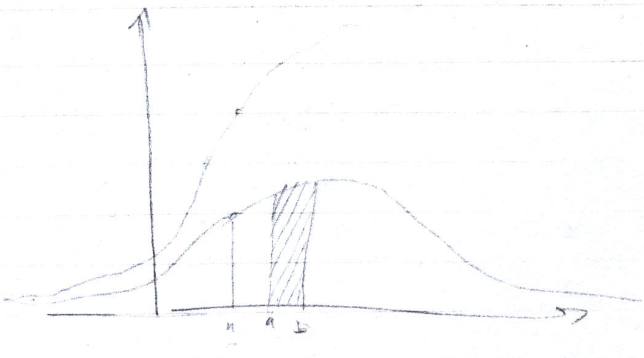
Aula 19/03/2013



- $f(x) > 0$ ✓
- $\int_{-\infty}^{+\infty} f(x)\,dx = 1$

$E(X) = \int_{-\infty}^{+\infty} x f(x)\,dx$

$V(X) = E(X^2) - E^2(X)$

$x^2 \quad 2x$

Exemplo

1) $\int_{-\infty}^{+\infty} f(x)\,dx = 1$

$\Leftrightarrow \int_0^1 k - kx\,dx \Leftrightarrow \left[kx - \frac{kx^2}{2}\right]_0^1 = \frac{k}{1} - \frac{k}{2} - 0$

$P(X > 0{,}3 \mid X < 0{,}5) = \frac{P(0{,}3 < x < 0{,}5)}{P(X < 0{,}5)}$

$= \frac{\int_{0,3}^{0,5} 2(1-x)\,dx}{\int_0^{0,5} 2(1-x)\,dx}$

$P(X < t) = P(X \leq t) = \int_{-\infty}^{t} f(x)\,dx = 0{,}5$

66 a) $\int_{-\infty}^{+\infty} f_X(x)\,dx = 1 \Leftrightarrow \int_0^1 a\,dx + \int_2^4 \frac{1}{4}\,dx = 1$

$\Leftrightarrow a + \left(1 - \frac{1}{2}\right) = 1 \Leftrightarrow a = \frac{1}{2}$

b) $P(1 < x \leq 3) = P(1 \leq X \leq 3)$

$P(1 < X < 3)$

$= \int_1^3 f_X(x)\,dx$

$(x^2)' = 2x$

$= \int_1^2 0\,dx + \int_2^3 \frac{1}{4}\,dx = \frac{1}{4}$

c) $E(x) = \int_{-\infty}^{+\infty} x f_X(x)\,dx$

$= \int_0^1 \frac{x}{2}\,dx + \int_2^4 \frac{x}{4} = \left(\frac{1}{4}\int_0^1 2x\,dx + \right) = \left[\frac{x^2}{4}\right]_0^1 + \left[\frac{x^2}{8}\right]_2^4 = \frac{1}{4} + 2 - \frac{1}{2}$

$= \frac{7}{4}$

$$V(X) = E(X^2) - E^2(X) = \frac{29}{6_{(\times 8)}} - \frac{49}{16_{(\times 3)}} = \frac{232 - 147}{48} = \frac{85}{48}$$

C.A.:

$$E(X^2) = \int_{-\infty}^{+\infty} x^2 f_X(x)\,dx$$

$$\sigma(x) = \sqrt{\frac{85}{48}} \simeq 1{,}3331$$

$$= \int_0^1 \frac{x^2}{2}\,dx + \int_2^4 \frac{x^2}{4}\,dx$$

$$= \left[\frac{x^3}{6}\right]_0^1 + \left[\frac{x^3}{12}\right]_2^4 = \frac{1}{6} + \frac{16}{3} - \frac{2}{3} \cdot \frac{2}{3} = \frac{29}{6}$$

