Teoria da Computação

(Theoretical Computer Science)
Licenciatura em Engenharia Informática
Lecture Notes 2012-2013

Luis Caires

version of September 20, 2012

1 Review of Set Theory, Modeling with Sets

The basic goal of this chapter is to help you learn how to:

- model data spaces and data structures using basic Set Theory
- specify properties of states of a computing system and of elements within a data structures using Logic

1.1 Basic Set Theory

1. Sets, Everything is a Set and ZFC

   Set theory was invented to provide a foundation to model ALL mathematical concepts. In turn mathematical concepts can be used to model most concepts of scientific and technological disciplines. Informatics and computer science are not an exception. It turns out that set theory and mathematical logic are particularly convenient tools to model concepts in informatics and computer science.

   Set theory and logic play for informatics the same basic role as mathematical analysis (calculus) plays for disciplines such as physics or electronic engineering.

   We will base our presentation on ZFC (Zermelo-Fraenkel-Cantor) Set Theory, due to these three famous mathematicians. Set theory was also developed by pioneers of computer science, for example, John Von Neumann.

Set theory is based on the idea that "Everything is a set". Actually, this means "Everything can be modeled by a set". ZFC models things such as boolean values, natural numbers, relations, functions, databases, and even algorithms, just based on the fundamental notion of set.

2. Emptyset

   The empty set is the "simplest" set we may think of. It is the set without elements. It is represented $\emptyset$.

3. Membership

   The fundamental form of statement in set theory is

   $$x \in y$$

   which means "x is a member of y", or "x belongs to y".

4. Extensionality

   The "Extensionality Principle" of set theory means that sets are determined uniquely by their elements. If two sets (finite or infinite) have exactly the same elements, then they are actually the same set. For example, we may think of two Java vectors with exactly the same elements, without being the same vector. This is not the case with sets.

   Practically, if we want to check if two sets A and B are actually the same set, if is enough to check that every element of A also belongs to B, and that every element of B also belongs to A.

   $$A = B \Leftrightarrow (\forall x.x \in A \Leftrightarrow x \in B)$$

   Extensionality also implies that there is just one empty set.

5. Subset

   A set x is a subset of a set y if all elements of x belong to y. Formally, we have

   $$A \subseteq B \Leftrightarrow \forall x.(x \in A \Rightarrow x \in B)$$

   Note that $A \subseteq A$ for all sets A, and $\emptyset \subseteq A$ for all sets A. Sometimes we use $A \subset B$ to say that A is a strict subset of B. A strict subset of a set B is a subset that is not the trivial subset B.

   $$A \subset B \Leftrightarrow (A \subseteq B) \wedge A \neq B$$

6. Enumeration

   We can define sets in various ways.

   The simplest way is by exhaustively enumerating all the elements in the set you want to specify

$$
\begin{array}{lcl}
BOOL & \triangleq & \{FALSE, TRUE\} \\
DWARFS & \triangleq & \{\text{“Sneezy”}, \text{“Sleepy”}, \text{“Dopey”}, \text{“Doc”}, \text{“Happy”}, \\
 & & \text{“Bashful”}, \text{“Grumpy”}\} \\
LAMPSTATES & \triangleq & \{ON, OFF\}
\end{array}
$$

   Obviously, this only works for specifying finite sets.

   When we define a set by enumerating its elements, the order or presentation does not matter! So, the following enumerations define the same set:

$$\{1, 2, 3\}$$
$$\{2, 1, 3\}$$
$$\{3, 1, 2\}$$

7. Sets, Sets of Sets, Sets of Sets of Sets, ...

   An set can also be an element of another set, and so on. This is useful to describe structured entities, with several components

$$STACK \triangleq \{0, \{2, \{3\}\}\}$$
$$BOOLS \triangleq \{\emptyset, \{TRUE\}, \{FALSE\}, BOOL\}$$

8. Comprehension

   We may define a new set using a logical property to select the elements we want to collect. For example

   The set of even natural numbers:

$$EVEN \triangleq \{n \in NAT \mid n\%2 = 0\}$$

   The set of non empty sets:

$$NOTEMPTY \triangleq \{s \mid s \neq \emptyset\}$$

   The general form of the “naive” comprehension principle allows us to define a new set given any property P expressed in the logic of set theory.

$$\{x \mid P(x)\}$$

The logic of set theory is essentially first-order logic enriched with several constants and operators that talk about sets, for example, the empty set, the membership relation, equality, etc, etc.

