector moment of **F** about the point P to the nearest degree.

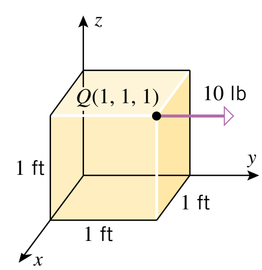


Figure Ex-33

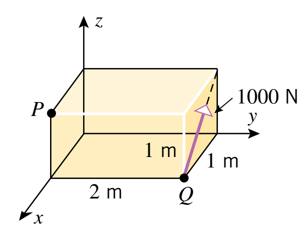


Figure Ex-34

35. As shown in the accompanying figure, a force of 200 N is applied at an angle of 18° to a point near the end of a monkey wrench. Find the scalar moment of the force about the center of the bolt. [Treat this as a problem in two dimensions.]

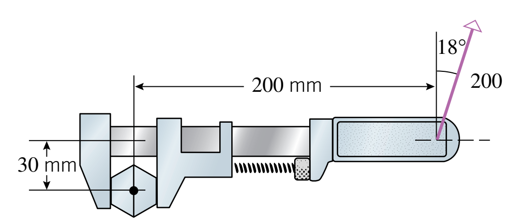


Figure Ex-35

36. Prove parts (*b*) and (*c*) of Theorem 12.4.3.

37. Prove parts (*d*) and (*e*) of Theorem 12.4.3.

38. Prove part (*b*) of Theorem 12.4.1 for 3 × 3 determinants. [Just give the proof for the first two rows.] Then use (*b*) to prove (*a*).

39. Expressions of the form

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \quad\text{and}\quad (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$$

are called ***vector triple products***. It can be proved with some effort that

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$$
$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{w} \cdot \mathbf{u})\mathbf{v} - (\mathbf{w} \cdot \mathbf{v})\mathbf{u}$$

These expressions can be summarized with the following mnemonic rule:

vector triple product = (outer · remote)adjacent − (outer · adjacent)remote

See if you can figure out what the expressions "outer," "remote," and "adjacent" mean in this rule, and then use the rule to find the two vector triple products of the vectors

$$\mathbf{u} = \mathbf{i} + 3\mathbf{j} - \mathbf{k}, \quad \mathbf{v} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}, \quad \mathbf{w} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$$

40. (a) Use the result in Exercise 39 to show that:
$\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ lies in the same plane as **v** and **w**
$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ lies in the same plane as **u** and **v**.
(b) Use a geometrical argument to justify the results in part (a).

41. Prove: If **a**, **b**, **c**, and **d** lie in the same plane when positioned with a common initial point, then

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = \mathbf{0}$$

C **42.** Use a CAS to approximate the minimum area of a triangle if two of its vertices are $(2, -1, 0)$ and $(3, 2, 2)$ and its third vertex is on the curve $y = \ln x$ in the xy-plane.

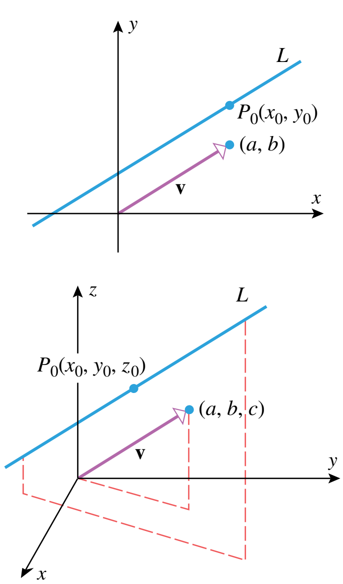
43. If a force **F** is applied to an object at a point Q, then the line through Q parallel to **F** is called the ***line of action*** of the force. We defined the vector moment of **F** about a point P to be $\overrightarrow{PQ} \times \mathbf{F}$. Show that if Q' is any point on the line of action of **F**, then $\overrightarrow{PQ} \times \mathbf{F} = \overrightarrow{PQ'} \times \mathbf{F}$; that is, it is not essential to use the point of application to compute the vector moment—any point on the line of action will do. [*Hint:* Write $\overrightarrow{PQ'} = \overrightarrow{PQ} + \overrightarrow{QQ'}$ and use properties of the cross product.]

12.5 PARAMETRIC EQUATIONS OF LINES

In this section we will discuss parametric equations of lines in 2-space and 3-space. In 3-space, parametric equations of lines are especially important because they generally provide the most convenient form for representing lines algebraically.

LINES DETERMINED BY A POINT AND A VECTOR

A line in 2-space or 3-space can be determined uniquely by specifying a point on the line and a nonzero vector parallel to the line (Figure 12.5.1). The following theorem gives parametric equations of the line through a point P_0 and parallel to a nonzero vector **v**:



A unique line L passes through P_0 and is parallel to **v**.

Figure 12.5.1

12.5.1 THEOREM.

(a) *The line in 2-space that passes through the point $P_0(x_0, y_0)$ and is parallel to the nonzero vector $\mathbf{v} = \langle a, b \rangle = a\mathbf{i} + b\mathbf{j}$ has parametric equations*

$$x = x_0 + at, \quad y = y_0 + bt \tag{1}$$

(b) *The line in 3-space that passes through the point $P_0(x_0, y_0, z_0)$ and is parallel to the nonzero vector $\mathbf{v} = \langle a, b, c \rangle = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ has parametric equations*

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct \tag{2}$$

We will prove part (b). The proof of (a) is similar.

Proof (b). If L is the line in 3-space that passes through the point $P_0(x_0, y_0, z_0)$ and is parallel to the nonzero vector $\mathbf{v} = \langle a, b, c \rangle$, then L consists precisely of those points $P(x, y, z)$ for which the vector $\overrightarrow{P_0P}$ is parallel to **v** (Figure 12.5.2). In other words, the point $P(x, y, z)$ is on L if and only if $\overrightarrow{P_0P}$ is a scalar multiple of **v**, say

$$\overrightarrow{P_0P} = t\mathbf{v}$$

This equation can be written as

$$\langle x - x_0, y - y_0, z - z_0 \rangle = \langle ta, tb, tc \rangle$$

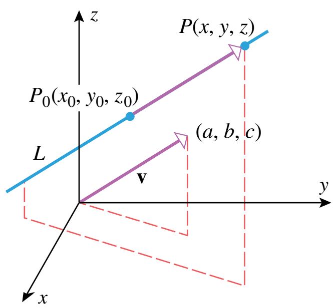


Figure 12.5.2

which implies that

$$x - x_0 = ta, \quad y - y_0 = tb, \quad z - z_0 = tc$$

from which (2) follows.

REMARK. Although it is not stated explicitly, it is understood in Equations (1) and (2) that $-\infty < t < +\infty$, which reflects the fact that lines extend indefinitely.

Example 1 Find parametric equations of the line

(a) passing through (4, 2) and parallel to $\mathbf{v} = \langle -1, 5 \rangle$;

(b) passing through (1, 2, −3) and parallel to $\mathbf{v} = 4\mathbf{i} + 5\mathbf{j} - 7\mathbf{k}$;

(c) passing through the origin in 3-space and parallel to $\mathbf{v} = \langle 1, 1, 1 \rangle$.

Solution (a). From (1) with $x_0 = 4$, $y_0 = 2$, $a = -1$, and $b = 5$ we obtain

$$x = 4 - t, \quad y = 2 + 5t$$

Solution (b). From (2) we obtain

$$x = 1 + 4t, \quad y = 2 + 5t, \quad z = -3 - 7t$$

Solution (c). From (2) with $x_0 = 0$, $y_0 = 0$, $z_0 = 0$, $a = 1$, $b = 1$, and $c = 1$ we obtain

$$x = t, \quad y = t, \quad z = t$$ ◀

Example 2

(a) Find parametric equations of the line L passing through the points $P_1(2, 4, -1)$ and $P_2(5, 0, 7)$.

(b) Where does the line intersect the xy-plane?

Solution (a). The vector $\overrightarrow{P_1P_2} = \langle 3, -4, 8 \rangle$ is parallel to L and the point $P_1(2, 4, -1)$ lies on L, so it follows from (2) that L has parametric equations

$$x = 2 + 3t, \quad y = 4 - 4t, \quad z = -1 + 8t \tag{3}$$

Had we used P_2 as the point on L rather than P_1, we would have obtained the equations

$$x = 5 + 3t, \quad y = -4t, \quad z = 7 + 8t$$

Although these equations look different from those obtained using P_1, the two sets of equations are actually equivalent in that both generate L as t varies from $-\infty$ to $+\infty$. To see this, note that if t_1 gives a point

$$(x, y, z) = (2 + 3t_1, 4 - 4t_1, -1 + 8t_1)$$

on L using the first set of equations, then $t_2 = t_1 - 1$ gives the *same* point

$$\begin{aligned}(x, y, z) &= (5 + 3t_2, -4t_2, 7 + 8t_2)\\ &= (5 + 3(t_1 - 1), -4(t_1 - 1), 7 + 8(t_1 - 1))\\ &= (2 + 3t_1, 4 - 4t_1, -1 + 8t_1)\end{aligned}$$

on L using the second set of equations. Conversely, if t_2 gives a point on L using the second set of equations, then $t_1 = t_2 + 1$ gives the same point using the first set.

Solution (b). It follows from (3) in part (a) that the line intersects the xy-plane at the point where $z = -1 + 8t = 0$, that is, when $t = \frac{1}{8}$. Substituting this value of t in (3) yields the point of intersection $(x, y, z) = \left(\frac{19}{8}, \frac{7}{2}, 0\right)$. ◀

Example 3 Let L_1 and L_2 be the lines

$$L_1\colon x = 1 + 4t, \quad y = 5 - 4t, \quad z = -1 + 5t$$

$$L_2\colon x = 2 + 8t, \quad y = 4 - 3t, \quad z = 5 + t$$

(a) Are the lines parallel?

(b) Do the lines intersect?

Solution (a). The line L_1 is parallel to the vector $4\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$, and the line L_2 is parallel to the vector $8\mathbf{i} - 3\mathbf{j} + \mathbf{k}$. These vectors are not parallel since neither is a scalar multiple of the other. Thus, the lines are not parallel.

Solution (b). For L_1 and L_2 to intersect at some point (x_0, y_0, z_0) these coordinates would have to satisfy the equations of both lines. In other words, there would have to exist values t_1 and t_2 for the parameters such that

$$x_0 = 1 + 4t_1, \quad y_0 = 5 - 4t_1, \quad z_0 = -1 + 5t_1$$

and

$$x_0 = 2 + 8t_2, \quad y_0 = 4 - 3t_2, \quad z_0 = 5 + t_2$$

This leads to three conditions on t_1 and t_2,

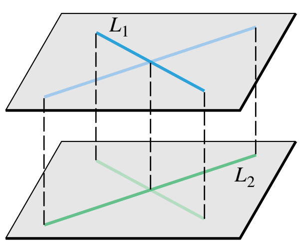
$$\begin{aligned} 1 + 4t_1 &= 2 + 8t_2 \\ 5 - 4t_1 &= 4 - 3t_2 \\ -1 + 5t_1 &= 5 + t_2 \end{aligned} \tag{4}$$

Thus, the lines intersect if there are values of t_1 and t_2 that satisfy all three equations, and the lines do not intersect if there are no such values. You should be familiar with methods for solving systems of two linear equations in two unknowns; however, this is a system of three linear equations in two unknowns. To determine whether this system has a solution we will solve the first two equations for t_1 and t_2 and then check whether these values satisfy the third equation.

We will solve the first two equations by the method of elimination. We can eliminate the unknown t_1 by adding the equations. This yields the equation

$$6 = 6 + 5t_2$$

from which we obtain $t_2 = 0$. We can now find t_1 by substituting this value of t_2 in either the first or second equation. This yields $t_1 = \frac{1}{4}$. However, the values $t_1 = \frac{1}{4}$ and $t_2 = 0$ do not satisfy the third equation in (4), so the lines do not intersect. ◀



Parallel planes containing skew lines L_1 and L_2 can be determined by translating each line until it intersects the other.

Figure 12.5.3

Two lines in 3-space that are not parallel and do not intersect (such as those in Example 3) are called ***skew*** lines. As illustrated in Figure 12.5.3, any two skew lines lie in parallel planes.

LINE SEGMENTS

Sometimes one is not interested in an entire line, but rather some *segment* of a line. Parametric equations of a line segment can be obtained by finding parametric equations for the entire line, then restricting the parameter appropriately so that only the desired segment is generated.

Example 4 Find parametric equations for the line segment that joins the points $P_1(2, 4, -1)$ and $P_2(5, 0, 7)$.

Solution. From Example 2, the line through the points P_1 and P_2 has parametric equations $x = 2 + 3t$, $y = 4 - 4t$, $z = -1 + 8t$. With these equations, the point P_1 corresponds to $t = 0$ and P_2 to $t = 1$. Thus, the line segment that joins P_1 and P_2 is given by

$$x = 2 + 3t, \quad y = 4 - 4t, \quad z = -1 + 8t \qquad (0 \le t \le 1)$$ ◀

VECTOR EQUATIONS OF LINES

We will now show how vector notation can be used to express the parametric equations of a line more compactly. Because two vectors are equal if and only if their components are equal, (1) and (2) can be written in vector form as

$$\begin{aligned} \langle x, y \rangle &= \langle x_0 + at, y_0 + bt \rangle \\ \langle x, y, z \rangle &= \langle x_0 + at, y_0 + bt, z_0 + ct \rangle \end{aligned}$$

or, equivalently, as

$$\langle x, y\rangle = \langle x_0, y_0\rangle + t\langle a, b\rangle \tag{5}$$

$$\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle \tag{6}$$

For the equation in 2-space we define the vectors $\mathbf{r}$, $\mathbf{r}_0$, and $\mathbf{v}$ as

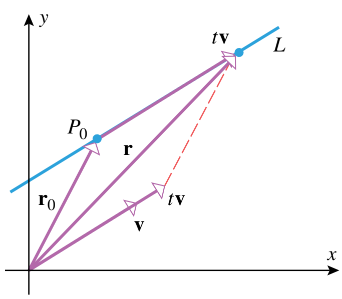
$$\mathbf{r} = \langle x, y\rangle, \quad \mathbf{r}_0 = \langle x_0, y_0\rangle, \quad \mathbf{v} = \langle a, b\rangle \tag{7}$$

and for the equation in 3-space we define them as

$$\mathbf{r} = \langle x, y, z\rangle, \quad \mathbf{r}_0 = \langle x_0, y_0, z_0\rangle, \quad \mathbf{v} = \langle a, b, c\rangle \tag{8}$$

Substituting (7) and (8) in (5) and (6), respectively, yields the equation

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v} \tag{9}$$



$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$

Figure 12.5.4

in both cases. We call this the ***vector equation of a line*** in 2-space or 3-space. In this equation, $\mathbf{v}$ is a nonzero vector parallel to the line, and $\mathbf{r}_0$ is a vector whose components are the coordinates of a point on the line.

We can interpret Equation (9) geometrically by positioning the vectors $\mathbf{r}_0$ and $\mathbf{v}$ with their initial points at the origin and the vector $t\mathbf{v}$ with its initial point at P_0 (Figure 12.5.4). The vector $t\mathbf{v}$ is a scalar multiple of $\mathbf{v}$ and hence is parallel to $\mathbf{v}$ and L. Moreover, since the initial point of $t\mathbf{v}$ is at the point P_0 on L, this vector actually runs along L; hence, the vector $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$ can be interpreted as the vector from the origin to a point on L. As the parameter t varies from 0 to $+\infty$, the terminal point of $\mathbf{r}$ traces out the portion of L that extends from P_0 in the direction of $\mathbf{v}$, and as t varies from 0 to $-\infty$, the terminal point of $\mathbf{r}$ traces out the portion of L that extends from P_0 in the direction that is opposite to $\mathbf{v}$. Thus, the entire line is traced as t varies over the interval $(-\infty, +\infty)$, and it is traced in the direction of $\mathbf{v}$ as t increases.

Example 5 The equation

$$\langle x, y, z\rangle = \langle -1, 0, 2\rangle + t\langle 1, 5, -4\rangle$$

is of form (9) with

$$\mathbf{r}_0 = \langle -1, 0, 2\rangle \quad \text{and} \quad \mathbf{v} = \langle 1, 5, -4\rangle$$

Thus, the equation represents the line in 3-space that passes through the point $(-1, 0, 2)$ and is parallel to the vector $\langle 1, 5, -4\rangle$. ◀

Example 6 Find an equation of the line in 3-space that passes through the points $P_1(2, 4, -1)$ and $P_2(5, 0, 7)$.

Solution. The vector

$$\overrightarrow{P_1P_2} = \langle 3, -4, 8\rangle$$

is parallel to the line, so it can be used as $\mathbf{v}$ in (9). For $\mathbf{r}_0$ we can use either the vector from the origin to P_1 or the vector from the origin to P_2. Using the former yields

$$\mathbf{r}_0 = \langle 2, 4, -1\rangle$$

Thus, a vector equation of the line through P_1 and P_2 is

$$\langle x, y, z\rangle = \langle 2, 4, -1\rangle + t\langle 3, -4, 8\rangle$$

If needed, we can express the line parametrically by equating corresponding components on the two sides of this vector equation, in which case we obtain the parametric equations in Example 2 (verify). ◀

EXERCISE SET 12.5 Graphing Utility C CAS

1. (a) Find parametric equations for the lines through the corner of the unit square shown in part (*a*) of the accompanying figure.
(b) Find parametric equations for the lines through the corner of the unit cube shown in part (*b*) of the accompanying figure.