(67)

$$f_X(x) = \begin{cases} 0 & , x < a \\ \frac{24}{x^4} & , x \geq a \end{cases}$$

$$\int_{-\infty}^{+\infty} f_X(x)\,dx = 1 \Rightarrow \cancel{\int_{-\infty}^{a} 0\,dx} + \int_a^{+\infty} \frac{24}{x^4}\,dx = 1$$

$$(=) \left[-\frac{8}{x^3}\right]_a^{+\infty} = 1 \;(=)\; \frac{8}{a^3} = 1 \;(=)\; a = 2$$

a) i) $P(X>10) = \int_{10}^{+\infty} f_X(x)\,dx = \left[-\frac{8}{x^3}\right]_{10}^{+\infty} = \frac{8}{10^3} = 0{,}008$

ii) $P(4 < X < 5) = \int_4^5 f_X(x)\,dx = \left[\frac{-8}{x^3}\right]_4^5 = \frac{-8}{5^3} \cancel{+} \frac{8}{4^3} = \frac{-8}{125} + \frac{8}{64}$

b) $E(X) = \int_{-\infty}^{+\infty} x\, f_X(x)\,dx = \int_2^{+\infty} \frac{24}{x^3} = \left[-\frac{12}{x^2}\right]_2^{+\infty} = 3$

$$V(X) = E(X^2) - E^2(X) = 12 - 9 = 3$$

$$E(X^2) = \int_{-\infty}^{+\infty} x^2 f_X(x)\,dx = \int_2^{+\infty} \frac{24}{x^2}\,dx = \left[-\frac{24}{x}\right]_2^{+\infty} = 12$$

$$\sigma(X) = \sqrt{V(X)} = \sqrt{3} \simeq 1{,}7\ldots$$

d) i) $\bar{X} = \frac{X_1 + \ldots + X_{100}}{100} = \frac{\sum_{i=1}^{100} X_i}{100}$

ii) $E(\bar{X}) = E\left(\frac{1}{100}\sum_{i=1}^{100} X_i\right) = \frac{1}{100} E\left(\sum_{i=1}^{100} X_i\right) = \frac{1}{100}\sum_{i=1}^{100} E(X_i) = \frac{1}{100}\sum_{i=1}^{100} 3 = \frac{300}{100} = 3$

$$V(\bar{X}) = V\left(\frac{1}{100}\sum_{i=1}^{100} X_i\right) = \frac{1}{10000} V\left(\sum_{i=1}^{100} X_i\right)$$ Como X_i's são independentes

$$= \frac{1}{10000}\sum_{i=1}^{100} V(X_i)$$

$$= \frac{1}{10000}\sum_{i=1}^{100} 3 = \frac{300}{10000} = 0{,}03 = \frac{3}{100} = \frac{V(X)}{100}$$

Aula 26/3/2012

(71)

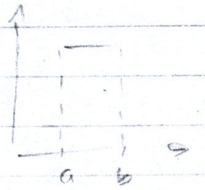
a) $\int_{-\infty}^{+\infty} f_X(x)\,dx = 1 \Leftrightarrow a = 1/5$

b) $E(X) = \int_{-\infty}^{+\infty} x\, f_X(x)\, dx = 5/2$

c) $V(X) = E(X^2) - E^2(X) = \frac{25}{3} - \frac{25}{4} = \frac{25}{12}$

$E(X^2) = \int_{-\infty}^{+\infty} x^2 f_X(x)\,dx = \frac{25}{3}$

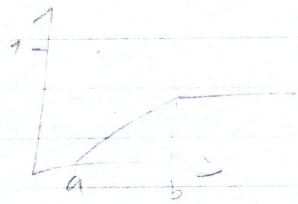
Nota: $f_X(x) \begin{cases} \frac{1}{b-a}, & x \in [a,b] \\ 0, & x \notin [a,b] \end{cases}$ $\quad \frac{1}{b-a}$



$E(X) = \frac{a+b}{2}$

$V(X) = \frac{(b-a)^2}{12}$ $\quad U(a,b)$

$F(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a < x < b \\ 1, & x \geq b \end{cases}$



· Números aleatórios

$x_0 = 5$

$(a x_{i-1} + c) \bmod m$

$(5 x_{i-1} + 7) \bmod 8$

$n = 1 \rightarrow x_1 = (5\times 5 + 7) \bmod 8 = 32 \bmod 8 = 0$

$x_2 = (5\times 0 + 7) \bmod 8 = 7 \bmod 8 = 7$

$F^{-1}(u) = u = \frac{x^2}{2} \Leftrightarrow 2u = x^2 \Leftrightarrow x = \sqrt{2y}$

(82)

a) Será nec. encontrar a função inversa de $F_X(x)$

$y = 1 - e^{-Kx^\beta} \Leftrightarrow y = 1 - e^{-Kx^\beta} \Leftrightarrow y = 1 - e^{-Kx^\beta} \Leftrightarrow$

$\Leftrightarrow 1 - y = e^{-Kx^\beta} \Leftrightarrow \ln(1-y) = -K x^\beta \Leftrightarrow$