9. Russell's Paradox

   In 1901 Bertrand Russell discovered an inconsistency of Cantor-Frege set theory, by considering the set

   $$R \triangleq \{x \mid x \notin x\}$$

   Intuitively (so to speak), R is the set of all sets that are not members of themselves. Being not a member of itself is a property that make sense, in principle. We can think of many sets that enjoy this property, for example, the empty set is not a member of itself. The set of boolean values is not itself a boolean value.

   Since there are so many examples of sets that are not members of themselves, the set R as defined above, if it exists, must be not empty! We may even naively think that R contains all the sets that exists, since perhaps no set can be a member of itself.

   But a paradox (or inconsistency) arises! Consider the meaning of the proposition

   $$R \in R$$

   By definition of the "set" R, $R \in R$ means that $R \notin R$.

   Likewise, if we assume $R \notin R$, then it cannot be the case that $R \notin R$. So $R \notin R$ implies $R \in R$.

   Even if surprised at first, we must conclude that, according to the definition of R, we have

   $$R \in R \text{ if and only if } R \notin R$$

   which is obviously an absurd.

   Since we arrived to an absurd statement only by following the basic rules of logic and the definition of R, Russell concluded, rightly, that an expression like $\{x \mid x \notin x\}$ must be meaningless, and cannot be used to define a set. Such meaningless expressions should not be accepted by the language of set theory.

10. Separation

    To avoid confusions like Russell's paradox, we will always use Comprehension in a refined form, using the Separation principle of ZFC.

The general idea of the separation principle is that we may define a new set given any property P expressed in the logic of set theory, to select elements from some **already well defined** set S.

$$\{x \in S \mid P(x)\}$$

So, according to this principle, we have the right to write

$$\{n \in NAT \mid n\%2 = 0\}$$

a well defined set, but not an expression such as $\{s \mid s \neq \emptyset\}$.

Be careful to always use the separation principle when defining sets by comprehension in this course!

11. Union

    Besides Enumeration and Separation, we may define sets using the Union operation

    $$A \cup B$$

    Intuitively $A \cup B$ denotes the set that contains exactly the elements in A and B.

    $$\forall x.(x \in A \cup B) \Leftrightarrow (x \in A) \vee (x \in B)$$

    Given a set of sets S we also define the union $\bigcup S$ to mean the union of all sets which are elements of S. More precisely, we have

    $$\forall x.(x \in \bigcup S) \Leftrightarrow \exists y.(y \in S \wedge x \in y)$$

12. Intersection / Disjointness

    We may define sets using the Intersection operation

    $$A \cap B$$

    Intuitively $A \cap B$ denotes the set that contains exactly the elements that belong both to A and to B.

    $$\forall x.(x \in A \cap B) \Leftrightarrow (x \in A) \wedge (x \in B)$$

    We may also see that

    $$A \cap B = \{x \in A \mid x \in B\}$$

Given a set of sets S we also define the intersection $\bigcap S$ to mean the intersection of all sets which are elements of S. More precisely, we have

$$\forall x.(x \in \bigcap S) \Leftrightarrow \forall y.(y \in S \Rightarrow x \in y)$$

Two sets A and B are said to be disjoint, in symbols $A\#B$, if they do not contain any common member. We have

$$A\#B \Leftrightarrow (A \cap B) = \emptyset$$

We say that a collection S of sets is pairwise disjoint if all pairs of sets in the collection are disjoint. More precisely

$$\#S \Leftrightarrow \forall x.\forall y.(x \in S \wedge y \in S \wedge x \neq y \Rightarrow x\#y)$$

13. Relative Complement

    Given a sets A and B, the relative complement $A \setminus B$ denotes the set of all elements of A that do not belong to B. Formally

    $$A \setminus B = \{x \in A \mid x \notin B\}$$

    The "absolute complement" of a set A, written $\overline{A}$ is not definable in ZFC, due to the Russell paradox.