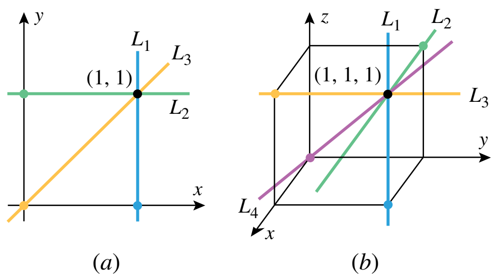


Figure Ex-1

2. (a) Find parametric equations for the line segments on the unit square in part (*a*) of the accompanying figure.
(b) Find parametric equations for the line segments in the unit cube shown in part (*b*) of the accompanying figure.

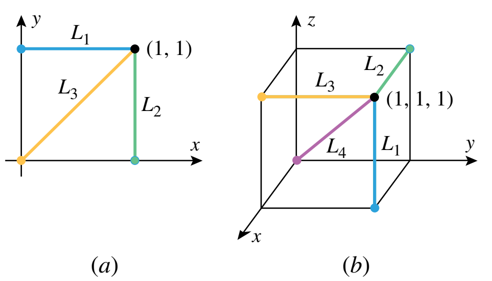


Figure Ex-2

In Exercises 3 and 4, find parametric equations for the line through P_1 and P_2 and also for the line segment joining those points.

3. (a) $P_1(3, -2)$, $P_2(5, 1)$ (b) $P_1(5, -2, 1)$, $P_2(2, 4, 2)$

4. (a) $P_1(0, 1)$, $P_2(-3, -4)$ (b) $P_1(-1, 3, 5)$, $P_2(-1, 3, 2)$

In Exercises 5 and 6, find parametric equations for the line whose vector equation is given.

5. (a) $\langle x, y\rangle = \langle 2, -3\rangle + t\langle 1, -4\rangle$
(b) $x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = \mathbf{k} + t(\mathbf{i} - \mathbf{j} + \mathbf{k})$

6. (a) $x\mathbf{i} + y\mathbf{j} = (3\mathbf{i} - 4\mathbf{j}) + t(2\mathbf{i} + \mathbf{j})$
(b) $\langle x, y, z\rangle = \langle -1, 0, 2\rangle + t\langle -1, 3, 0\rangle$

In Exercises 7 and 8, find a point P on the line and a vector $\mathbf{v}$ parallel to the line by inspection.

7. (a) $x\mathbf{i} + y\mathbf{j} = (2\mathbf{i} - \mathbf{j}) + t(4\mathbf{i} - \mathbf{j})$
(b) $\langle x, y, z\rangle = \langle -1, 2, 4\rangle + t\langle 5, 7, -8\rangle$

8. (a) $\langle x, y\rangle = \langle -1, 5\rangle + t\langle 2, 3\rangle$
(b) $x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (\mathbf{i} + \mathbf{j} - 2\mathbf{k}) + t\mathbf{j}$

In Exercises 9 and 10, express the given parametric equations of a line in vector form using bracket notation and also using **i**, **j**, **k** notation.

9. (a) $x = -3 + t,\ y = 4 + 5t$
(b) $x = 2 - t,\ y = -3 + 5t,\ z = t$

10. (a) $x = t,\ y = -2 + t$
(b) $x = 1 + t,\ y = -7 + 3t,\ z = 4 - 5t$

In Exercises 11–18, find parametric equations of the line that satisfies the stated conditions.

11. The line through $(-5, 2)$ that is parallel to $2\mathbf{i} - 3\mathbf{j}$.

12. The line through $(0, 3)$ that is parallel to the line $x = -5+t$, $y = 1 - 2t$.

13. The line that is tangent to the circle $x^2 + y^2 = 25$ at the point $(3, -4)$.

14. The line that is tangent to the parabola $y = x^2$ at the point $(-2, 4)$.

15. The line through $(-1, 2, 4)$ that is parallel to $3\mathbf{i} - 4\mathbf{j} + \mathbf{k}$.

16. The line through $(2, -1, 5)$ that is parallel to $\langle -1, 2, 7\rangle$.

17. The line through $(-2, 0, 5)$ that is parallel to the line $x = 1 + 2t$, $y = 4 - t$, $z = 6 + 2t$.

18. The line through the origin that is parallel to the line $x = t$, $y = -1 + t$, $z = 2$.

19. Where does the line $x = 1 + 3t$, $y = 2 - t$ intersect
(a) the x-axis (b) the y-axis (c) the parabola $y = x^2$?

20. Where does the line $\langle x, y\rangle = \langle 4t, 3t\rangle$ intersect the circle $x^2 + y^2 = 25$?

In Exercises 21 and 22, find the intersections of the lines with the xy-plane, the xz-plane, and the yz-plane.

21. $x = -2,\ y = 4 + 2t,\ z = -3 + t$

22. $x = -1 + 2t,\ y = 3 + t,\ z = 4 - t$

23. Where does the line $x = 1 + t$, $y = 3 - t$, $z = 2t$ intersect the cylinder $x^2 + y^2 = 16$?

24. Where does the line $x = 2 - t$, $y = 3t$, $z = -1 + 2t$ intersect the plane $2y + 3z = 6$?

In Exercises 25 and 26, show that the lines L_1 and L_2 intersect, and find their point of intersection.

25. L_1: $x = 2 + t,\ y = 2 + 3t,\ z = 3 + t$
L_2: $x = 2 + t,\ y = 3 + 4t,\ z = 4 + 2t$

26. $L_1: x+1=4t,\ y-3=t,\ z-1=0$
$L_2: x+13=12t,\ y-1=6t,\ z-2=3t$

In Exercises 27 and 28, show that the lines L_1 and L_2 are skew.

27. $L_1: x=1+7t,\ y=3+t,\ z=5-3t$
$L_2: x=4-t,\ y=6,\ z=7+2t$

28. $L_1: x=2+8t,\ y=6-8t,\ z=10t$
$L_2: x=3+8t,\ y=5-3t,\ z=6+t$

In Exercises 29 and 30, determine whether the lines L_1 and L_2 are parallel.

29. $L_1: x=3-2t,\ y=4+t,\ z=6-t$
$L_2: x=5-4t,\ y=-2+2t,\ z=7-2t$

30. $L_1: x=5+3t,\ y=4-2t,\ z=-2+3t$
$L_2: x=-1+9t,\ y=5-6t,\ z=3+8t$

In Exercises 31 and 32, determine whether the points P_1, P_2, and P_3 lie on the same line.

31. $P_1(6,9,7),\ P_2(9,2,0),\ P_3(0,-5,-3)$

32. $P_1(1,0,1),\ P_2(3,-4,-3),\ P_3(4,-6,-5)$

In Exercises 33 and 34, show that the lines L_1 and L_2 are the same.

33. $L_1: x=3-t,\ y=1+2t$
$L_2: x=-1+3t,\ y=9-6t$

34. $L_1: x=1+3t,\ y=-2+t,\ z=2t$
$L_2: x=4-6t,\ y=-1-2t,\ z=2-4t$

In Exercises 35 and 36, describe the line segment represented by the vector equation.

35. $\langle x,y\rangle=\langle 1,0\rangle+t\langle -2,3\rangle \quad (0\le t\le 2)$

36. $\langle x,y,z\rangle=\langle -2,1,4\rangle+t\langle 3,0,-1\rangle \quad (0\le t\le 3)$

In Exercises 37 and 38, use the method in Exercise 25 of Section 12.3 to find the distance from the point P to the line L, and then check your answer using the method in Exercise 26 of Section 12.4.

37. $P(-2,1,1)$
$L: x=3-t,\ y=t,\ z=1+2t$

38. $P(1,4,-3)$
$L: x=2+t,\ y=-1-t,\ z=3t$

In Exercises 39 and 40, show that the lines L_1 and L_2 are parallel, and find the distance between them.

39. $L_1: x=2-t,\ y=2t,\ z=1+t$
$L_2: x=1+2t,\ y=3-4t,\ z=5-2t$

40. $L_1: x=2t,\ y=3+4t,\ z=2-6t$
$L_2: x=1+3t,\ y=6t,\ z=-9t$

41. (a) Find parametric equations for the line through the points (x_0, y_0, z_0) and (x_1, y_1, z_1).
(b) Find parametric equations for the line through the point (x_1, y_1, z_1) and parallel to the line

$$x=x_0+at, \quad y=y_0+bt, \quad z=z_0+ct$$

42. Let L be the line that passes through the point (x_0, y_0, z_0) and is parallel to the vector $\mathbf{v}=\langle a,b,c\rangle$, where a, b, and c are nonzero. Show that a point (x,y,z) lies on the line L if and only if

$$\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$$

These equations, which are called the ***symmetric equations*** of L, provide a nonparametric representation of L.

43. (a) Describe the line whose symmetric equations are

$$\frac{x-1}{2}=\frac{y+3}{4}=z-5$$

[See Exercise 42.]
(b) Find parametric equations for the line in part (a).

44. Find the point on the line segment joining $P_1(1,4,-3)$ and $P_2(1,5,-1)$ that is $\frac{2}{3}$ of the way from P_1 to P_2.

45. Let L_1 and L_2 be the lines whose parametric equations are

$$L_1: x=1+2t, \quad y=2-t, \quad z=4-2t$$
$$L_2: x=9+t, \quad y=5+3t, \quad z=-4-t$$

(a) Show that L_1 and L_2 intersect at the point $(7,-1,-2)$.
(b) Find, to the nearest degree, the acute angle between L_1 and L_2 at their intersection.
(c) Find parametric equations for the line that is perpendicular to L_1 and L_2 and passes through their point of intersection.

46. Let L_1 and L_2 be the lines whose parametric equations are

$$L_1: x=4t, \quad y=1-2t, \quad z=2+2t$$
$$L_2: x=1+t, \quad y=1-t, \quad z=-1+4t$$

(a) Show that L_1 and L_2 intersect at the point $(2,0,3)$.
(b) Find, to the nearest degree, the acute angle between L_1 and L_2 at their intersection.
(c) Find parametric equations for the line that is perpendicular to L_1 and L_2 and passes through their point of intersection.

In Exercises 47 and 48, find parametric equations of the line that contains the point P and intersects the line L at a right angle.

47. $P(0,2,1)$
$L: x=2t,\ y=1-t,\ z=2+t$

48. $P(3,1,-2)$
$L: x=-2+2t,\ y=4+2t,\ z=2+t$

49. Two bugs are walking along lines in 3-space. At time t bug 1 is at the point (x, y, z) on the line

$$x = 4 - t, \quad y = 1 + 2t, \quad z = 2 + t$$

and at the same time t bug 2 is at the point (x, y, z) on the line

$$x = t, \quad y = 1 + t, \quad z = 1 + 2t$$

Assume that distance is in centimeters and that time is in minutes.

(a) Find the distance between the bugs at time $t = 0$.

(b) Use a graphing utility to graph the distance between the bugs as a function of time from $t = 0$ to $t = 5$.

(c) What does the graph tell you about the distance between the bugs?

(d) How close do the bugs get?

50. Suppose that the temperature T at a point (x, y, z) on the line $x = t$, $y = 1 + t$, $z = 3 - 2t$ is $T = 25x^2yz$. Use a CAS or a calculating utility with a root-finding capability to approximate the maximum temperature on that portion of the line that extends from the xz-plane to the xy-plane.

12.6 PLANES IN 3-SPACE

In this section we will use vectors to derive equations of planes in 3-space, and then we will use these equations to solve various geometric problems.

PLANES PARALLEL TO THE COORDINATE PLANES

The graph of the equation $x = a$ in an xyz-coordinate system consists of all points of the form (a, y, z), where y and z are arbitrary. One such point is $(a, 0, 0)$, and all others are in the plane that passes through this point and is parallel to the yz-plane (Figure 12.6.1). Similarly, the graph of $y = b$ is the plane through $(0, b, 0)$ that is parallel to the xz-plane, and the graph of $z = c$ is the plane through $(0, 0, c)$ that is parallel to the xy-plane.

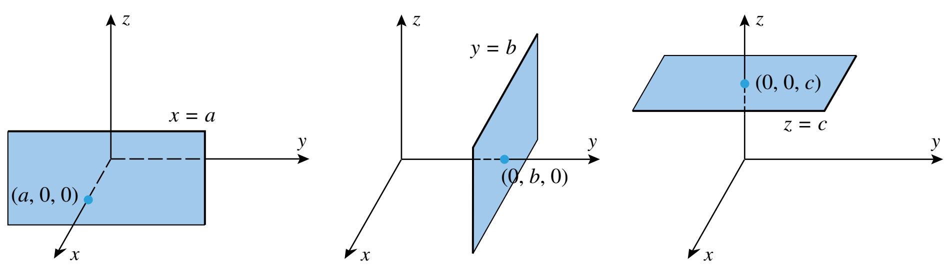


Figure 12.6.1

PLANES DETERMINED BY A POINT AND A NORMAL VECTOR

A plane in 3-space can be determined uniquely by specifying a point in the plane and a vector perpendicular to the plane (Figure 12.6.2). A vector perpendicular to a plane is called a ***normal*** to the plane.

Suppose that we want to find an equation of the plane passing through $P_0(x_0, y_0, z_0)$ and perpendicular to the vector $\mathbf{n} = \langle a, b, c \rangle$. Define the vectors $\mathbf{r}_0$ and $\mathbf{r}$ as

$$\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle \quad \text{and} \quad \mathbf{r} = \langle x, y, z \rangle$$

It should be evident from Figure 12.6.3 that the plane consists precisely of those points $P(x, y, z)$ for which the vector $\mathbf{r} - \mathbf{r}_0$ is orthogonal to $\mathbf{n}$; or, expressed as an equation,

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0 \tag{1}$$

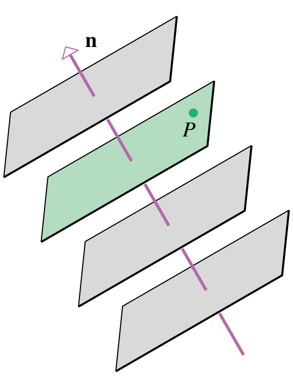
If preferred, we can express this vector equation in terms of components as

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0 \tag{2}$$

from which we obtain

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \tag{3}$$

This is called the ***point-normal form*** of the equation of a plane. Formulas (1) and (2) are vector versions of this formula.



The colored plane is uniquely determined by the point P and the vector $\mathbf{n}$ perpendicular to the plane.

Figure 12.6.2

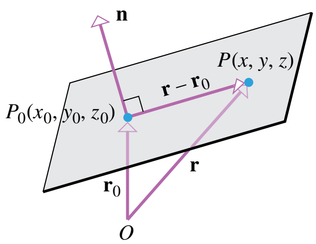


Figure 12.6.3

FOR THE READER. What does Equation (1) represent if $\mathbf{n} = \langle a, b\rangle$, $\mathbf{r}_0 = \langle x_0, y_0\rangle$, and $\mathbf{r} = \langle x, y\rangle$ are vectors in an *xy*-plane in 2-space? Draw a picture.

Example 1 Find an equation of the plane passing through the point $(3, -1, 7)$ and perpendicular to the vector $\mathbf{n} = \langle 4, 2, -5\rangle$.

Solution. From (3), a point-normal form of the equation is

$$4(x-3) + 2(y+1) - 5(z-7) = 0 \tag{4}$$

If preferred, this equation can be written in vector form as

$$\langle 4, 2, -5\rangle \cdot \langle x-3, y+1, z-7\rangle = 0$$ ◀

Observe that if we multiply out the terms in (3) and simplify, we obtain an equation of the form

$$ax + by + cz + d = 0 \tag{5}$$

For example, Equation (4) in Example 1 can be rewritten as

$$4x + 2y - 5z + 25 = 0$$

The following theorem shows that every equation of form (5) represents a plane in 3-space.

12.6.1 THEOREM. *If a, b, c, and d are constants, and a, b, and c are not all zero, then the graph of the equation*

$$ax + by + cz + d = 0 \tag{6}$$

is a plane that has the vector $\mathbf{n} = \langle a, b, c\rangle$ as a normal.

Proof. Since a, b, and c are not all zero, there is at least one point (x_0, y_0, z_0) whose coordinates satisfy Equation (6). For example, if $a \neq 0$, then such a point is $(-d/a, 0, 0)$, and similarly if $b \neq 0$ or $c \neq 0$ (verify). Thus, let (x_0, y_0, z_0) be any point whose coordinates satisfy (6); that is,

$$ax_0 + by_0 + cz_0 + d = 0$$

Subtracting this equation from (6) yields

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

which is the point-normal form of a plane with normal $\mathbf{n} = \langle a, b, c\rangle$. ∎

Equation (6) is called the ***general form*** of the equation of a plane.

Example 2 Determine whether the planes

$$3x - 4y + 5z = 0 \quad \text{and} \quad -6x + 8y - 10z - 4 = 0$$

are parallel.