$\Leftrightarrow -\ln(1-y) = K x^\beta \Leftrightarrow \frac{-\ln(1-y)}{K} = x^\beta \Leftrightarrow$

$\Leftrightarrow x = \sqrt[\beta]{\frac{-\ln(1-y)}{K}}$

• Tendo os valores de u gerados aleatoriamente de $U(0,1)$

$$\boxed{x = 5\sqrt{\frac{-\ln(1-u)}{3}}}$$

$x_1 = 5\sqrt{\frac{-\ln(1-0,7517)}{3}} = 0,6267$

(23) a) $F_X(x) = \begin{cases} 0 & , x < 0 \\ 0,4 & , 0 \leq x < 1 \\ 0,6 & , 1 \leq x < 2 \\ 0,7 & , 2 \leq x < 3 \\ 1 & , x \geq 3 \end{cases}$

$w = \begin{cases} 0, & 0 \leq u < 0,4 \\ 1, & 0,4 \leq u \leq 0,6 \\ 2, & 0,6 \leq u < 0,7 \\ 3, & 0,7 \leq u \leq 1 \end{cases}$

ii)

$E(L) = \int_{-\infty}^{+\infty} \alpha\, f_X(x)\, dx =$

$\int_0^5 \alpha \frac{1}{5} dx = \int_0^y 400x - 100(y-x) dx + \int_y^5 400 x\, dx =$

$= \ldots = 2000y - 250y^2$

iii) $E(L(X,y)) = 2000y - 250y^2$

$\frac{\partial}{\partial y} E(L(x,y)) = 0 \Leftrightarrow$

$\Leftrightarrow 2000 - 500y = 0 \Leftrightarrow y = 4$//

b) 3 2 0 1 3 0 1

i) $L(x,y) = \begin{cases} 2000x - 500(y-x) & , x < y \\ 2000x & , x \geq y \end{cases}$

2/4/2013

Binomial

$n \rightarrow$ nº de experiências

$p \rightarrow$ prob. de sucesso

$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$

$E(X) = np$

$V(X) = np(1-p)$

$X \sim Bin(n, p)$

Hipergeométrica

$N \rightarrow$ população

$M \rightarrow$ característica

$n \rightarrow$ amostra

$P(X=k) = \frac{\binom{M}{k}\binom{N-M}{n-k}}{\binom{N}{n}}$

$\max(0, M+n-N) \leq k \leq \min(M, n)$

$E(X) = \frac{nM}{N}$; $V(X) = \frac{nM}{N^2(N-1)}(N-M)(N-n)$

$X \sim H(N, M, n)$

Uniforme

1	2	...	m
$\frac{1}{m}$	$\frac{1}{m}$	...	$\frac{1}{m}$

$E(X) = \frac{m+1}{2}$

$V(X) = \frac{m^2-1}{12}$

$X \sim U(m)$

(54) a) $X \sim H(N, M, n)$

$P(X=k) = \frac{\binom{M}{k}\binom{N-M}{n-k}}{\binom{N}{n}}$

Com:

$N = 20$ (nº chips no lote)

$M = 4$ (nº defeituosos no lote)

$n = 5$ (nº chips retirados) , $0 \leq k \leq 4$

b) $E(X) = \frac{nM}{N} \Leftrightarrow \frac{5 \times 4}{20} = 1$; c) $V(X) = \frac{nM}{N^2(N-1)}(N-M)(N-n) = \frac{5 \times 4}{20^2(20-1)}(20-4)(20-5) = \frac{16 \times 15}{\ldots \times 19}$ ...

b) Y = "nº chamadas (recepção) em 30 min"

$Y \sim P(3 \times 1{,}5) \equiv P(4{,}5)$

$P(Y=y) = \frac{e^{-4,5}\, 4{,}5^{y}}{y!}$

ii) $P(Y<2) = P(Y=0) + P(Y=1)$

$= e^{-4,5} + 4{,}5\, e^{-4,5}$

$= 5{,}5\, e^{-4,5} = \ldots$

$E(Y) = 4{,}5 = V(Y)$

$\sigma(Y) = \sqrt{4{,}5} \simeq \ldots$

c) Seja Z = "nº chamadas recebidas pelo director em 30 mins"

Temos que: $Z \sim P(2)$

Seja $S = Z + Y$ como Z e Y são independentes e ambas seguem dist. de Poisson, então, pelo teorema da aditividade da Poisson:

$S \sim P(2 + 4{,}5) \equiv P(6{,}5)$

(54) Y = "nº empregados em 100 que contem 1 peça num dia"

$Y \sim Bin(100; 0{,}005)$

Como $n = 100 \geq 50 \wedge np = 0{,}5 \leq 5$

então, $Bin(100; 0{,}005) \overset{d}{\simeq} P(0{,}5)$

...