14. Pairs

    For structuring information we need some kind of construction to aggregate data. The simplest one is the pair. We may form e.g., a pair consisting of a team and the size of the team.

    $$\mathit{daltons} \triangleq (\{\text{“}\mathit{jack}\text{”}, \text{“}\mathit{joe}\text{”}, \text{“}\mathit{averell}\text{”}, \text{“}\mathit{william}\text{”}\}, 4)$$

    This corresponds to the well known notion of **ordered pair**. In set theory, everything is a set, and in fact an ordered pair such as the one above may be encoded in a set, using the scheme

    $$(x, y) \triangleq \{x, \{x, y\}\}$$

    This encoding of pairs is a variant of one Kuratowski proposed in 1921.

    In practice, we will simply use the standard notation (x, y) to represent ordered pairs.

15. Products

    The product of to sets A and B, written $A \times B$ is the set of all ordered pairs whose first element belongs to A and the second element belongs to B.

    We have

    $$\forall x.(x \in A \times B) \Leftrightarrow \exists a.\exists b.(a \in A \wedge b \in B \wedge x = (a, b))$$

    This operation is also called the "cartesian" product. The name "cartesian" derives from the name of René Descartes, the mathematician-philosoper that invented the related concept of cartesian plane, where one conceive points with two coordinates (x, y) (even if it is best known by his famous punchline "I think therefore I am" :-).

16. Fixed Sequences and n-tuples.

    We may represent tuples of more than 2 elements by iterating the product. For example $STRING \times NAT \times STRING$ denotes the set of all triples (a, b, c) where $a \in STRING$, $b \in NAT$ and $c \in STRING$.

    This idea of forming sets of tuples of any fixed arbitrary length works by considering the operation $A \times B$ to be right associative, so $A \times B \times C$ is actually an abbreviation of $A \times (B \times C)$.

    In the same way a triple such as (a, b, c) is actually an abbreviation of a pair $(a, (b, c))$.

    So we can say, for example, that the first component of (a, b, c) is a and the second component of (a, b, c) is (b, c).

    Note however that a sequence such as $((a, b), c)$ is different from the sequence (a, b, c). The first is a sequence of two elements, namely the pair (a, b) and c, while the second sequence contains three elements, a, b and c.

    This reasoning applies to sequences of elements of arbitrary finite length.

17. Relations

    A (binary) relation between elements of a set A and elements of a set B is modeled as a subset of the product $A \times B$. For example, the relation $SAMEPAR$ that holds between two natural numbers if and only if they have the same parity (odd or even) is defined as follows

    $$SAMEPAR \triangleq \{(x, y) \in NAT \times NAT \mid x\%2 = y\%2\}$$

For example, $(2, 8) \in SAMEPAR$ and $(9, 1) \in SAMEPAR$ but $(191, 256) \notin SAMEPAR$.

When R is supposed to denote a relation, we write $a\, R\, b$ for $(a, b) \in R$, to make it more readable. For example, we may write $2\ SAMEPAR\ 8$.

Here some other examples of binary relations:

$$x\ FATHER\_OF\ y$$

$$n\ ANCESTOR\_OF\ y$$

$$n\ LINKED\_TO\ y$$

We can also define relations between more than 2 elements. For that, we just iterate the constructions above, using products and n-tuples. For example, a phone list may be seen as a relation

$$PHONELIST \subset FIRSTNAME \times LASTNAME \times PHONENUM$$

where we may set $FIRSTNAME \triangleq STRING$, $LASTNAME \triangleq STRING$ and $PHONENUM \triangleq NAT$. For example, we may consider

$$(\text{“Luis”}, \text{“Caires”}, 218402825) \in PHONELIST$$

Relations are an extremely important concept in informatics and computer science. For example, it is pervasive in databases theory and practice, which are based in the so called relational data model, invented by Edgar Codd in 1970. Codd won the 1981 ACM Turing Award for this key contribution to Informatics. The relational model is the basis of most modern database systems, which use the query language SQL. You will learn more about this in the Databases course.