Solution. It is clear geometrically that two planes are parallel if and only if their normals are parallel vectors. A normal to the first plane is

$$\mathbf{n}_1 = \langle 3, -4, 5\rangle$$

and a normal to the second plane is

$$\mathbf{n}_2 = \langle -6, 8, -10\rangle$$

Since $\mathbf{n}_2$ is a scalar multiple of $\mathbf{n}_1$, the normals are parallel, and hence so are the planes. ◀

We have seen that a unique plane is determined by a point in the plane and a nonzero vector normal to the plane. In contrast, a unique plane is not determined by a point in the plane and a nonzero vector *parallel* to the plane (Figure 12.6.4). However, a unique plane is determined by a point in the plane and two nonparallel vectors that are parallel to the plane (Figure 12.6.5). A unique plane is also determined by three noncollinear points that lie in the plane (Figure 12.6.6).

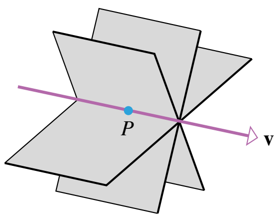


Figure 12.6.4

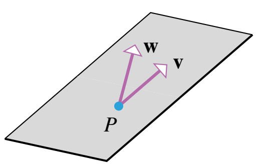


Figure 12.6.5

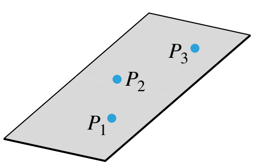


Figure 12.6.6

Example 3 Find an equation of the plane through the points $P_1(1, 2, -1)$, $P_2(2, 3, 1)$, and $P_3(3, -1, 2)$.

Solution. Since the points P_1, P_2, and P_3 lie in the plane, the vectors $\overrightarrow{P_1P_2} = \langle 1, 1, 2\rangle$ and $\overrightarrow{P_1P_3} = \langle 2, -3, 3\rangle$ are parallel to the plane. Therefore,

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 2 \\ 2 & -3 & 3 \end{vmatrix} = 9\mathbf{i} + \mathbf{j} - 5\mathbf{k}$$

is normal to the plane, since it is orthogonal to both $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$. By using this normal and the point $P_1(1, 2, -1)$ in the plane, we obtain the point-normal form

$$9(x-1) + (y-2) - 5(z+1) = 0$$

which can be rewritten as

$$9x + y - 5z - 16 = 0$$ ◀

Example 4 Determine whether the line

$$x = 3 + 8t, \quad y = 4 + 5t, \quad z = -3 - t$$

is parallel to the plane $x - 3y + 5z = 12$.

Solution. The vector $\mathbf{v} = \langle 8, 5, -1\rangle$ is parallel to the line and the vector $\mathbf{n} = \langle 1, -3, 5\rangle$ is normal to the plane. For the line and plane to be parallel, the vectors $\mathbf{v}$ and $\mathbf{n}$ must be orthogonal. But this is not so, since the dot product

$$\mathbf{v} \cdot \mathbf{n} = (8)(1) + (5)(-3) + (-1)(5) = -12$$

is nonzero. Thus, the line and plane are not parallel. ◀

Example 5 Find the intersection of the line and plane in Example 4.

Solution. If we let (x_0, y_0, z_0) be the point of intersection, then the coordinates of this point satisfy both the equation of the plane and the parametric equations of the line. Thus,

$$x_0 - 3y_0 + 5z_0 = 12 \tag{7}$$

and for some value of t, say $t = t_0$,

$$x_0 = 3 + 8t_0, \quad y_0 = 4 + 5t_0, \quad z_0 = -3 - t_0 \tag{8}$$

Substituting (8) in (7) yields

$$(3 + 8t_0) - 3(4 + 5t_0) + 5(-3 - t_0) = 12$$

Solving for t_0 yields $t_0 = -3$ and on substituting this value in (8), we obtain

$$(x_0, y_0, z_0) = (-21, -11, 0)$$ ◀

ANGLES BETWEEN PLANES

Two distinct intersecting planes determine two positive angles of intersection—an (acute) angle θ that satisfies the condition $0 \le \theta \le \pi/2$ and the supplement of that angle (Figure 12.6.7*a*). If $\mathbf{n}_1$ and $\mathbf{n}_2$ are normals to the planes, then depending on the directions of $\mathbf{n}_1$

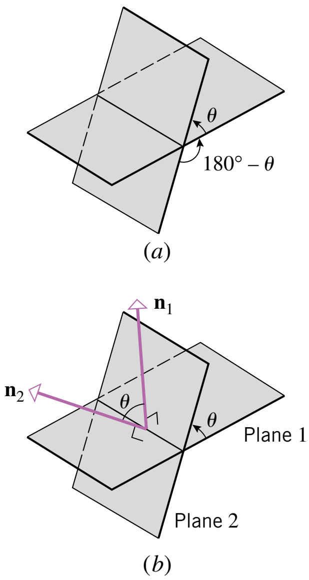


Figure 12.6.7

and $\mathbf{n}_2$, the angle θ is either the angle between $\mathbf{n}_1$ and $\mathbf{n}_2$ or the angle between $\mathbf{n}_1$ and $-\mathbf{n}_2$ (Figure 12.6.7*b*). In both cases, Theorem 12.3.3 yields the following formula for the acute angle θ between the planes:

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|} \tag{9}$$

Example 6 Find the acute angle of intersection between the two planes

$$2x - 4y + 4z = 7 \quad \text{and} \quad 6x + 2y - 3z = 2$$

Solution. The given equations yield the normals $\mathbf{n}_1 = \langle 2, -4, 4\rangle$ and $\mathbf{n}_2 = \langle 6, 2, -3\rangle$. Thus, Formula (9) yields

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|} = \frac{|-8|}{\sqrt{36}\sqrt{49}} = \frac{4}{21}$$

from which we obtain

$$\theta = \cos^{-1}\left(\frac{4}{21}\right) \approx 79^\circ$$

◀

DISTANCE PROBLEMS INVOLVING PLANES

Next we will consider three basic "distance problems" in 3-space:

- Find the distance between a point and a plane.
- Find the distance between two parallel planes.
- Find the distance between two skew lines.

The three problems are related. If we can find the distance between a point and a plane, then we can find the distance between parallel planes by computing the distance between one of the planes and an arbitrary point P_0 in the other plane (Figure 12.6.8*a*). Moreover, we can find the distance between two skew lines by computing the distance between parallel planes containing them (Figure 12.6.8*b*).

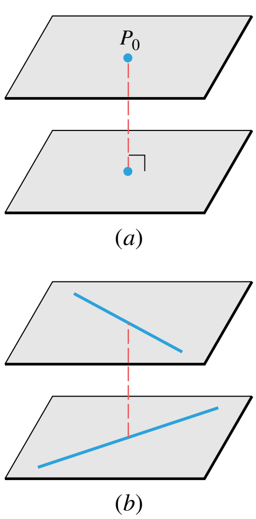


Figure 12.6.8

12.6.2 THEOREM. *The distance D between a point $P_0(x_0, y_0, z_0)$ and the plane $ax + by + cz + d = 0$ is*

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}} \tag{10}$$

Proof. Let $Q(x_1, y_1, z_1)$ be any point in the plane, and position the normal $\mathbf{n} = \langle a, b, c\rangle$ so that its initial point is at Q. As illustrated in Figure 12.6.9, the distance D is equal to the length of the orthogonal projection of $\overrightarrow{QP_0}$ on $\mathbf{n}$. Thus, from (12) of Section 12.3,

$$D = \|\text{proj}_{\mathbf{n}}\overrightarrow{QP_0}\| = \left\|\frac{\overrightarrow{QP_0} \cdot \mathbf{n}}{\|\mathbf{n}\|^2}\mathbf{n}\right\| = \frac{|\overrightarrow{QP_0} \cdot \mathbf{n}|}{\|\mathbf{n}\|^2}\|\mathbf{n}\| = \frac{|\overrightarrow{QP_0} \cdot \mathbf{n}|}{\|\mathbf{n}\|}$$

But

$$\overrightarrow{QP_0} = \langle x_0 - x_1, y_0 - y_1, z_0 - z_1\rangle$$
$$\overrightarrow{QP_0} \cdot \mathbf{n} = a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1)$$
$$\|\mathbf{n}\| = \sqrt{a^2 + b^2 + c^2}$$

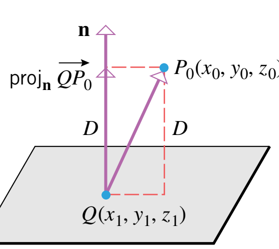


Figure 12.6.9

Thus,

$$D = \frac{|a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1)|}{\sqrt{a^2 + b^2 + c^2}} \tag{11}$$

Since the point $Q(x_1, y_1, z_1)$ lies in the plane, its coordinates satisfy the equation of the plane; that is,

$$ax_1 + by_1 + cz_1 + d = 0$$

or

$$d = -ax_1 - by_1 - cz_1$$

Combining this expression with (11) yields (10). ▮

Example 7 Find the distance D between the point $(1, -4, -3)$ and the plane

$$2x - 3y + 6z = -1$$

Solution. Formula (10) requires the plane to be rewritten in the form $ax+by+cz+d = 0$. Thus, we rewrite the equation of the given plane as

$$2x - 3y + 6z + 1 = 0$$

from which we obtain $a = 2$, $b = -3$, $c = 6$, and $d = 1$. Substituting these values and the coordinates of the given point in (10), we obtain

$$D = \frac{|(2)(1) + (-3)(-4) + 6(-3) + 1|}{\sqrt{2^2 + (-3)^2 + 6^2}} = \frac{|-3|}{7} = \frac{3}{7}$$ ◀

REMARK. See Exercise 48 for an analog of Formula (10) in 2-space that can be used to compute the distance between a point and a line.

Example 8 The planes

$$x + 2y - 2z = 3 \quad \text{and} \quad 2x + 4y - 4z = 7$$

are parallel since their normals, $\langle 1, 2, -2\rangle$ and $\langle 2, 4, -4\rangle$, are parallel vectors. Find the distance between these planes.

Solution. To find the distance D between the planes, we can select an arbitrary point in one of the planes and compute its distance to the other plane. By setting $y = z = 0$ in the equation $x + 2y - 2z = 3$, we obtain the point $P_0(3, 0, 0)$ in this plane. From (10), the distance from P_0 to the plane $2x + 4y - 4z = 7$ is

$$D = \frac{|(2)(3) + 4(0) + (-4)(0) - 7|}{\sqrt{2^2 + 4^2 + (-4)^2}} = \frac{1}{6}$$ ◀

Example 9 It was shown in Example 3 of Section 12.5 that the lines

$$L_1\colon x = 1 + 4t, \quad y = 5 - 4t, \quad z = -1 + 5t$$
$$L_2\colon x = 2 + 8t, \quad y = 4 - 3t, \quad z = 5 + t$$

are skew. Find the distance between them.

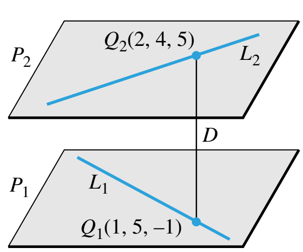


Figure 12.6.10

Solution. Let P_1 and P_2 denote parallel planes containing L_1 and L_2, respectively (Figure 12.6.10). To find the distance D between L_1 and L_2, we will calculate the distance from a point in P_1 to the plane P_2. Since L_1 lies in plane P_1, we can find a point in P_1 by finding a point on the line L_1; we can do this by substituting any convenient value of t in the parametric equations of L_1. The simplest choice is $t = 0$, which yields the point $Q_1(1, 5, -1)$.

The next step is to find an equation for the plane P_2. For this purpose, observe that the vector $\mathbf{u}_1 = \langle 4, -4, 5\rangle$ is parallel to line L_1, and therefore also parallel to planes P_1 and P_2. Similarly, $\mathbf{u}_2 = \langle 8, -3, 1\rangle$ is parallel to L_2 and hence parallel to P_1 and P_2. Therefore, the cross product

$$\mathbf{n} = \mathbf{u}_1 \times \mathbf{u}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -4 & 5 \\ 8 & -3 & 1 \end{vmatrix} = 11\mathbf{i} + 36\mathbf{j} + 20\mathbf{k}$$

is normal to both P_1 and P_2. Using this normal and the point $Q_2(2, 4, 5)$ found by setting $t = 0$ in the equations of L_2, we obtain an equation for P_2:

$$11(x-2) + 36(y-4) + 20(z-5) = 0$$

or

$$11x + 36y + 20z - 266 = 0$$

The distance between $Q_1(1, 5, -1)$ and this plane is

$$D = \frac{|(11)(1) + (36)(5) + (20)(-1) - 266|}{\sqrt{11^2 + 36^2 + 20^2}} = \frac{95}{\sqrt{1817}}$$

which is also the distance between L_1 and L_2. ◀

Exercise Set 12.6

1. Find equations of the planes P_1, P_2, and P_3 that are parallel to the coordinate planes and pass through the corner $(3, 4, 5)$ of the box shown in the accompanying figure.

2. Find equations of the planes P_1, P_2, and P_3 that are parallel to the coordinate planes and pass through the corner (x_0, y_0, z_0) of the box shown in the accompanying figure.

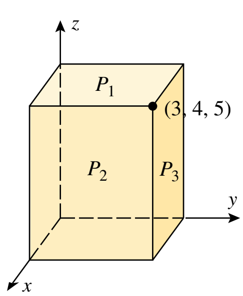


Figure Ex-1

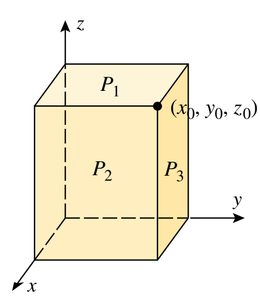


Figure Ex-2

In Exercises 3–6, find an equation of the plane that passes through the point P and has the vector $\mathbf{n}$ as a normal.

3. $P(2, 6, 1)$; $\mathbf{n} = \langle 1, 4, 2\rangle$

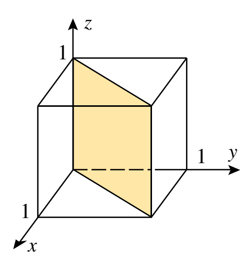
4. $P(-1, -1, 2)$; $\mathbf{n} = \langle -1, 7, 6\rangle$

5. $P(1, 0, 0)$; $\mathbf{n} = \langle 0, 0, 1\rangle$

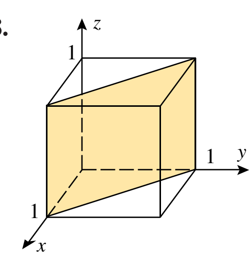
6. $P(0, 0, 0)$; $\mathbf{n} = \langle 2, -3, -4\rangle$

In Exercises 7–10, find an equation of the plane indicated in the figure.

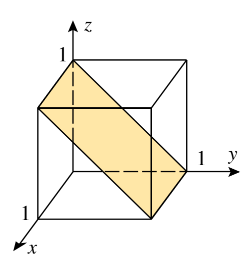
7.



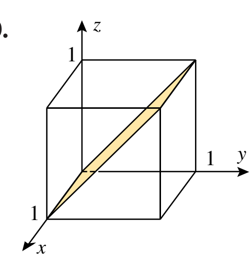
8.



9.



10.



In Exercises 11 and 12, find an equation of the plane that passes through the given points.

11. $(-2, 1, 1)$, $(0, 2, 3)$, and $(1, 0, -1)$

12. $(3, 2, 1)$, $(2, 1, -1)$, and $(-1, 3, 2)$

In Exercises 13 and 14, determine whether the planes are parallel, perpendicular, or neither.

13. (a) $2x - 8y - 6z - 2 = 0$
$-x + 4y + 3z - 5 = 0$
(b) $3x - 2y + z = 1$
$4x + 5y - 2z = 4$
(c) $x - y + 3z - 2 = 0$
$2x + z = 1$

14. (a) $3x - 2y + z = 4$
$6x - 4y + 3z = 7$
(b) $y = 4x - 2z + 3$
$x = \frac{1}{4}y + \frac{1}{2}z$
(c) $x + 4y + 7z = 3$
$5x - 3y + z = 0$

In Exercises 15 and 16, determine whether the line and plane are parallel, perpendicular, or neither.

15. (a) $x = 4 + 2t,\ y = -t,\ z = -1 - 4t;$
$3x + 2y + z - 7 = 0$
(b) $x = t,\ y = 2t,\ z = 3t;$
$x - y + 2z = 5$
(c) $x = -1 + 2t,\ y = 4 + t,\ z = 1 - t;$
$4x + 2y - 2z = 7$

16. (a) $x = 3 - t,\ y = 2 + t,\ z = 1 - 3t;$
$2x + 2y - 5 = 0$
(b) $x = 1 - 2t,\ y = t,\ z = -t;$
$6x - 3y + 3z = 1$
(c) $x = t,\ y = 1 - t,\ z = 2 + t;$
$x + y + z = 1$

In Exercises 17 and 18, determine whether the line and plane intersect; if so, find the coordinates of the intersection.

17. (a) $x = t,\ y = t,\ z = t;$
$3x - 2y + z - 5 = 0$
(b) $x = 2 - t,\ y = 3 + t,\ z = t;$
$2x + y + z = 1$

18. (a) $x = 3t,\ y = 5t,\ z = -t;$
$2x - y + z + 1 = 0$
(b) $x = 1 + t,\ y = -1 + 3t,\ z = 2 + 4t;$
$x - y + 4z = 7$

In Exercises 19 and 20, find the acute angle of intersection of the planes to the nearest degree.