(62) $X \sim P(2)$

capacidade = 3/dia

a) $P(X>3) = 1 - P(X \leq 3)$

$= 1 - P(X=0) - P(X=1) - P(X=2) - P(X=3)$

b) $P(X \leq 4) = 0{,}857123 + \underbrace{P(X=4)}_{0,0902} = 0{,}940323$

c) $E(X) = 2$

d) Y = "nº automóveis atendidos por dia"

Y	0	1	2	3
	$P(X=0)$	$P(X=1)$	$P(X=2)$	$P(X\geq 3)$
	0,1353	0,2707	0,2707	0,3233

e) $E(Y) = \sum_{i=1}^{4} x_i \cdot P(Y=x_i)$

$= 0 \times 0{,}1353 + \ldots + 3 \times 0{,}3233 \simeq 1{,}782$

f) $E(X-Y) = E(X) - E(Y) = 0{,}218$

g) Z = "nº automóveis que chegam em 5 dias"

$Z \sim P(5 \times 2) \equiv P(10)$

$P(Z=9) = \frac{e^{-10}\, 10^{9}}{9!} = 0{,}125110$

Aula 9/4/2013

$$f_X(x) = \begin{cases} 0 & , x < \lambda \\ \frac{1}{\delta} e^{-\frac{x-\lambda}{\delta}} & , x \geq \lambda \end{cases}$$

logo :

$$F_X(x) \begin{cases} 0 & , x < \lambda \\ 1 - e^{-\frac{x-\lambda}{\delta}} & , x \geq \lambda \end{cases}$$

$E(X) = \lambda + \delta$
$V(X) = \delta^2$
$E(\lambda, \delta)$
Prop falta de memória
$P(X \geq s+t \mid X \geq s) = P(X \geq t)$

Como
$\lambda = 0$ e $\delta = 1000$

Então
$$F_X(x) = \begin{cases} 0 & , x < 0 \\ 1 - e^{-\frac{x}{1000}} & , x \geq 0 \end{cases}$$

$X \sim N(0,1)$

como tab.

$P(X \leq 0{,}25) = \Phi(0{,}25) = 0{,}5987$

$P(X > -0{,}2) = P(X < 0{,}2) = P(X \leq 0{,}2) = \Phi(0{,}2) = 0{,}5793$

$P(X \leq -0{,}25) = 1 - P(X > -0{,}25)$
ou $= P(X \geq 0{,}25)$
$= 1 - P(X < 0{,}25)$
$= 1 - P(X \leq 0{,}25) = 1 - \Phi(0{,}25)$

73
75
79
80

$P(X > 0{,}2) \overset{?}{=} 1 - P(X \leq 0{,}2)$

$P(0{,}1 \leq X \leq 0{,}2) = \Phi(0{,}2) - \Phi(0{,}1)$

$$\boxed{Z = \frac{X - \mu}{\sigma} \sim N(0,1)}$$

Exemplo: $X \sim N(2,4)$ $\rightarrow Z = \frac{X-2}{2} \sim N(0,1)$

μ $\sigma = 2$

$P(X > 3) = P\left(\frac{X-2}{2} > \frac{3-2}{2}\right)$
$= P\left(Z > \frac{1}{2}\right)$

73 a)

b) $P(X < 1000) = F(1000^-) = 1 - e^{-1} = 0{,}632$

c) $P(1500 \leq X < 2000)$
$= F(2000^-) - F(1500^-)$
$= e^{-1{,}5} - e^{-2}$
$\simeq 0{,}0878$

d) $E(X) = \lambda + \delta = 1000$

$V(X) = \delta^2 = 1000^2$

$\left\{ \begin{array}{cc} -1 & 4 \\ 0{,}632 & 0{,}368 \end{array} \right.$

e) $E(L) = 0{,}632 \times (-1) + \overbrace{(1-0{,}632)}^{0{,}368} \times 4 = \ldots$