18. PowerSet

    We often need to define the set of all subsets of a given set. For example, we may want to consider a specific phonelist, as defined above. To what set does such phonelist belong? Well, a single phonelist is a set of triples (each one representing a record) where each triple belongs to the set

    $$FIRSTNAME \times LASTNAME \times PHONENUM$$

    The set of all sets of records of these kind is denoted by the powerset

    $$\wp(FIRSTNAME \times\ LASTNAME \times PHONENUM)$$

    In general, for any sets A and S we have that

    $$A \in \wp(S) \Leftrightarrow A \subseteq S$$

19. Functions

A function is modeled in set theory just as a special kind of relation, a relation between arguments and the corresponding results. Since a function cannot give two different results for the same argument, we impose the following condition for a binary relation R to be considered a function

$$function(R) \triangleq \forall (x, y) \in R, \forall (x', y') \in R \,.\, (x = x') \Rightarrow (y = y')$$

This means that if F is a function such that $(\text{“}luis\text{”}, a) \in F$ and $(\text{“}luis\text{”}, b) \in F$ then $a = b$, for example $a = b = 45$. There cannot be two different pairs with the same first component!

We may think of F as the AGE function that assigns to a person its (unique) age.

Since the result b of a function relation F is unique for any given argument, we denote such result by $F(a)$ where a is the first element of the pair $(a, b) \in F$. In the example above, we have, say $F(\text{“}luis\text{”}) = 45$.

So, note that, in the end, a function in set theory is nothing but a set of ordered pairs!

To highlight the use of ordered pairs in the context of functions, we also use the following alternative notation for ordered pairs

$$x \mapsto y \triangleq (x, y)$$

The notation $x \mapsto y$ reads “x is mapped to y” (“x é aplicado em y”).

Given a function as a set (of ordered pairs) we also call such set (of ordered pairs) the **extension** of the function.

For example, the extension of the NOT function on booleans may be represented by:

$$NOT \triangleq \{TRUE \mapsto FALSE, FALSE \mapsto TRUE\}$$

Then, we have $NOT(TRUE) = FALSE$, and $(FALSE, TRUE) \in NOT$.

The set of all subsets of $A \times B$ which are functions is denoted by

$$A \to B$$

In other words,

$$A \to B \triangleq \{R \in \wp(A \times B) \mid function(R)\}$$

We may then write, as usual

$$NOT \in BOOL \to BOOL$$

$F \in A \to B$ means that F is a function that sends elements of A into elements of B.

The set A (in $A \to B$) is called the **domain** of the function F, and B the **codomain** of the function F.

There are several ways of defining functions in set theory. An convenient way we will often use is to follow the pattern

$$F \triangleq \{x \mapsto y \in D \times C \mid P(x, y)\}$$

where $P(x, y)$ is a logical condition between the argument x and the result y, D is the domain and C is the codomain. For example,

$$DOUBLE \triangleq \{x \mapsto y \in NAT \times NAT \mid y = 2 \times x\}$$

Then $DOUBLE(2) = 4$, etc...

20. Identity Function

    For any set A there is the identity function on A, that maps each $e \in A$ into itself. The identity on A is noted Id_A. We have

    $$Id_A = \{a \mapsto b \in A \times A \mid a = b\}$$

    so that $Id_A(a) = a$ for all $a \in A$.

21. Projections

    Projections are useful functions that may be used to select elements from pairs and n-tuples.

    Given any product $A \times B$ we define the functions

    $$\pi_1 \triangleq \{((a, b) \mapsto a) \in (A \times B) \times A \mid (a, b) \in A \times B\}$$

    $$\pi_2 \triangleq \{((a, b) \mapsto b) \in (A \times B) \times B \mid (a, b) \in A \times B\}$$

    You may check that for the functions π_1 and π_2 just defined we have

    $$\pi_1 \in (A \times B) \to A$$

    $$\pi_2 \in (A \times B) \to B$$

    For example, $\pi_1((\text{“}luis\text{”}, 45)) = \text{“}luis\text{”}$, and $\pi_2((\text{“}luis\text{”}, 45)) = 45$.

    Projections generalize to n-tuples, for example, we may define the projections π_3, π_4, etc, which operate on triples, 4-tuples, etc.