19. $x = 0$ and $2x - y + z - 4 = 0$

20. $x + 2y - 2z = 5$ and $6x - 3y + 2z = 8$

In Exercises 21–30, find an equation of the plane that satisfies the stated conditions.

21. The plane through the origin that is parallel to the plane $4x - 2y + 7z + 12 = 0$.

22. The plane that contains the line $x = -2 + 3t$, $y = 4 + 2t$, $z = 3 - t$ and is perpendicular to the plane $x - 2y + z = 5$.

23. The plane through the point $(-1, 4, 2)$ that contains the line of intersection of the planes $4x - y + z - 2 = 0$ and $2x + y - 2z - 3 = 0$.

24. The plane through $(-1, 4, -3)$ that is perpendicular to the line $x - 2 = t$, $y + 3 = 2t$, $z = -t$.

25. The plane through $(1, 2, -1)$ that is perpendicular to the line of intersection of the planes $2x + y + z = 2$ and $x + 2y + z = 3$.

26. The plane through the points $P_1(-2, 1, 4)$, $P_2(1, 0, 3)$ that is perpendicular to the plane $4x - y + 3z = 2$.

27. The plane through $(-1, 2, -5)$ that is perpendicular to the planes $2x - y + z = 1$ and $x + y - 2z = 3$.

28. The plane that contains the point $(2, 0, 3)$ and the line $x = -1 + t$, $y = t$, $z = -4 + 2t$.

29. The plane whose points are equidistant from $(2, -1, 1)$ and $(3, 1, 5)$.

30. The plane that contains the line $x = 3t$, $y = 1 + t$, $z = 2t$ and is parallel to the intersection of the planes $2x - y + z = 0$ and $y + z + 1 = 0$.

31. Find parametric equations of the line through the point $(5, 0, -2)$ that is parallel to the planes $x - 4y + 2z = 0$ and $2x + 3y - z + 1 = 0$.

32. Do the points $(1, 0, -1)$, $(0, 2, 3)$, $(-2, 1, 1)$, and $(4, 2, 3)$ lie in the same plane? Justify your answer two different ways.

33. Show that the line $x = 0$, $y = t$, $z = t$
(a) lies in the plane $6x + 4y - 4z = 0$
(b) is parallel to and below the plane $5x - 3y + 3z = 1$
(c) is parallel to and above the plane $6x + 2y - 2z = 3$.

34. Show that if a, b, and c are nonzero, then the plane whose intercepts with the coordinate axes are $x = a$, $y = b$, and $z = c$ is given by the equation
$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

35. Show that the lines
$$x = -2 + t, \quad y = 3 + 2t, \quad z = 4 - t$$
$$x = 3 - t, \quad y = 4 - 2t, \quad z = t$$
are parallel and find an equation of the plane they determine.

36. Show that the lines
$$L_1\colon x + 1 = 4t, \quad y - 3 = t, \quad z - 1 = 0$$
$$L_2\colon x + 13 = 12t, \quad y - 1 = 6t, \quad z - 2 = 3t$$
intersect and find an equation of the plane they determine.

In Exercises 37 and 38, find parametric equations of the line of intersection of the planes.

37. $-2x + 3y + 7z + 2 = 0$
$x + 2y - 3z + 5 = 0$

38. $3x - 5y + 2z = 0$
$z = 0$

In Exercises 39 and 40, find the distance between the point and the plane.

39. $(1, -2, 3)$; $2x - 2y + z = 4$

40. $(0, 1, 5)$; $3x + 6y - 2z - 5 = 0$

In Exercises 41 and 42, find the distance between the given parallel planes.

41. $-2x + y + z = 0$
$6x - 3y - 3z - 5 = 0$

42. $x + y + z = 1$
$x + y + z = -1$

In Exercises 43 and 44, find the distance between the given skew lines.

43. $x = 1 + 7t,\ y = 3 + t,\ z = 5 - 3t$
$x = 4 - t,\ y = 6,\ z = 7 + 2t$

44. $x = 3 - t,\ y = 4 + 4t,\ z = 1 + 2t$
$x = t,\ y = 3,\ z = 2t$

45. Find an equation of the sphere with center $(2, 1, -3)$ that is tangent to the plane $x - 3y + 2z = 4$.

46. Locate the point of intersection of the plane $2x + y - z = 0$ and the line through $(3, 1, 0)$ that is perpendicular to the plane.

47. Show that the line $x = -1 + t$, $y = 3 + 2t$, $z = -t$ and the plane $2x - 2y - 2z + 3 = 0$ are parallel, and find the distance between them.

48. Formulas (1), (2), (3), (5), and (10), which apply to planes in 3-space, have analogs for lines in 2-space.

(a) Draw an analog of Figure 12.6.3 in 2-space to illustrate that the equation of the line that passes through the point $P(x_0, y_0)$ and is perpendicular to the vector $\mathbf{n} = \langle a, b\rangle$ can be expressed as

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

where $\mathbf{r} = \langle x, y\rangle$ and $\mathbf{r}_0 = \langle x_0, y_0\rangle$.

(b) Show that the vector equation in part (a) can be expressed as

$$a(x - x_0) + b(y - y_0) = 0$$

This is called the ***point-normal form of a line***.

(c) Using the proof of Theorem 12.6.1 as a guide, show that if a and b are not both zero, then the graph of the equation

$$ax + by + c = 0$$

is a line that has $\mathbf{n} = \langle a, b\rangle$ as a normal.

(d) Using the proof of Theorem 12.6.2 as a guide, show that the distance D between a point $P(x_0, y_0)$ and the line $ax + by + c = 0$ is

$$D = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

49. Use the formula in part (d) of Exercise 48 to find the distance between the point $P(-3, 5)$ and the line $y = -2x + 1$.

50. (a) Show that the distance D between parallel planes

$$ax + by + cz + d_1 = 0$$
$$ax + by + cz + d_2 = 0$$

is

$$D = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$$

(b) Use the formula in part (a) to solve Exercise 41.

12.7 QUADRIC SURFACES

In this section we will study an important class of surfaces that are the three-dimensional analogs of the conic sections.

TRACES OF SURFACES

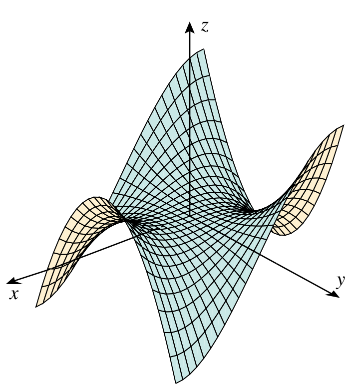


Figure 12.7.1

Although the general shape of a curve in 2-space can be obtained by plotting points, this method is not usually helpful for surfaces in 3-space because too many points are required. It is more common to build up the shape of a surface with a network of ***mesh lines***, which are curves obtained by cutting the surface with well-chosen planes. For example, Figure 12.7.1, which was generated by a CAS, shows the graph of $z = x^3 - 3xy^2$ rendered with a combination of mesh lines and colorization to produce the surface detail. This surface is called a "monkey saddle" because a monkey sitting astride the surface has a place for its two legs and tail.

The mesh line that results when a surface is cut by a plane is called the ***trace*** of the surface in the plane (Figure 12.7.2). Usually, surfaces are built up from traces in planes that are parallel to the coordinate planes, so we will begin by showing how the equations of such traces can be obtained. For this purpose, we will consider the surface

$$z = x^2 + y^2 \tag{1}$$

shown in Figure 12.7.3*a*.

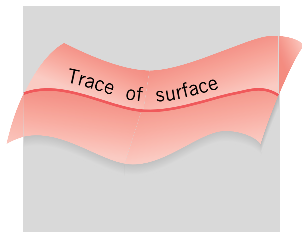


Figure 12.7.2

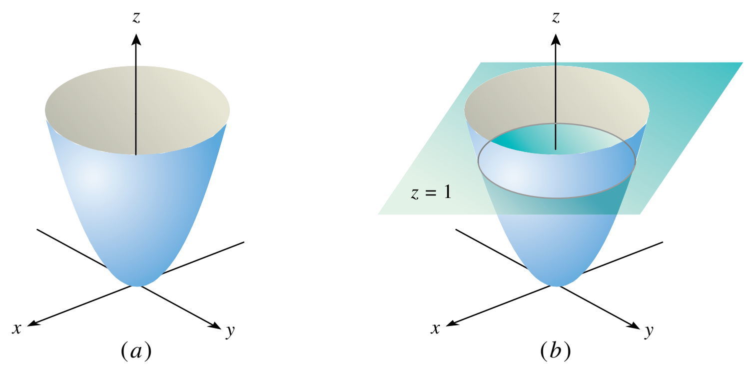


Figure 12.7.3

The basic procedure for finding the equation of a trace is to substitute the equation of the plane into the equation of the surface. For example, to find the trace of the surface $z = x^2 + y^2$ in the plane $z = 1$, we substitute $z = 1$ in (1), which yields

$$x^2 + y^2 = 1 \qquad (z = 1) \tag{2}$$

This is a circle of radius 1 centered at the point (0, 0, 1) (Figure 12.7.3*b*).

REMARK. The parenthetical part of Equation (2) is a reminder that the z-coordinate of all points on the trace is $z = 1$. This needs to be stated explicitly because z does not appear in the equation $x^2 + y^2 = 1$.

Figure 12.7.4*a* suggests that the traces of (1) in planes that are parallel to and above the xy-plane form a family of circles that are centered on the z-axis and whose radii increase with z. To confirm this, let us consider the trace in a general plane $z = k$ that is parallel to the xy-plane. The equation of the trace is

$$x^2 + y^2 = k \qquad (z = k)$$

If $k \geq 0$, then the trace is a circle of radius $\sqrt{k}$ centered at the point $(0, 0, k)$. In particular, if $k = 0$, then the radius is zero, so the trace in the xy-plane is the single point (0, 0, 0). Thus, for nonnegative values of k the traces parallel to the xy-plane form a family of circles, centered on the z-axis, whose radii start at zero and increase with k. This confirms our conjecture. If $k < 0$, then the equation $x^2 + y^2 = k$ has no graph, which means that there is no trace.

Now let us examine the traces of (1) in planes parallel to the yz-plane. Such planes have equations of the form $x = k$, so we substitute this in (1) to obtain

$$z = k^2 + y^2 \qquad (x = k)$$

which we can rewrite as

$$z - k^2 = y^2 \qquad (x = k) \tag{3}$$

For simplicity, let us start with the case where $k = 0$ (the trace in the yz-plane), in which case the trace has the equation

$$z = y^2 \qquad (x = 0)$$

You should be able to recognize that this is a parabola that has its vertex at the origin, opens in the positive z-direction, and is symmetric about the z-axis (Figure 12.7.4*b* shows a two-dimensional view). You should also be able to recognize that the $-k^2$ term in (3) has the effect of translating the parabola $z = y^2$ in the positive z-direction, so the new vertex falls at $(k, 0, k^2)$. Thus, the traces parallel to the yz-plane form a family of parabolas whose vertices move upward as k^2 increases. This is consistent with Figure 12.7.4*c*. Similarly, the traces in planes parallel to the xz-plane have equations of the form

$$z - k^2 = x^2 \qquad (y = k)$$

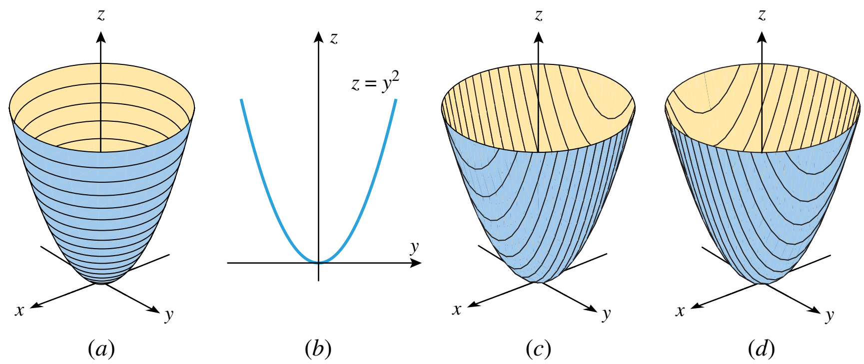


Figure 12.7.4

which again is a family of parabolas whose vertices move upward as k^2 increases (Figure 12.7.4*d*).

THE QUADRIC SURFACES

In the discussion of Formula (2) in Section 11.5 we noted that a second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

represents a conic section (possibly degenerate). The analog of this equation in an xyz-coordinate system is

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0 \tag{4}$$

which is called a ***second-degree equation in x, y, and z***. The graphs of such equations are called ***quadric surfaces*** or sometimes ***quadrics***.

The six nondegenerate types of quadric surfaces are shown in Table 12.7.1—*ellipsoids*, *hyperboloids of one sheet*, *hyperboloids of two sheets*, *elliptic cones*, *elliptic paraboloids*, and *hyperbolic paraboloids*. (The constants a, b, and c that appear in the equations in the table are assumed to be positive.) Observe that none of the quadric surfaces in the table have cross-product terms in their equations. This is because of their orientations relative to the coordinate axes. Later in this section we will discuss other possible orientations that produce equations of the quadric surfaces with no cross-product terms. In the special case where the elliptic cross sections of an elliptic cone or an elliptic paraboloid are circles, the terms *circular cone* and *circular paraboloid* are used.

TECHNIQUES FOR GRAPHING QUADRIC SURFACES

Accurate graphs of quadric surfaces are best left for graphing utilities. However, the techniques that we will now discuss can be used to generate rough sketches of these surfaces that are useful for various purposes.

A rough sketch of an ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad (a > 0, b > 0, c > 0) \tag{5}$$

can be obtained by first plotting the intersections with the coordinate axes, then sketching the elliptical traces in the coordinate planes, and then sketching the surface itself using the traces as a guide. Example 1 illustrates this technique.

Example 1 Sketch the ellipsoid

$$\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{9} = 1 \tag{6}$$

Solution. The x-intercepts can be obtained by setting $y = 0$ and $z = 0$ in (6). This yields $x = \pm 2$. Similarly, the y-intercepts are $y = \pm 4$, and the z-intercepts are $z = \pm 3$. From these intercepts we obtain the elliptical traces and the ellipsoid sketched in Figure 12.7.5. ◀

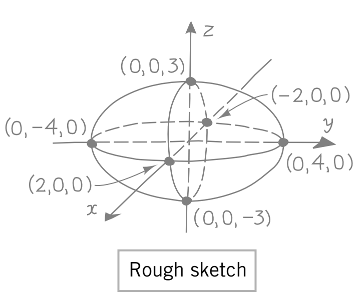
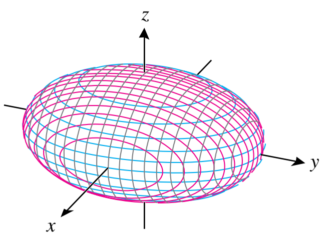
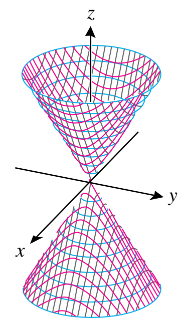
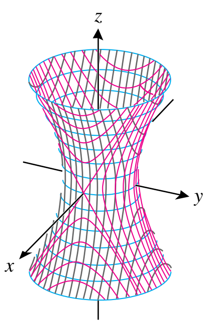
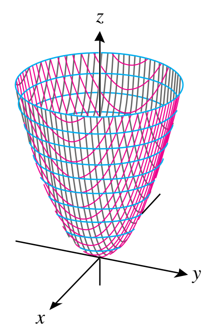
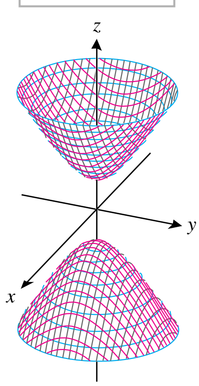
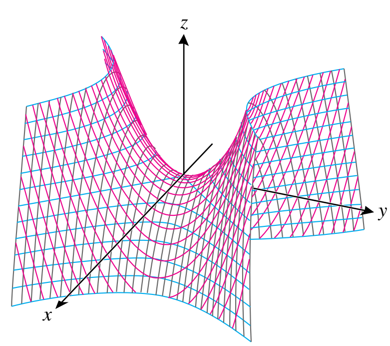


Figure 12.7.5

Table 12.7.1

<table>
<tr><th>SURFACE</th><th>EQUATION</th><th>SURFACE</th><th>EQUATION</th></tr>
<tr>
<td>ELLIPSOID
</td>
<td>$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
The traces in the coordinate planes are ellipses, as are the traces in those planes that are parallel to the coordinate planes and intersect the surface in more than one point.</td>
<td>ELLIPTIC CONE
</td>
<td>$z^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$
The trace in the xy-plane is a point (the origin), and the traces in planes parallel to the xy-plane are ellipses. The traces in the yz- and xz-planes are pairs of lines intersecting at the origin. The traces in planes parallel to these are hyperbolas.</td>
</tr>
<tr>
<td>HYPERBOLOID OF ONE SHEET
</td>
<td>$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
The trace in the xy-plane is an ellipse, as are the traces in planes parallel to the xy-plane. The traces in the yz-plane and xz-plane are hyperbolas, as are the traces in those planes that are parallel to these and do not pass through the x- or y-intercepts. At these intercepts the traces are pairs of intersecting lines.</td>
<td>ELLIPTIC PARABOLOID
</td>
<td>$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$
The trace in the xy-plane is a point (the origin), and the traces in planes parallel to and above the xy-plane are ellipses. The traces in the yz- and xz-planes are parabolas, as are the traces in planes parallel to these.</td>
</tr>
<tr>
<td>HYPERBOLOID OF TWO SHEETS
</td>
<td>$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
There is no trace in the xy-plane. In planes parallel to the xy-plane that intersect the surface in more than one point the traces are ellipses. In the yz- and xz-planes, the traces are hyperbolas, as are the traces in those planes that are parallel to these.</td>
<td>HYPERBOLIC PARABOLOID
</td>
<td>$z = \frac{y^2}{b^2} - \frac{x^2}{a^2}$
The trace in the xy-plane is a pair of lines intersecting at the origin. The traces in planes parallel to the xy-plane are hyperbolas. The hyperbolas above the xy-plane open in the y-direction, and those below in the x-direction. The traces in the yz- and xz-planes are parabolas, as are the traces in planes parallel to these.</td>
</tr>
</table>

A rough sketch of a hyperboloid of one sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \qquad (a > 0, b > 0, c > 0) \tag{7}$$

can be obtained by first sketching the elliptical trace in the xy-plane, then the elliptical traces in the planes $z = \pm c$, and then the hyperbolic curves that join the endpoints of the axes of these ellipses. The next example illustrates this technique.