$V(L) = E(L^2) - E^2(L) \ldots$

$\sigma(L) \ldots$

(75) $Y(t) = \text{"} \ldots \text{"}$

$Y(t) \sim P(\delta t)$

a) $P(T > t) = P(X(t) = 0)$

$= \dfrac{e^{-\delta t} \times (\delta t)^0}{0!} = e^{-\delta t}$

b) $P(T \leq t) = 1 - e^{\boxed{-\delta t}} \Rightarrow -\delta t = -\dfrac{t - \lambda}{\delta}\ (?)$

$\dfrac{t - \lambda}{\delta^{-1}} = -\dfrac{t - 0}{\delta^{-1}} \Rightarrow \lambda = 0$

$\delta = \dfrac{1}{\delta}$

$E(\lambda, \delta) = E(0, 1/\delta)$

c) $P(T > s+t \mid T > s) = \dfrac{P(T > s+t)}{P(T > s)} = \ldots \overset{??\ \text{rec}}{=} P(T > t)$

(79) X: "duração da viagem"

$X \sim N(20, 4^2)$

Por um teorema:

$Z = \dfrac{X - 20}{4} \sim N(0,1)$

$\boxed{N(\mu, \sigma^2)}$

a) $P(X > 25) = P\left(\dfrac{X-20}{4} > \dfrac{25-20}{4}\right) = P\left(Z > \dfrac{5}{4}\right) = 1 - P(Z \leq 1{,}25) = 1 - \Phi(1{,}25) =$

$= 1 - 0{,}8944 = \ldots$

b) $P(X > 15) = P\left(\dfrac{X-20}{4} > \dfrac{15-20}{4}\right) = P\left(Z > -\dfrac{5}{4}\right) = P\left(Z < \dfrac{5}{4}\right)$

$= P\left(Z \leq \dfrac{5}{4}\right) = \Phi(1{,}25)$

$= 0{,}8944$

c) $P(15 \leq X \leq 25)$

$= P(X \leq 25) - P(X < 15)$

$= P\left(\dfrac{X-20}{4} \leq \dfrac{25-20}{4}\right) - P\left(\dfrac{X-20}{4} < \dfrac{15-20}{4}\right)$

$= P\left(Z \leq \dfrac{5}{4}\right) - P\left(Z < -\dfrac{5}{4}\right)$

$= \Phi(1{,}25) - \left(1 - P\left(Z \geq -\dfrac{5}{4}\right)\right)$

$= 0{,}8944 - 1 + P\left(Z > \dfrac{5}{4}\right)$ pela alínea a)

$= 0{,}8944 - 1 + 0{,}8944 = 0{,}7888$

$P(\text{"de não conseguir chegar entre as 8:50 e as 9 h"}) = 1 - 0{,}7888 = 0{,}2112$

d) $P(X > t) = 0{,}2 \Leftrightarrow$

$\Leftrightarrow P\left(\dfrac{X-20}{4} > \dfrac{t-20}{4}\right) = 0{,}2$

$\Leftrightarrow P\left(Z > \dfrac{t-20}{4}\right) = 0{,}2$

$\Leftrightarrow P\left(Z \leq \dfrac{t-20}{4}\right) = 0{,}8$

$\Leftrightarrow \Phi^{-1}(0{,}8) = \dfrac{t-20}{4} \Leftrightarrow 0{,}84 = \dfrac{t-20}{4} \Leftrightarrow 0{,}84 \times 4 + 20 = t$

$\Leftrightarrow t = 23{,}36$

Aula Prática 11/4/2013

e) P("duas de 3 demoram + de 25 min")