1.2 Solved modeling problems

1. Model the following system with a structure.

   A lamp with two states `ON` and `OFF`.

   (a) Model the set of states of a lamp with a set $SLAMP$.

   (b) Define a function in $SLAMP \to SLAMP$ that models the "turn on" operation.

   (c) Define a function in $SLAMP \to SLAMP$ that models the "turn off" operation.

   (d) Define a function in $SLAMP \to BOOL$ that returns the current state of the lamp.

   Solution The set of states:

   $$SLAMP = \{0, 1\}$$

   The function of (b)

   $$turn\_on \triangleq \{0 \mapsto 1, 1 \mapsto 1\}$$

   The function of (c)

   $$turn\_off \triangleq \{0 \mapsto 0, 1 \mapsto 0\}$$

   The function of (d)

   $$status \triangleq \{0 \mapsto FALSE, 1 \mapsto TRUE\}$$

   The structure modeling the system:

   $$LAMP \triangleq (SLAMP, turn\_on, turn\_off, status)$$

2. Model the following system with a structure.

   A counter keeps the count of cars inside a tunnel by keeping track if cars entering the tunnel and cars exiting the tunnel.

   (a) Model the set of states of a counter with a set $SCOUNTER$.

   (b) Define a function in $SCOUNTER \to SCOUNTER$ that models the "car enter" operation.

   (c) Define a partial function in $SCOUNTER \to SCOUNTER$ that models the "car exit" operation.

(d) Define a function in $SCOUNTER \to NAT$ that yields the number of cars currently inside the tunnel.

Solution The set of states:

$$SCOUNTER \triangleq NAT$$

The function of (b)

$$car\_enter \triangleq \{n \mapsto m \in NAT \times NAT \mid m = n + 1\}$$

The function of (c)

$$car\_exit \triangleq \{n \mapsto m \in NAT \times NAT \mid n = m + 1\}$$

The function of (d)

$$cars\_inside \triangleq id_{NAT}$$

The structure modeling the system:

$$COUNTER \triangleq (SCOUNTER, car\_enter, car\_exit, cars\_inside)$$

3. Model the following data with sets

   (a) The set of all bank accounts, where each bank account includes the owner name, the account number, and the balance.

   (b) Define a function *JOIN* that given a set of bank accounts B without repeated account numbers, and two account numbers in B, yields a set of bank accounts identical to the given one, except that the two given accounts are merged in a new account, under the number of (and owner of) smallest account number.

   (c) To what set belongs the function *JOIN* ?

   Solution We may first define the sets, just for convenience,

$$\begin{array}{lcl} NAME & \triangleq & STRING \\ ACCNUM & \triangleq & NAT \\ AMOUNT & \triangleq & NAT \end{array}$$

   (a) The set of all bank accounts

$$ACC \triangleq NAME \times ACCNUM \times AMOUNT$$

   An example of a bank account

$$(\text{“luis”}, 1024, 80000000000)$$

   We have $(\text{“luis”}, 1024, 80000000000) \in ACC$

(b) Any set of bank accounts B is a subset of ACC, in other words, a member of $\wp(ACC)$.

For any set $B \in \wp(ACC)$ and account numbers n_1 and n_2 in B, we define the set

$$
\begin{array}{l}
merge(B, n_1, n_2) \\
\triangleq \\
\{c \in B \mid \pi_2(c) \neq n_1 \wedge \pi_2(c) \neq n_2\} \\
\cup \\
\{(o, n, b) \in ACC \mid \\
\qquad n = min(n_1, n_2) \wedge \exists b_1.\exists b_2.(o, n_1, b_1) \in B \wedge (o, n_2, b_2) \in B \wedge b = b_1 + b_2\}
\end{array}
$$

The first part of the union contains the accounts in B that are not the accounts with numbers n_1 or n_2.

The second part of the union contains the "joined" account.

The function $JOIN$ can then be defined

$$JOIN \triangleq \{(S, n_1, n_2) \mapsto M \mid M = merge(S, n_1, n_2)\}$$

(c) We have

$$JOIN \in (\wp(ACC) \times ACCNUM \times ACCNUM) \to \wp(ACC)$$