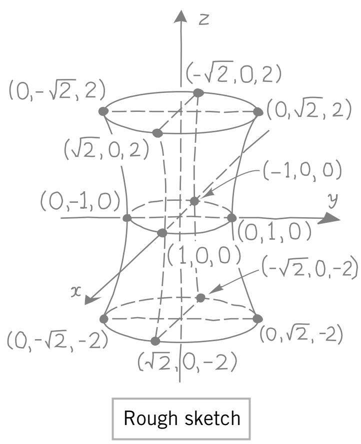


Figure 12.7.6

Example 2 Sketch the graph of the hyperboloid of one sheet

$$x^2 + y^2 - \frac{z^2}{4} = 1 \tag{8}$$

Solution. The trace in the xy-plane, obtained by setting $z = 0$ in (8), is

$$x^2 + y^2 = 1 \qquad (z = 0)$$

which is a circle of radius 1 centered on the z-axis. The traces in the planes $z = 2$ and $z = -2$, obtained by setting $z = \pm 2$ in (8), are given by

$$x^2 + y^2 = 2 \qquad (z = \pm 2)$$

which are circles of radius $\sqrt{2}$ centered on the z-axis. Joining these circles by the hyperbolic traces in the vertical coordinate planes yields the graph in Figure 12.7.6. ◀

A rough sketch of the hyperboloid of two sheets

$$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad (a > 0, b > 0, c > 0) \tag{9}$$

can be obtained by first plotting the intersections with the z-axis, then sketching the elliptical traces in the planes $z = \pm 2c$, and then sketching the hyperbolic traces that connect the z-axis intersections and the endpoints of the axes of the ellipses. (It is not essential to use the planes $z = \pm 2c$, but these are good choices since they simplify the calculations slightly and have the right spacing for a good sketch.) The next example illustrates this technique.

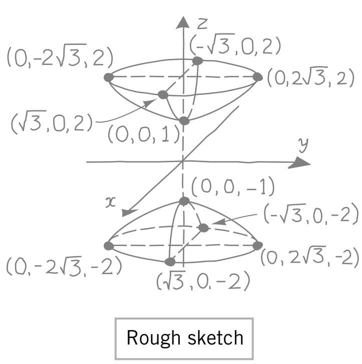


Figure 12.7.7

Example 3 Sketch the graph of the hyperboloid of two sheets

$$z^2 - x^2 - \frac{y^2}{4} = 1 \tag{10}$$

Solution. The z-intercepts, obtained by setting $x = 0$ and $y = 0$ in (10), are $z = \pm 1$. The traces in the planes $z = 2$ and $z = -2$, obtained by setting $z = \pm 2$ in (10), are given by

$$\frac{x^2}{3} + \frac{y^2}{12} = 1 \qquad (z = \pm 2)$$

Sketching these ellipses and the hyperbolic traces in the vertical coordinate planes yields Figure 12.7.7. ◀

A rough sketch of the elliptic cone

$$z^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2} \qquad (a > 0, b > 0) \tag{11}$$

can be obtained by first sketching the elliptical traces in the planes $z = \pm 1$ and then sketching the linear traces that connect the endpoints of the axes of the ellipses. The next example illustrates this technique.

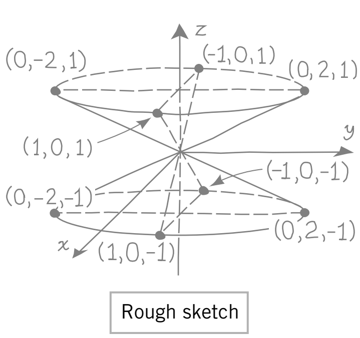


Figure 12.7.8

Example 4 Sketch the graph of the elliptic cone

$$z^2 = x^2 + \frac{y^2}{4} \tag{12}$$

Solution. The traces of (12) in the planes $z = \pm 1$ are given by

$$x^2 + \frac{y^2}{4} = 1 \qquad (z = \pm 1)$$

Sketching these ellipses and the linear traces in the vertical coordinate planes yields the graph in Figure 12.7.8. ◀

REMARK. Observe that if $a = b$ in (11), then the traces parallel to the xy-plane are circles, in which case we call the surface a *circular cone*.

A rough sketch of the elliptic paraboloid

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \qquad (a > 0, b > 0) \tag{13}$$

can be obtained by first sketching the elliptical trace in the plane $z = 1$ and then sketching the parabolic traces in the vertical coordinate planes to connect the origin to the ends of the axes of the ellipse. The next example illustrates this technique.

Example 5 Sketch the graph of the elliptic paraboloid

$$z = \frac{x^2}{4} + \frac{y^2}{9} \tag{14}$$

Solution. The trace of (14) in the plane $z = 1$ is

$$\frac{x^2}{4} + \frac{y^2}{9} = 1 \qquad (z = 1)$$

Sketching this ellipse and the parabolic traces in the vertical coordinate planes yields the graph in Figure 12.7.9. ◀

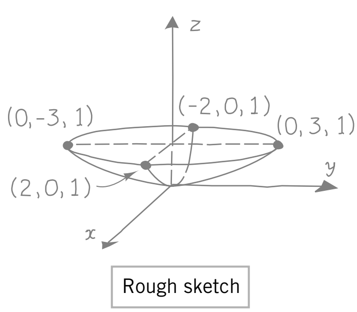


Figure 12.7.9

A rough sketch of the hyperbolic paraboloid

$$z = \frac{y^2}{b^2} - \frac{x^2}{a^2} \qquad (a > 0, b > 0) \tag{15}$$

can be obtained by first sketching the two parabolic traces that pass through the origin (one in the plane $x = 0$ and the other in the plane $y = 0$). After the parabolic traces are drawn, sketch the hyperbolic traces in the planes $z = \pm 1$ and then fill in any missing edges. The next example illustrates this technique.

Example 6 Sketch the graph of the hyperbolic paraboloid

$$z = \frac{y^2}{4} - \frac{x^2}{9} \tag{16}$$

Solution. Setting $x = 0$ in (16) yields

$$z = \frac{y^2}{4} \qquad (x = 0)$$

which is a parabola in the yz-plane with vertex at the origin and opening in the positive z-direction (since $z \geq 0$), and setting $y = 0$ yields

$$z = -\frac{x^2}{9} \qquad (y = 0)$$

which is a parabola in the xz-plane with vertex at the origin and opening in the negative z-direction.

The trace in the plane $z = 1$ is

$$\frac{y^2}{4} - \frac{x^2}{9} = 1 \qquad (z = 1)$$

which is a hyperbola that opens along a line parallel to the y-axis (verify), and the trace in

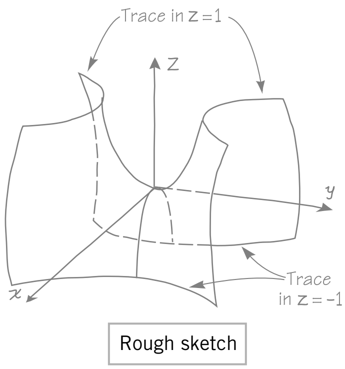


Figure 12.7.10

the plane $z = -1$ is

$$\frac{x^2}{9} - \frac{y^2}{4} = 1 \qquad (z = -1)$$

which is a hyperbola that opens along a line parallel to the x-axis. Combining all of the above information leads to the sketch in Figure 12.7.10. ◀

REMARK. The hyperbolic paraboloid in Figure 12.7.10 has an interesting behavior at the origin—the trace in the xz-plane has a relative maximum at (0, 0, 0), and the trace in the yz-plane has a relative minimum at (0, 0, 0). Thus, a bug walking on the surface may view the origin as a highest point if traveling along one path, or may view the origin as a lowest point if traveling along a different path. A point with this property is commonly called a ***saddle point*** or a ***minimax point***.

Figure 12.7.11 shows two computer-generated views of the hyperbolic paraboloid in Example 6. The first view, which is much like our rough sketch in Figure 12.7.10, has cuts at the top and bottom that are hyperbolic traces parallel to the xy-plane. In the second view the top horizontal cut has been omitted; this helps to emphasize the parabolic traces parallel to the xz-plane.

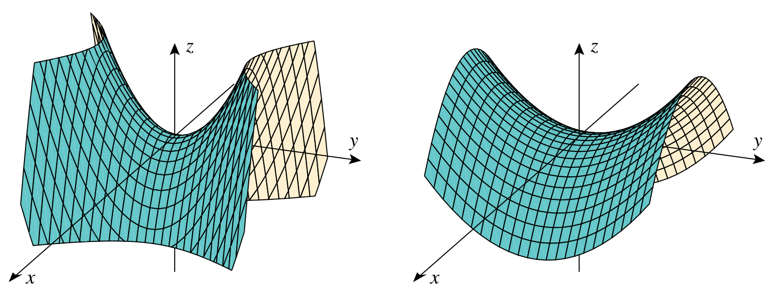
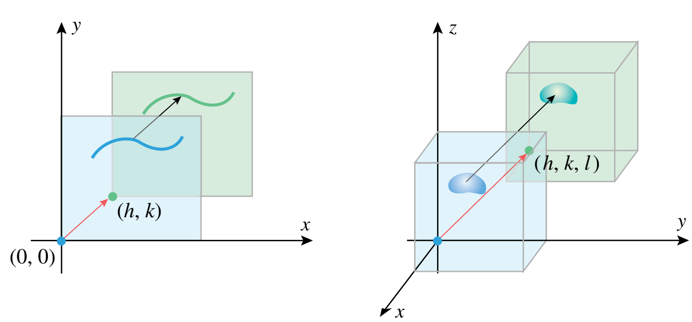


Figure 12.7.11

TRANSLATIONS OF QUADRIC SURFACES

In Section 11.4 we saw that a conic in an xy-coordinate system can be translated by substituting $x - h$ for x and $y - k$ for y in its equation. To understand why this works, think of the xy-axes as fixed, and think of the plane as a transparent sheet of plastic on which all graphs are drawn. When the coordinates of points are modified by substituting $(x-h, y-k)$ for (x, y), the geometric effect is to translate the sheet of plastic (and hence all curves) so that the point on the plastic that was initially at (0, 0) is moved to the point (h, k) (see Figure 12.7.12a).



(a) (b)

Figure 12.7.12

For the analog in three dimensions, think of the xyz-axes as fixed, and think of 3-space as a transparent block of plastic in which all surfaces are embedded. When the coordinates of points are modified by substituting $(x-h, y-k, z-\ell)$ for (x, y, z), the geometric effect is to translate the block of plastic (and hence all surfaces) so that the point in the plastic block that was initially at $(0, 0, 0)$ is moved to the point (h, k, ℓ) (see Figure 12.7.12*b*).

Example 7 Describe the surface $z=(x-1)^2+(y+2)^2+3$.

Solution. The equation can be rewritten as

$$z-3=(x-1)^2+(y+2)^2$$

This surface is the paraboloid that results by translating the paraboloid

$$z=x^2+y^2$$

in Figure 12.7.3 so that the new "vertex" is at the point $(1, -2, 3)$. A rough sketch of this paraboloid is shown in Figure 12.7.13. ◀

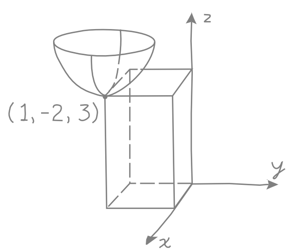


Figure 12.7.13

Example 8 Describe the surface

$$4x^2+4y^2+z^2+8y-4z=-4$$

Solution. Completing the squares yields

$$4x^2+4(y+1)^2+(z-2)^2=-4+4+4$$

or

$$x^2+(y+1)^2+\frac{(z-2)^2}{4}=1$$

Thus, the surface is the ellipsoid that results when the ellipsoid

$$x^2+y^2+\frac{z^2}{4}=1$$

is translated so that the new "center" is at the point $(0, -1, 2)$. A rough sketch of this ellipsoid is shown in Figure 12.7.14. ◀

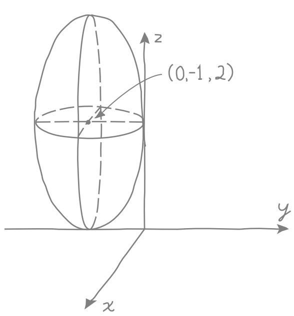


Figure 12.7.14

FOR THE READER. The ellipsoid in Figure 12.7.14 was sketched with its cross section in the *yz*-plane tangent to the *y*- and *z*-axes. Confirm that this is correct.

REFLECTIONS OF SURFACES IN 3-SPACE

Recall that in an *xy*-coordinate system a point (x, y) is reflected about the x-axis if y is replaced by $-y$, and it is reflected about the y-axis if x is replaced by $-x$. In an *xyz*-coordinate system, a point (x, y, z) is reflected about the *xy*-plane if z is replaced by $-z$, it is reflected about the *yz*-plane if x is replaced by $-x$, and it is reflected about the *xz*-plane if y is replaced by $-y$ (Figure 12.7.15). It follows that *replacing a variable by its negative in the equation of a surface causes that surface to be reflected about a coordinate plane*.

Recall also that in an *xy*-coordinate system a point (x, y) is reflected about the line $y=x$ if x and y are interchanged. However, in an *xyz*-coordinate system, interchanging x and y reflects the point (x, y, z) about the plane $y=x$ (Figure 12.7.16). Similarly, interchanging

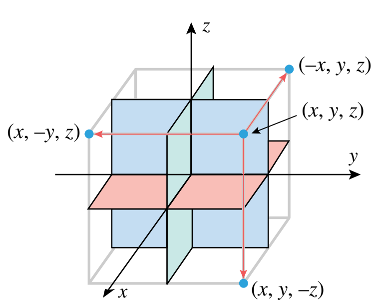


Figure 12.7.15

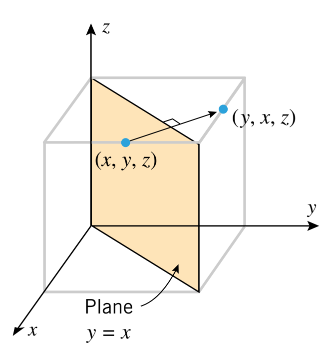


Figure 12.7.16

x and z reflects the point about the plane $x = z$, and interchanging y and z reflects it about the plane $y = z$. Thus, it follows that *interchanging two variables in the equation of a surface reflects that surface about a plane that makes a* $45°$ *angle with two of the coordinate planes*.

Example 9 Describe the surfaces

(a) $y^2 = x^2 + z^2$ (b) $z = -(x^2 + y^2)$

Solution (a). The graph of the equation $y^2 = x^2 + z^2$ results from interchanging y and z in the equation $z^2 = x^2 + y^2$. Thus, the graph of the equation $y^2 = x^2 + z^2$ can be obtained by reflecting the graph of $z^2 = x^2 + y^2$ about the plane $y = z$. Since the graph of $z^2 = x^2 + y^2$ is a circular cone opening along the z-axis (see Table 12.7.1), it follows that the graph of $y^2 = x^2 + z^2$ is a circular cone opening along the y-axis (Figure 12.7.17).