$= {}^3C_2\ 0,1856^2 \times 0,8794 \cong 0,0299$

79
80
81
78

(80) X = "peso do passageiro" $\qquad X \sim N(75, c)$

Y = "peso da bagagem" $\qquad Y \sim N(15, 2^2)$

W = "peso de outra carga" $\qquad W \sim N(6000, 1000^2)$

Por um teorema,

$$Z_1 = \frac{X-75}{\sqrt{c}} \sim N(0,1)$$

$$Z_2 = \frac{Y-15}{\sqrt{c}} \sim N(0,1)$$

$$Z_3 = \frac{W-6000}{1000} \sim N(0,1)$$

a) $P(Y > 20) = P(Z_2 > 2,5)$

$= 1 - P(Z_2 \leq 2,5)$

$= 1 - \Phi(2,5)$

$= 0,0062$

b) $P(14 \leq Y \leq 20) = P(Y \leq 20) - P(Y < 14)$

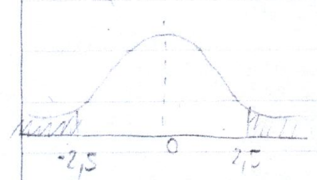
$= P(Z_2 \leq 2,5) - P(Z_2 < -0,5)$

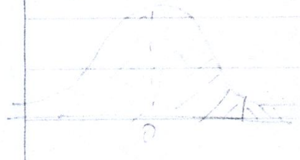
$= \Phi(2,5) - P(Z_2 > 0,5)$

$= \Phi(2,5) - (1 - P(Z_2 \leq 0,5))$

$= \Phi(2,5) + \Phi(0,5) - 1$

$= 0,6853$





c) $P(X < 72) \overset{?}{=} 0,1587$

$\Leftrightarrow P\left(Z_1 < -\frac{3}{\sqrt{c}}\right) = 0,1587$

$\Leftrightarrow P\left(Z_1 > \frac{3}{\sqrt{c}}\right) = 0,1587$

$\Leftrightarrow 1 - P\left(Z_1 \leq \frac{3}{\sqrt{c}}\right) = 0,1587$

$\Leftrightarrow P\left(Z_1 < \frac{3}{\sqrt{c}}\right) = 0,8413 \Leftrightarrow \Phi^{-1}(0,8413) = \frac{3}{\sqrt{c}}$

$\Leftrightarrow 1 = \frac{3}{\sqrt{c}}$

$\Leftrightarrow \sqrt{c} = \frac{3}{1} \Leftrightarrow c = 9$

d) Sabendo que, para $i = 1, \dots, 20$
$X_i \sim N(75, 4^2)$ são ind.,
$Y_i \sim N(15, 2^2)$ são ind,
e $W \sim N(6000, 1000^2)$ . .

então a variável que define o peso total do avião é dada por:

$$S = \sum_{i=1}^{20} X_i + \sum_{i=1}^{20} Y_i + W$$

Como as variáveis aleatórias são independentes e todas têm dist.ª normal, então, por teorema:

$$S \sim N\Big(\underbrace{\sum_{i=1}^{20} 75 + \sum_{i=1}^{20} 15 + 6000}_{7800},\ \underbrace{\sum_{i=1}^{20} 4^2 + \sum_{i=1}^{20} 2^2 + 1000^2}_{1000400}\Big)$$

Por outro teorema:

$$Z_4 = \frac{S - 7800}{\sqrt{1000400}} \sim N(0,1)$$

$\approx (0,2)$

Queremos:

$$P(S < 8000) = P\left(\frac{S - 7800}{\sqrt{1000400}} < \frac{200}{\sqrt{1000400}}\right) = P(Z_4 < 0,2) = \Phi(0,2)$$
$$= 0,5793$$

//

(81) A = "diâmetro das esferas"
$A \sim N(5; 0,1^2)$
Exigência: $4,9 \leq A \leq 5,1$

Prejuízo sucata: -1 u.m.
Lucro venda : 2 u.m.

a) $P(\text{"vendida"}) = P(4,9 \leq A \leq 5,1) = \dots$ (similar ao 80b) ⇒ por teorema: $Z = \frac{A-5}{0,1} \sim N(0,1)$ etc) $\dots = 68,26\ \%$

$P(\text{"sucata"}) = (100 - 68,26)\% = 31,74\%$

b) Se produzir K esferas, o número de esferas que satisfazem o requisito é $K \times 0,6826 = m \Leftrightarrow K = \frac{m}{0,6826}$

c) Lucro: $2 \times m - 1 \times (K - m) = 2m - \left(\frac{m}{0,6826} - m\right)$

$= \dots =$

$= 1,535\, m$

d)

$$L(A) \begin{cases} -1 & 2 \\ 0{,}3174 & 0{,}6826 \end{cases}$$

$$E(L(A)) = -1 \times 0{,}3174 + 2 \times 0{,}6826$$
$$= 1{,}0478$$

B: "diâmetro da esfera dp do aperfeiçoamento"

$B \sim N(5,\ 0{,}05^2)$

$P(4{,}9 \leq B \leq 5{,}1) = 0{,}9544$

TPC ??

$$L(B) = \begin{cases} -1 & 2 \\ 0{,}0456 & 0{,}9544 \end{cases}$$

$$E(L(B)) = -1 \times 0{,}0456 + 2 \times 0{,}9544 \Rightarrow \frac{1{,}8632}{1{,}0478} = 1{,}778202$$
$$= 1{,}8632$$