Solution (b). The graph of the equation $z = -(x^2 + y^2)$ can be written as $-z = x^2 + y^2$, which can be obtained by replacing z with $-z$ in the equation $z = x^2 + y^2$. Since the graph of $z = x^2 + y^2$ is a circular paraboloid opening in the positive z-direction (see Table 12.7.1), it follows that the graph of $z = -(x^2 + y^2)$ is a circular paraboloid opening in the negative z-direction (Figure 12.7.18). ◀

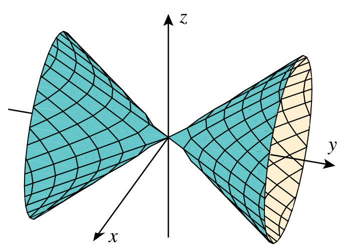


Figure 12.7.17

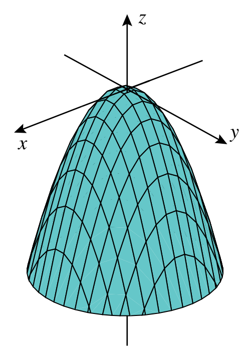


Figure 12.7.18

A TECHNIQUE FOR IDENTIFYING QUADRIC SURFACES

The equations of the quadric surfaces in Table 12.7.1 have certain characteristics that make it possible to identify quadric surfaces that are derived from these equations by reflections. These identifying characteristics, which are shown in Table 12.7.2, are based on writing the equation of the quadric surface so that all of the variable terms are on the left side of the equation and there is a 1 or a 0 on the right side. When there is a 1 on the right side the surface is an ellipsoid, hyperboloid of one sheet, or a hyperboloid of two sheets, and when there is a 0 on the right side it is an elliptic cone, an elliptic paraboloid, or a hyperbolic paraboloid. Within the group with a 1 on the right side, ellipsoids have no minus signs, hyperboloids of one sheet have one minus sign, and hyperboloids of two sheets have two minus signs. Within the group with a 0 on the right side, elliptic cones have no linear terms, elliptic paraboloids have one linear term and two quadratic terms with the same sign, and hyperbolic paraboloids have one linear term and two quadratic terms with opposite signs. These characteristics do not change when the surface is reflected about a coordinate plane or planes of the form $x = y$, $x = z$, or $y = z$, thereby making it possible to identify the reflected quadric surface from the form of its equation.

Example 10 Identify the surfaces

(a) $3x^2 - 4y^2 + 12z^2 + 12 = 0$ (b) $4x^2 - 4y + z^2 = 0$

Table 12.7.2

EQUATION	CHARACTERISTIC	CLASSIFICATION
$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	No minus signs	Ellipsoid
$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$	One minus sign	Hyperboloid of one sheet
$\frac{z^2}{c^2}-\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	Two minus signs	Hyperboloid of two sheets
$z^2-\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$	No linear terms	Elliptic cone
$z-\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$	One linear term; two quadratic terms with the same sign	Elliptic paraboloid
$z-\frac{y^2}{b^2}+\frac{x^2}{a^2}=0$	One linear term; two quadratic terms with opposite signs	Hyperbolic paraboloid

Solution (a). The equation can be rewritten as

$$\frac{y^2}{3}-\frac{x^2}{4}-z^2=1$$

This equation has a 1 on the right side and two negative terms on the left side, so its graph is a hyperboloid of two sheets.

Solution (b). The equation has one linear term and two quadratic terms with the same sign, so its graph is an elliptic paraboloid. ◀

EXERCISE SET 12.7

In Exercises 1 and 2, identify the quadric surface as an ellipsoid, hyperboloid of one sheet, hyperboloid of two sheets, elliptic cone, elliptic paraboloid, or hyperbolic paraboloid by matching the equation with one of the forms given in Table 12.7.1. State the values of a, b, and c in each case.

1. (a) $z=\frac{x^2}{4}+\frac{y^2}{9}$ (b) $z=\frac{y^2}{25}-x^2$
(c) $x^2+y^2-z^2=16$ (d) $x^2+y^2-z^2=0$
(e) $4z=x^2+4y^2$ (f) $z^2-x^2-y^2=1$

2. (a) $6x^2+3y^2+4z^2=12$ (b) $y^2-x^2-z=0$
(c) $9x^2+y^2-9z^2=9$ (d) $4x^2+y^2-4z^2=-4$
(e) $2z-x^2-4y^2=0$ (f) $12z^2-3x^2=4y^2$

3. Find an equation for and sketch the surface that results when the circular paraboloid $z=x^2+y^2$ is reflected about the plane
(a) $z=0$ (b) $x=0$ (c) $y=0$
(d) $y=x$ (e) $x=z$ (f) $y=z$.

4. Find an equation for and sketch the surface that results when the hyperboloid of one sheet $x^2+y^2-z^2=1$ is reflected about the plane
(a) $z=0$ (b) $x=0$ (c) $y=0$
(d) $y=x$ (e) $x=z$ (f) $y=z$.

5. The given equations represent quadric surfaces whose orientations are different from those in Table 12.7.1. In each part, identify the quadric surface, and give a verbal description of its orientation (e.g., an elliptic cone opening along the z-axis or a hyperbolic paraboloid straddling the y-axis).

(a) $\frac{z^2}{c^2}-\frac{y^2}{b^2}+\frac{x^2}{a^2}=1$ (b) $\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$

(c) $x=\frac{y^2}{b^2}+\frac{z^2}{c^2}$ (d) $x^2=\frac{y^2}{b^2}+\frac{z^2}{c^2}$

(e) $y=\frac{z^2}{c^2}-\frac{x^2}{a^2}$ (f) $y=-\left(\frac{x^2}{a^2}+\frac{z^2}{c^2}\right)$

6. For each of the surfaces in Exercise 5, find the equation of the surface that results if the given surface is reflected about the xz-plane and that surface is then reflected about the plane $z = 0$.

In Exercises 7 and 8, find equations of the traces in the coordinate planes, and sketch the traces in an xyz-coordinate system. [*Suggestion:* If you have trouble sketching a trace directly in three dimensions, start with a sketch in two dimensions by placing the coordinate plane in the plane of the paper; then transfer that sketch to three dimensions.]

7. (a) $\dfrac{x^2}{9} + \dfrac{y^2}{25} + \dfrac{z^2}{4} = 1$ (b) $z = x^2 + 4y^2$
(c) $\dfrac{x^2}{9} + \dfrac{y^2}{16} - \dfrac{z^2}{4} = 1$

8. (a) $y^2 + 9z^2 = x$ (b) $4x^2 - y^2 + 4z^2 = 4$
(c) $z^2 = x^2 + \dfrac{y^2}{4}$

In Exercises 9 and 10, the traces of the surfaces in the planes are conic sections. In each part, find an equation of the trace, and state whether it is an ellipse, a parabola, or a hyperbola.

9. (a) $4x^2 + y^2 + z^2 = 4;\ y = 1$
(b) $4x^2 + y^2 + z^2 = 4;\ x = \frac{1}{2}$
(c) $9x^2 - y^2 - z^2 = 16;\ x = 2$
(d) $9x^2 - y^2 - z^2 = 16;\ z = 2$
(e) $z = 9x^2 + 4y^2;\ y = 2$
(f) $z = 9x^2 + 4y^2;\ z = 4$

10. (a) $9x^2 - y^2 + 4z^2 = 9;\ x = 2$
(b) $9x^2 - y^2 + 4z^2 = 9;\ y = 4$
(c) $x^2 + 4y^2 - 9z^2 = 0;\ y = 1$
(d) $x^2 + 4y^2 - 9z^2 = 0;\ z = 1$
(e) $z = x^2 - 4y^2;\ x = 1$
(f) $z = x^2 - 4y^2;\ z = 4$

In Exercises 11–22, identify and sketch the quadric surface.

11. $x^2 + \dfrac{y^2}{4} + \dfrac{z^2}{9} = 1$ **12.** $x^2 + 4y^2 + 9z^2 = 36$

13. $\dfrac{x^2}{4} + \dfrac{y^2}{9} - \dfrac{z^2}{16} = 1$ **14.** $x^2 + y^2 - z^2 = 9$

15. $4z^2 = x^2 + 4y^2$ **16.** $9x^2 + 4y^2 - 36z^2 = 0$

17. $9z^2 - 4y^2 - 9x^2 = 36$ **18.** $y^2 - \dfrac{x^2}{4} - \dfrac{z^2}{9} = 1$

19. $z = y^2 - x^2$ **20.** $16z = y^2 - x^2$

21. $4z = x^2 + 2y^2$ **22.** $z - 3x^2 - 3y^2 = 0$

In Exercises 23–28, the given equations represent quadric surfaces whose orientations are different from those in Table 12.7.1. Identify and sketch the surface.

23. $x^2 - 3y^2 - 3z^2 = 0$ **24.** $x - y^2 - 4z^2 = 0$

25. $2y^2 - x^2 + 2z^2 = 8$ **26.** $x^2 - 3y^2 - 3z^2 = 9$

27. $z = \dfrac{x^2}{4} - \dfrac{y^2}{9}$ **28.** $4x^2 - y^2 + 4z^2 = 16$

In Exercises 29–32, sketch the surface.

29. $z = \sqrt{x^2 + y^2}$ **30.** $z = \sqrt{1 - x^2 - y^2}$

31. $z = \sqrt{x^2 + y^2 - 1}$ **32.** $z = \sqrt{1 + x^2 + y^2}$

In Exercises 33–36, identify the surface, and make a rough sketch that shows its position and orientation.

33. $z = (x + 2)^2 + (y - 3)^2 - 9$

34. $4x^2 - y^2 + 16(z - 2)^2 = 100$

35. $9x^2 + y^2 + 4z^2 - 18x + 2y + 16z = 10$

36. $z^2 = 4x^2 + y^2 + 8x - 2y + 4z$

Exercises 37 and 38 are concerned with the ellipsoid $4x^2 + 9y^2 + 18z^2 = 72$.

37. (a) Find an equation of the elliptical trace in the plane $z = \sqrt{2}$.
(b) Find the lengths of the major and minor axes of the ellipse in part (a).
(c) Find the coordinates of the foci of the ellipse in part (a).
(d) Describe the orientation of the focal axis of the ellipse in part (a) relative to the coordinate axes.

38. (a) Find an equation of the elliptical trace in the plane $x = 3$.
(b) Find the lengths of the major and minor axes of the ellipse in part (a).
(c) Find the coordinates of the foci of the ellipse in part (a).
(d) Describe the orientation of the focal axis of the ellipse in part (a) relative to the coordinate axes.

Exercises 39–42 refer to the hyperbolic paraboloid $z = y^2 - x^2$.

39. (a) Find an equation of the hyperbolic trace in the plane $z = 4$.
(b) Find the vertices of the hyperbola in part (a).
(c) Find the foci of the hyperbola in part (a).
(d) Describe the orientation of the focal axis of the hyperbola in part (a) relative to the coordinate axes.

40. (a) Find an equation of the hyperbolic trace in the plane $z = -4$.
(b) Find the vertices of the hyperbola in part (a).
(c) Find the foci of the hyperbola in part (a).
(d) Describe the orientation of the focal axis of the hyperbola in part (a) relative to the coordinate axes.

41. (a) Find an equation of the parabolic trace in the plane $x = 2$.
(b) Find the vertex of the parabola in part (a).
(c) Find the focus of the parabola in part (a).
(d) Describe the orientation of the focal axis of the parabola in part (a) relative to the coordinate axes.

42. (a) Find an equation of the parabolic trace in the plane $y = 2$.
(b) Find the vertex of the parabola in part (a).
(c) Find the focus of the parabola in part (a).
(d) Describe the orientation of the focal axis of the parabola in part (a) relative to the coordinate axes.

In Exercises 43 and 44, sketch the region enclosed between the surfaces and describe their curve of intersection.

43. The paraboloids $z = x^2 + y^2$ and $z = 4 - x^2 - y^2$

44. The hyperbolic paraboloid $x^2 = y^2 + z$ and the ellipsoid $x^2 = 4 - 2y^2 - 2z$

In Exercises 45 and 46, find an equation for the surface generated by revolving the curve about the axis.

45. $y = 4x^2$ $(z = 0)$ about the y-axis

46. $y = 2x$ $(z = 0)$ about the y-axis

47. Find an equation of the surface consisting of all points $P(x, y, z)$ that are equidistant from the point $(0, 0, 1)$ and the plane $z = -1$. Identify the surface.

48. Find an equation of the surface consisting of all points $P(x, y, z)$ that are twice as far from the plane $z = -1$ as from the point $(0, 0, 1)$. Identify the surface.

49. If a sphere

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{a^2} = 1$$

of radius a is compressed in the z-direction, then the resulting surface, called an ***oblate spheroid***, has an equation of the form

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} = 1$$

where $c < a$. Show that the oblate spheroid has a circular trace of radius a in the xy-plane and an elliptical trace in the xz-plane with major axis of length $2a$ along the x-axis and minor axis of length $2c$ along the z-axis.

50. The Earth's rotation causes a flattening at the poles, so its shape is often modeled as an oblate spheroid rather than a sphere (see Exercise 49 for terminology). One of the models used by global positioning satellites is the ***World Geodetic System of 1984*** (WGS-84), which treats the Earth as an oblate spheroid whose equatorial radius is 6378.1370 km and whose polar radius (the distance from the Earth's center to the poles) is 6356.5231 km. Use the WGS-84 model to find an equation for the surface of the Earth relative to the coordinate system shown in the accompanying figure.

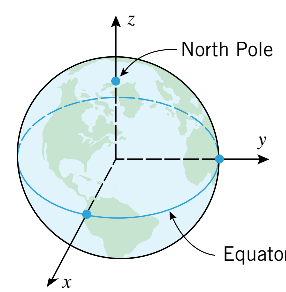


Figure Ex-50

51. Use the method of slicing to show that the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is $\frac{4}{3}\pi abc$.

12.8 CYLINDRICAL AND SPHERICAL COORDINATES

In this section we will discuss two new types of coordinate systems in 3-space that are often more useful than rectangular coordinate systems for studying surfaces with symmetries. These new coordinate systems also have important applications in navigation, astronomy, and the study of rotational motion about an axis.

CYLINDRICAL AND SPHERICAL COORDINATE SYSTEMS

Three coordinates are required to establish the location of a point in 3-space. We have already done this using rectangular coordinates. However, Figure 12.8.1 shows two other possibilities: part (*a*) of the figure shows the ***rectangular coordinates*** (x, y, z) of a point P, part (*b*) shows the ***cylindrical coordinates*** (r, θ, z) of P, and part (*c*) shows the ***spherical coordinates*** (ρ, θ, ϕ) of P. In a rectangular coordinate system the coordinates can be any real numbers, but in cylindrical and spherical coordinate systems there are restrictions on the allowable values of the coordinates (as indicated in Figure 12.8.1).

March 6, 2001 14:27 g65-ch12 Sheet number 62 Page number 852 cyan magenta yellow black

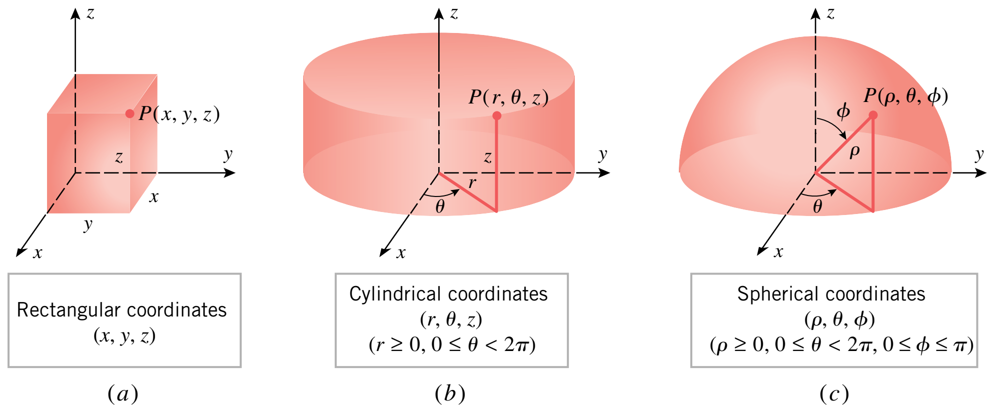


Figure 12.8.1

CONSTANT SURFACES

In rectangular coordinates the surfaces represented by equations of the form

$$x = x_0, \quad y = y_0, \quad \text{and} \quad z = z_0$$

where x_0, y_0, and z_0 are constants, are planes parallel to the yz-plane, xz-plane, and xy-plane, respectively (Figure 12.8.2*a*). In cylindrical coordinates the surfaces represented by equations of the form

$$r = r_0, \quad \theta = \theta_0, \quad \text{and} \quad z = z_0$$

where r_0, θ_0, and z_0 are constants, are shown in Figure 12.8.2*b*:

- The surface $r = r_0$ is a right circular cylinder of radius r_0 centered on the z-axis. At each point (r, θ, z) on this cylinder, r has the value r_0, but θ and z are unrestricted except for our general restriction that $0 \le \theta < 2\pi$.
- The surface $\theta = \theta_0$ is a half-plane attached along the z-axis and making an angle θ_0 with the positive x-axis. At each point (r, θ, z) on this surface, θ has the value θ_0, but r and z are unrestricted except for our general restriction that $r \ge 0$.
- The surface $z = z_0$ is a horizontal plane. At each point (r, θ, z) on this plane, z has the value z_0, but r and θ are unrestricted except for the general restrictions.

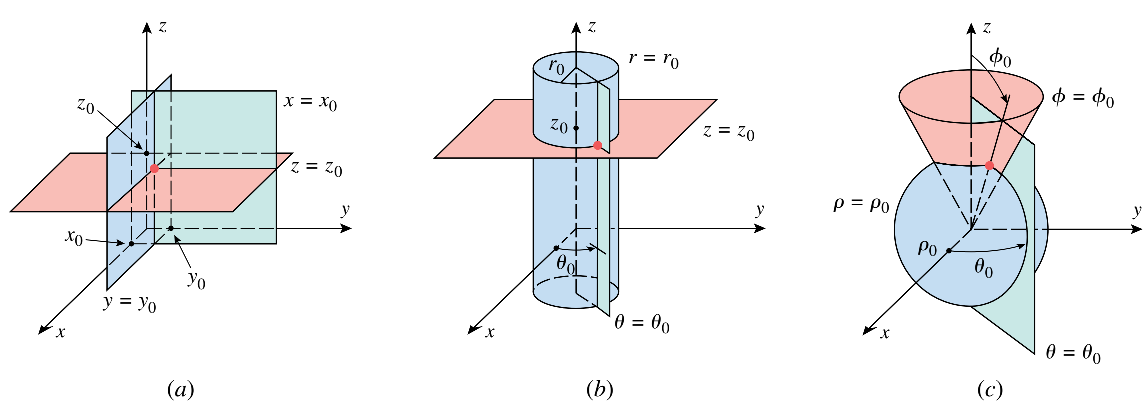


Figure 12.8.2

In spherical coordinates the surfaces represented by equations of the form

$$\rho = \rho_0, \quad \theta = \theta_0, \quad \text{and} \quad \phi = \phi_0$$

where ρ_0, θ_0, and ϕ_0 are constants, are shown in Figure 12.8.2*c*:

- The surface $\rho = \rho_0$ consists of all points whose distance ρ from the origin is ρ_0. Assuming ρ_0 to be nonnegative, this is a sphere of radius ρ_0 centered at the origin.
- As in cylindrical coordinates, the surface $\theta = \theta_0$ is a half-plane attached along the z-axis, making an angle of θ_0 with the positive x-axis.
- The surface $\phi = \phi_0$ consists of all points from which a line segment to the origin makes an angle of ϕ_0 with the positive z-axis. Depending on whether $0 < \phi_0 < \pi/2$ or $\pi/2 < \phi_0 < \pi$, this will be the nappe of a cone opening up or opening down. (If $\phi_0 = \pi/2$, then the cone is flat, and the surface is the xy-plane.)

CONVERTING COORDINATES

Just as we needed to convert between rectangular and polar coordinates in 2-space, so we will need to be able to convert between rectangular, cylindrical, and spherical coordinates in 3-space. Table 12.8.1 provides formulas for making these conversions.

Table 12.8.1

CONVERSION		FORMULAS	RESTRICTIONS
Cylindrical to rectangular	$(r, \theta, z) \to (x, y, z)$	$x = r\cos\theta,\quad y = r\sin\theta,\quad z = z$	$r \ge 0,\ \rho \ge 0$ $0 \le \theta < 2\pi$ $0 \le \phi \le \pi$
Rectangular to cylindrical	$(x, y, z) \to (r, \theta, z)$	$r = \sqrt{x^2+y^2},\ \tan\theta = y/x,\ z = z$	
Spherical to cylindrical	$(\rho, \theta, \phi) \to (r, \theta, z)$	$r = \rho\sin\phi,\quad \theta = \theta,\ z = \rho\cos\phi$	
Cylindrical to spherical	$(r, \theta, z) \to (\rho, \theta, \phi)$	$\rho = \sqrt{r^2+z^2},\ \theta = \theta,\ \tan\phi = r/z$	
Spherical to rectangular	$(\rho, \theta, \phi) \to (x, y, z)$	$x = \rho\sin\phi\cos\theta,\ y = \rho\sin\phi\sin\theta,\ z = \rho\cos\phi$	
Rectangular to spherical	$(x, y, z) \to (\rho, \theta, \phi)$	$\rho = \sqrt{x^2+y^2+z^2},\ \tan\theta = y/x,\ \cos\phi = z/\sqrt{x^2+y^2+z^2}$	

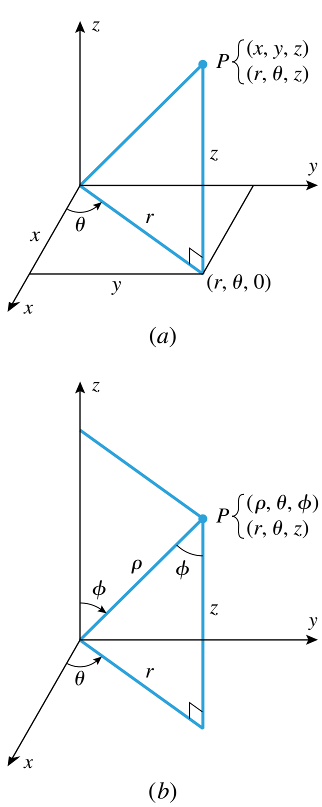


Figure 12.8.3

The diagrams in Figure 12.8.3 will help you to understand how the formulas in Table 12.8.1 are derived. For example, part (*a*) of the figure shows that in converting between rectangular coordinates (x, y, z) and cylindrical coordinates (r, θ, z), we can interpret (r, θ) as polar coordinates of (x, y). Thus, the polar-to-rectangular and rectangular-to-polar conversion formulas (1) and (2) of Section 11.1 provide the conversion formulas between rectangular and cylindrical coordinates in the table.

Part (*b*) of Figure 12.8.3 suggests that the spherical coordinates (ρ, θ, ϕ) of a point P can be converted to cylindrical coordinates (r, θ, z) by the conversion formulas

$$r = \rho\sin\phi, \quad \theta = \theta, \quad z = \rho\cos\phi \tag{1}$$

Moreover, since the cylindrical coordinates (r, θ, z) of P can be converted to rectangular coordinates (x, y, z) by the conversion formulas

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z \tag{2}$$

we can obtain direct conversion formulas from spherical coordinates to rectangular coordinates by substituting (1) in (2). This yields

$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi \tag{3}$$

The other conversion formulas in Table 12.8.1 are left as exercises.

Example 1

(a) Find the rectangular coordinates of the point with cylindrical coordinates

$$(r, \theta, z) = (4, \pi/3, -3)$$

(b) Find the rectangular coordinates of the point with spherical coordinates

$$(\rho, \theta, \phi) = (4, \pi/3, \pi/4)$$

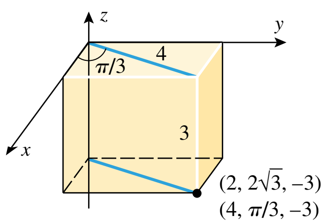


Figure 12.8.4

Solution (a). Applying the cylindrical-to-rectangular conversion formulas in Table 12.8.1 yields

$$x = r\cos\theta = 4\cos\frac{\pi}{3} = 2, \quad y = r\sin\theta = 4\sin\frac{\pi}{3} = 2\sqrt{3}, \quad z = -3$$

Thus, the rectangular coordinates of the point are $(x, y, z) = (2, 2\sqrt{3}, -3)$ (Figure 12.8.4).

Solution (b). Applying the spherical-to-rectangular conversion formulas in Table 12.8.1 yields

$$x = \rho\sin\phi\cos\theta = 4\sin\frac{\pi}{4}\cos\frac{\pi}{3} = \sqrt{2}$$

$$y = \rho\sin\phi\sin\theta = 4\sin\frac{\pi}{4}\sin\frac{\pi}{3} = \sqrt{6}$$

$$z = \rho\cos\phi = 4\cos\frac{\pi}{4} = 2\sqrt{2}$$

Thus, the rectangular coordinates of the point are $(x, y, z) = (\sqrt{2}, \sqrt{6}, 2\sqrt{2})$ (Figure 12.8.5). ◀

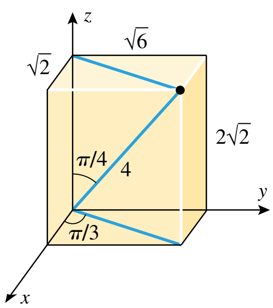


Figure 12.8.5

Example 2 Find the spherical coordinates of the point that has rectangular coordinates

$$(x, y, z) = (4, -4, 4\sqrt{6})$$

Solution. From the rectangular-to-spherical conversion formulas in Table 12.8.1 we obtain

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{16 + 16 + 96} = \sqrt{128} = 8\sqrt{2}$$

$$\tan\theta = \frac{y}{x} = -1$$

$$\cos\phi = \frac{z}{\sqrt{x^2 + y^2 + z^2}} = \frac{4\sqrt{6}}{8\sqrt{2}} = \frac{\sqrt{3}}{2}$$

From the restriction $0 \le \theta < 2\pi$ and the computed value of $\tan\theta$, the possibilities for θ are $\theta = 3\pi/4$ and $\theta = 7\pi/4$. However, the given point has a negative y-coordinate, so we must have $\theta = 7\pi/4$. Moreover, from the restriction $0 \le \phi \le \pi$ and the computed value of $\cos\phi$, the only possibility for ϕ is $\phi = \pi/6$. Thus, the spherical coordinates of the point are $(\rho, \theta, \phi) = (8\sqrt{2}, 7\pi/4, \pi/6)$ (Figure 12.8.6). ◀

EQUATIONS OF SURFACES IN CYLINDRICAL AND SPHERICAL COORDINATES

Surfaces of revolution about the z-axis of a rectangular coordinate system usually have simpler equations in cylindrical coordinates than in rectangular coordinates, and the equations of surfaces with symmetry about the origin are usually simpler in spherical coordinates than in rectangular coordinates. For example, consider the upper nappe of the circular cone whose equation in rectangular coordinates is

$$z = \sqrt{x^2 + y^2}$$

(Table 12.8.2). The corresponding equation in cylindrical coordinates can be obtained from the cylindrical-to-rectangular conversion formulas in Table 12.8.1. This yields

$$z = \sqrt{(r\cos\theta)^2 + (r\sin\theta)^2} = \sqrt{r^2} = |r| = r$$

so the equation of the cone in cylindrical coordinates is $z = r$. Going a step further, the equation of the cone in spherical coordinates can be obtained from the spherical-to-cylindrical conversion formulas from Table 12.8.1. This yields

$$\rho\cos\phi = \rho\sin\phi$$

which, if $\rho \ne 0$, can be rewritten as

$$\tan\phi = 1 \quad \text{or} \quad \phi = \frac{\pi}{4}$$

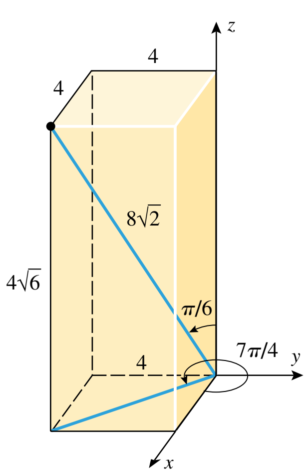
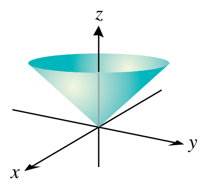
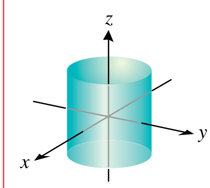
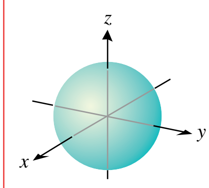
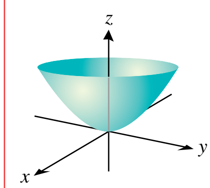
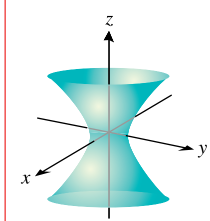


Figure 12.8.6

Geometrically, this tells us that the radial line from the origin to any point on the cone makes an angle of $\pi/4$ with the z-axis.

Table 12.8.2

	CONE	CYLINDER	SPHERE	PARABOLOID	HYPERBOLOID
					
RECTANGULAR	$z = \sqrt{x^2 + y^2}$	$x^2 + y^2 = 1$	$x^2 + y^2 + z^2 = 1$	$z = x^2 + y^2$	$x^2 + y^2 - z^2 = 1$
CYLINDRICAL	$z = r$	$r = 1$	$z^2 = 1 - r^2$	$z = r^2$	$z^2 = r^2 - 1$
SPHERICAL	$\phi = \pi/4$	$\rho = \csc\phi$	$\rho = 1$	$\rho = \cos\phi \csc^2\phi$	$\rho^2 = -\sec 2\phi$

Example 3 Find equations of the paraboloid $z = x^2 + y^2$ in cylindrical and spherical coordinates.

Solution. The rectangular-to-cylindrical conversion formulas in Table 12.8.1 yield

$$z = r^2 \tag{4}$$

which is the equation in cylindrical coordinates. Now applying the spherical-to-cylindrical conversion formulas to (4) yields

$$\rho \cos\phi = \rho^2 \sin^2\phi$$

which we can rewrite as

$$\rho = \cos\phi \csc^2\phi$$

Alternatively, we could have obtained this equation directly from the equation in rectangular coordinates by applying the spherical-to-rectangular conversion formulas (verify). ◀

FOR THE READER. Confirm that the equations for the cylinder and hyperboloid in cylindrical and spherical coordinates given in Table 12.8.2 are correct.

SPHERICAL COORDINATES IN NAVIGATION

Spherical coordinates are related to longitude and latitude coordinates used in navigation. To see why this is so, let us construct a right-hand rectangular coordinate system with its origin at the center of the Earth, its positive z-axis passing through the North Pole, and its positive x-axis passing through the prime meridian (Figure 12.8.7). If we assume the Earth to be a sphere of radius $\rho = 4000$ miles, then each point on the Earth has spherical coordinates of the form $(4000, \theta, \phi)$, where ϕ and θ determine the latitude and longitude of the point. It is common to specify longitudes in degrees east or west of the prime meridian

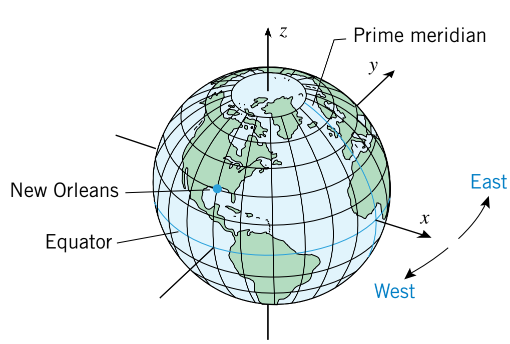


Figure 12.8.7

and latitudes in degrees north or south of the equator. However, the next example shows that it is a simple matter to determine ϕ and θ from such data.

Example 4 The city of New Orleans is located at 90° west longitude and 30° north latitude. Find its spherical and rectangular coordinates relative to the coordinate axes of Figure 12.8.7. (Assume that distance is in miles.)

Solution. A longitude of 90° west corresponds to $\theta = 360° - 90° = 270°$ or $\theta = 3\pi/2$ radians; and a latitude of 30° north corresponds to $\phi = 90° - 30° = 60°$ or $\phi = \pi/3$ radians. Thus, the spherical coordinates (ρ, θ, ϕ) of New Orleans are $(4000, 3\pi/2, \pi/3)$.

To find the rectangular coordinates we apply the spherical-to-rectangular conversion formulas in Table 12.8.1. This yields

$$x = 4000 \sin\frac{\pi}{3}\cos\frac{3\pi}{2} = 4000\frac{\sqrt{3}}{2}(0) = 0 \text{ mi}$$

$$y = 4000 \sin\frac{\pi}{3}\sin\frac{3\pi}{2} = 4000\frac{\sqrt{3}}{2}(-1) = -2000\sqrt{3} \text{ mi}$$

$$z = 4000 \cos\frac{\pi}{3} = 4000\left(\frac{1}{2}\right) = 2000 \text{ mi}$$

◀

EXERCISE SET 12.8 Graphing Utility C CAS

In Exercises 1 and 2, convert from rectangular to cylindrical coordinates.

1. (a) $(4\sqrt{3}, 4, -4)$ (b) $(-5, 5, 6)$
(c) $(0, 2, 0)$ (d) $(4, -4\sqrt{3}, 6)$

2. (a) $(\sqrt{2}, -\sqrt{2}, 1)$ (b) $(0, 1, 1)$
(c) $(-4, 4, -7)$ (d) $(2, -2, -2)$

In Exercises 3 and 4, convert from cylindrical to rectangular coordinates.

3. (a) $(4, \pi/6, 3)$ (b) $(8, 3\pi/4, -2)$
(c) $(5, 0, 4)$ (d) $(7, \pi, -9)$

4. (a) $(6, 5\pi/3, 7)$ (b) $(1, \pi/2, 0)$
(c) $(3, \pi/2, 5)$ (d) $(4, \pi/2, -1)$

In Exercises 5 and 6, convert from rectangular to spherical coordinates.

5. (a) $(1, \sqrt{3}, -2)$ (b) $(1, -1, \sqrt{2})$
(c) $(0, 3\sqrt{3}, 3)$ (d) $(-5\sqrt{3}, 5, 0)$

6. (a) $(4, 4, 4\sqrt{6})$ (b) $(1, -\sqrt{3}, -2)$
(c) $(2, 0, 0)$ (d) $(\sqrt{3}, 1, 2\sqrt{3})$

In Exercises 7 and 8, convert from spherical to rectangular coordinates.

7. (a) $(5, \pi/6, \pi/4)$ (b) $(7, 0, \pi/2)$
(c) $(1, \pi, 0)$ (d) $(2, 3\pi/2, \pi/2)$

8. (a) $(1, 2\pi/3, 3\pi/4)$ (b) $(3, 7\pi/4, 5\pi/6)$
(c) $(8, \pi/6, \pi/4)$ (d) $(4, \pi/2, \pi/3)$

In Exercises 9 and 10, convert from cylindrical to spherical coordinates.

9. (a) $(\sqrt{3}, \pi/6, 3)$ (b) $(1, \pi/4, -1)$
(c) $(2, 3\pi/4, 0)$ (d) $(6, 1, -2\sqrt{3})$

10. (a) $(4, 5\pi/6, 4)$ (b) $(2, 0, -2)$
(c) $(4, \pi/2, 3)$ (d) $(6, \pi, 2)$

In Exercises 11 and 12, convert from spherical to cylindrical coordinates.

11. (a) $(5, \pi/4, 2\pi/3)$ (b) $(1, 7\pi/6, \pi)$
(c) $(3, 0, 0)$ (d) $(4, \pi/6, \pi/2)$

12. (a) $(5, \pi/2, 0)$ (b) $(6, 0, 3\pi/4)$
(c) $(\sqrt{2}, 3\pi/4, \pi)$ (d) $(5, 2\pi/3, 5\pi/6)$

C **13.** Use a CAS or a programmable calculating utility to set up the conversion formulas in Table 12.8.1, and then use the CAS or calculating utility to solve the problems in Exercises 1, 3, 5, 7, 9, and 11.

C **14.** Use a CAS or a programmable calculating utility to set up the conversion formulas in Table 12.8.1, and then use the CAS or calculating utility to solve the problems in Exercises 2, 4, 6, 8, 10, and 12.

In Exercises 15–22, an equation is given in cylindrical coordinates. Express the equation in rectangular coordinates and sketch the graph.

15. $r = 3$ **16.** $\theta = \pi/4$ **17.** $z = r^2$
18. $z = r\cos\theta$ **19.** $r = 4\sin\theta$ **20.** $r = 2\sec\theta$
21. $r^2 + z^2 = 1$ **22.** $r^2\cos 2\theta = z$

In Exercises 23–30, an equation is given in spherical coordinates. Express the equation in rectangular coordinates and sketch the graph.

23. $\rho = 3$ **24.** $\theta = \pi/3$
25. $\phi = \pi/4$ **26.** $\rho = 2\sec\phi$
27. $\rho = 4\cos\phi$ **28.** $\rho\sin\phi = 1$
29. $\rho\sin\phi = 2\cos\theta$ **30.** $\rho - 2\sin\phi\cos\theta = 0$

In Exercises 31–42, an equation of a surface is given in rectangular coordinates. Find an equation of the surface in (a) cylindrical coordinates and (b) spherical coordinates.

31. $z = 3$ **32.** $y = 2$
33. $z = 3x^2 + 3y^2$ **34.** $z = \sqrt{3x^2 + 3y^2}$
35. $x^2 + y^2 = 4$ **36.** $x^2 + y^2 - 6y = 0$
37. $x^2 + y^2 + z^2 = 9$ **38.** $z^2 = x^2 - y^2$
39. $2x + 3y + 4z = 1$ **40.** $x^2 + y^2 - z^2 = 1$
41. $x^2 = 16 - z^2$ **42.** $x^2 + y^2 + z^2 = 2z$

In Exercises 43–46, describe the region in 3-space that satisfies the given inequalities.

43. $r^2 \le z \le 4$

44. $0 \le r \le 2\sin\theta, \quad 0 \le z \le 3$

45. $1 \le \rho \le 3$

46. $0 \le \phi \le \pi/6, \quad 0 \le \rho \le 2$

47. St. Petersburg (Leningrad), Russia, is located at 30° east longitude and 60° north latitude. Find its spherical and rectangular coordinates relative to the coordinate axes of Figure 12.8.7. Take miles as the unit of distance and assume the Earth to be a sphere of radius 4000 miles.

48. (a) Show that the curve of intersection of the surfaces $z = \sin\theta$ and $r = a$ (cylindrical coordinates) is an ellipse.
(b) Sketch the surface $z = \sin\theta$ for $0 \le \theta \le \pi/2$.

49. The accompanying figure shows a right circular cylinder of radius 10 cm spinning at 3 revolutions per minute about the z-axis. At time $t = 0$ s, a bug at the point (0, 10, 0) begins walking straight up the face of the cylinder at the rate of 0.5 cm/min.
(a) Find the cylindrical coordinates of the bug after 2 min.
(b) Find the rectangular coordinates of the bug after 2 min.
(c) Find the spherical coordinates of the bug after 2 min.

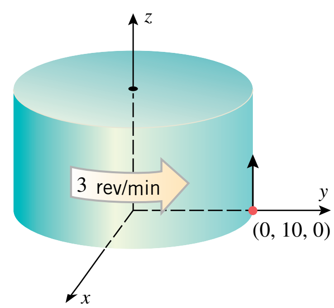


Figure Ex-49

50. Referring to Exercise 49, use a graphing utility to graph the bug's distance from the origin as a function of time.

51. A ship at sea is at point A that is 60° west longitude and 40° north latitude. The ship travels to point B that is 40° west longitude and 20° north latitude. Assuming that the Earth is a sphere with radius 6370 kilometers, find the shortest distance the ship can travel in going from A to B, given that the shortest distance between two points on a sphere is along the arc of the great circle joining the points. [*Suggestion:* Introduce an xyz-coordinate system as in Figure 12.8.7, and consider the angle between the vectors from the center of the Earth to the points A and B. If you are not familiar with the term "great circle," consult a dictionary.]

SUPPLEMENTARY EXERCISES

1. (a) What is the difference between a vector and a scalar? Give a physical example of each.
(b) How can you determine whether or not two vectors are orthogonal?
(c) How can you determine whether or not two vectors are parallel?
(d) How can you determine whether or not three vectors with a common initial point lie in the same plane in 3-space?

2. (a) Sketch vectors **u** and **v** for which **u** + **v** and **u** − **v** are orthogonal.
(b) How can you use vectors to determine whether four points in 3-space lie in the same plane?
(c) If forces $\mathbf{F}_1 = \mathbf{i}$ and $\mathbf{F}_2 = \mathbf{j}$ are applied at a point in 2-space, what force would you apply at that point to cancel the combined effect of $\mathbf{F}_1$ and $\mathbf{F}_2$?
(d) Write an equation of the sphere with center (1, −2, 2) that passes through the origin.

3. (a) Draw a picture that shows the direction angles α, β, and γ of a vector.
(b) What are the components of a unit vector in 2-space that makes an angle of 120° with the positive x-axis (two answers)?
(c) How can you use vectors to determine whether a triangle with known vertices P_1, P_2, and P_3 has an obtuse angle?
(d) True or false: The cross product of orthogonal unit vectors is a unit vector. Explain your reasoning.

4. (a) Make a table that shows all possible cross products of the vectors **i**, **j**, and **k**.
(b) Give a geometric interpretation of $\|\mathbf{u} \times \mathbf{v}\|$.
(c) Give a geometric interpretation of $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$.
(d) Write an equation of the plane that passes through the origin and is perpendicular to the line $x = t$, $y = 2t$, $z = -t$.

5. (a) List the six basic types of quadric surfaces, and describe their traces in planes parallel to the coordinate planes.
(b) Give the coordinates of the points that result when the point (x, y, z) is reflected about the plane $y = x$, the plane $y = z$, and the plane $x = z$.
(c) Describe the intersection of the surfaces $r = 5$ and $z = 1$ in cylindrical coordinates.
(d) Describe the intersection of the surfaces $\phi = \pi/4$ and $\theta = 0$ in spherical coordinates.

6. In each part, find an equation of the sphere with center $(-3, 5, -4)$ and satisfying the given condition.
(a) Tangent to the xy-plane
(b) Tangent to the xz-plane
(c) Tangent to the yz-plane

7. (a) Find the area of the triangle with vertices $A(1, 0, 1)$, $B(0, 2, 3)$, and $C(2, 1, 0)$.
(b) Use the result in part (a) to find the length of the altitude from vertex C to side AB.

8. Find the largest and smallest distances between the point $P(1, 1, 1)$ and the sphere

$$x^2 + y^2 + z^2 - 2y + 6z - 6 = 0$$

9. Let $\mathbf{a} = c\mathbf{i} + \mathbf{j}$ and $\mathbf{b} = 4\mathbf{i} + 3\mathbf{j}$. Find c so that
(a) **a** and **b** are orthogonal
(b) the angle between **a** and **b** is $\pi/4$
(c) the angle between **a** and **b** is $\pi/6$
(d) **a** and **b** are parallel.

10. Given the points $P(3, 4)$, $Q(1, 1)$, and $R(5, 2)$, use vector methods to find the coordinates of the fourth vertex of the parallelogram whose adjacent sides are $\overrightarrow{PQ}$ and $\overrightarrow{QR}$.

11. Let $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$ and $\mathbf{r} = \langle x, y, z \rangle$. Describe the set of all points (x, y, z) for which
(a) $\mathbf{r} \cdot \mathbf{r}_0 = 0$ (b) $(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{r}_0 = 0$.

12. What condition must the constants satisfy for the planes

$$a_1x + b_1y + c_1z = d_1 \quad \text{and} \quad a_2x + b_2y + c_2z = d_2$$

to be perpendicular?

13. Let A, B, C, and D be four distinct points in 3-space. Explain why the line through A and B must intersect the line through C and D if $\overrightarrow{AB} \times \overrightarrow{CD} \neq \mathbf{0}$ and $\overrightarrow{AC} \cdot (\overrightarrow{AB} \times \overrightarrow{CD}) = 0$.

14. Let A, B, and C be three distinct noncollinear points in 3-space. Describe the set of all points P that satisfy the vector equation $\overrightarrow{AP} \cdot (\overrightarrow{AB} \times \overrightarrow{AC}) = 0$.

15. True or false? Explain your reasoning.
(a) If $\mathbf{u} \cdot \mathbf{v} = 0$, then $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$.
(b) If $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, then $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$.
(c) If $\mathbf{u} \cdot \mathbf{v} = 0$ and $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, then $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$.

16. In each part, use the result in Exercise 39 of Section 12.4 to prove the vector identity.
(a) $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \times \mathbf{b} \cdot \mathbf{d})\mathbf{c} - (\mathbf{a} \times \mathbf{b} \cdot \mathbf{c})\mathbf{d}$
(b) $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} + (\mathbf{b} \times \mathbf{c}) \times \mathbf{a} + (\mathbf{c} \times \mathbf{a}) \times \mathbf{b} = \mathbf{0}$

17. Show that if **u** and **v** are unit vectors and θ is the angle between them, then $\|\mathbf{u} - \mathbf{v}\| = 2 \sin \frac{1}{2}\theta$.

18. Consider the points

$$A(1, -1, 2),\ B(2, -3, 0),\ C(-1, -2, 0),\ D(2, 1, -1)$$

(a) Find the volume of the parallelepiped that has the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$ as adjacent edges.
(b) Find the distance from D to the plane containing A, B, and C.

19. (a) Find parametric equations for the intersection of the planes $2x + y - z = 3$ and $x + 2y + z = 3$.
(b) Find the acute angle between the two planes.

20. A diagonal of a box makes angles of 50° and 70° with two of its edges. Find to the nearest degree the angle that it makes with the third edge.

21. Find the vector with length 5 and direction angles $\alpha = 60^\circ$, $\beta = 120^\circ$, $\gamma = 135^\circ$.

22. The accompanying figure shows a cube.
(a) Find the angle between the vectors **d** and **u** to the nearest degree.
(b) Make a conjecture about the angle between the vectors **d** and **v**, and confirm your conjecture by computing the angle.

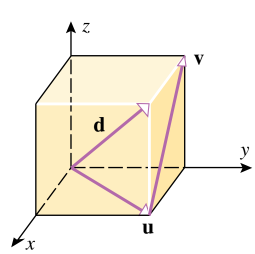


Figure Ex-22

23. In each part, identify the surface by completing the squares.
(a) $x^2 + 4y^2 - z^2 - 6x + 8y + 4z = 0$
(b) $x^2 + y^2 + z^2 + 6x - 4y + 12z = 0$
(c) $x^2 + y^2 - z^2 - 2x + 4y + 5 = 0$

24. In Exercise 42 of Section 12.5 we defined the symmetric equations of a line in 3-space. Consider the lines L_1 and L_2 whose symmetric equations are

$$L_1\colon \frac{x-1}{2} = \frac{y+\frac{3}{2}}{1} = \frac{z+1}{2}$$
$$L_2\colon \frac{x-4}{-1} = \frac{y-3}{-2} = \frac{z+4}{2}$$

(a) Are L_1 and L_2 parallel? Perpendicular?
(b) Find parametric equations for L_1 and L_2.
(c) Do L_1 and L_2 intersect? If so, where?

25. In each part, express the equation in cylindrical and spherical coordinates.
(a) $x^2 + y^2 = z$ (b) $x^2 - y^2 - z^2 = 0$

26. In each part, express the equation in rectangular coordinates.
(a) $z = r^2 \cos 2\theta$ (b) $\rho^2 \sin\phi \cos\phi \cos\theta = 1$

In Exercises 27 and 28, sketch the solid in 3-space that is described in spherical coordinates by the stated inequalities.

27. (a) $0 \le \rho \le 2$ (b) $0 \le \phi \le \pi/6$
(c) $0 \le \rho \le 2$ and $0 \le \phi \le \pi/6$

28. (a) $0 \le \rho \le 5$, $0 \le \phi \le \pi/2$, and $0 \le \theta \le \pi/2$
(b) $0 \le \phi \le \pi/3$ and $0 \le \rho \le 2\sec\phi$
(c) $0 \le \rho \le 2$ and $\pi/6 \le \phi \le \pi/3$

In Exercises 29 and 30, sketch the solid in 3-space that is described in cylindrical coordinates by the stated inequalities.

29. (a) $1 \le r \le 2$ (b) $2 \le z \le 3$ (c) $\pi/6 \le \theta \le \pi/3$
(d) $1 \le r \le 2$, $2 \le z \le 3$, and $\pi/6 \le \theta \le \pi/3$

30. (a) $r^2 + z^2 \le 4$ (b) $r \le 1$ (c) $r^2 + z^2 \le 4$ and $r > 1$

31. (a) The accompanying figure shows a surface of revolution that is generated by revolving the curve $y = f(x)$ in the xy-plane about the x-axis. Show that the equation of this surface is $y^2 + z^2 = [f(x)]^2$. [*Hint:* Each point on the curve traces a circle as it revolves about the x-axis.]
(b) Find an equation of the surface of revolution that is generated by revolving the curve $y = e^x$ in the xy-plane about the x-axis.
(c) Show that the ellipsoid $3x^2 + 4y^2 + 4z^2 = 16$ is a surface of revolution about the x-axis by finding a curve $y = f(x)$ in the xy-plane that generates it.

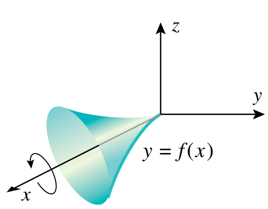


Figure Ex-31

32. In each part, use the idea in Exercise 31(a) to derive a formula for the stated surface of revolution.
(a) The surface generated by revolving the curve $x = f(y)$ in the xy-plane about the y-axis.
(b) The surface generated by revolving the curve $y = f(z)$ in the yz-plane about the z-axis.
(c) The surface generated by revolving the curve $z = f(x)$ in the xz-plane about the x-axis.

33. Sketch the surface whose equation in spherical coordinates is $\rho = a(1 - \cos\phi)$. [*Hint:* The surface is shaped like a familiar fruit.]

34. Assuming that force is in pounds and distance is in feet, find the work done by a constant force $\mathbf{F} = 3\mathbf{i} - 4\mathbf{j} + \mathbf{k}$ acting on a particle that moves on a straight line from $P(5, 7, 0)$ to $Q(6, 6, 6)$.

35. Assuming that force is in newtons and distance is in meters, find the work done by the resultant of the constant forces $\mathbf{F}_1 = \mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $\mathbf{F}_2 = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ acting on a particle that moves on a straight line from $P(-1, -2, 3)$ to $Q(0, 2, 0)$.

36. As shown in the accompanying figure, a force of 250 N is applied to a boat at an angle of 38° with the positive x-axis. What force $\mathbf{F}$ should be applied to the boat to produce a resultant force of 1000 N acting in the positive x-direction? State your answer by giving the magnitude of the force and its angle with the positive x-axis to the nearest degree.

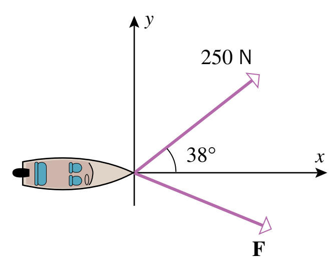


Figure Ex-36

37. Suppose that a force $\mathbf{F}$ with a magnitude of 9 lb is applied to the lever–shaft assembly shown in the accompanying figure.
(a) Express the force $\mathbf{F}$ in component form.
(b) Find the vector moment of $\mathbf{F}$ about the origin.

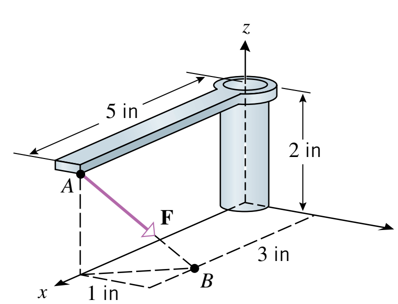


Figure Ex-37