Teoria da Computação

(Theoretical Computer Science)
Licenciatura em Engenharia Informática Exercises 2011-2012

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1 Modeling with Sets and Logic

1. Consider the language of set theory enriched with the following ingredients

   - a set $BOOL$ of all booleans {TRUE, FALSE};
   - a set NAT of all natural numbers $\{0, 1, 2, 3, \cdots, \}$;
   - the basic operations on natural numbers $(+, -, \times, \div, \%)$;
   - a set $CHAR$ of all UNICODE characters;
   - a set $STRING$ of all strings of characters ("hello", "tc", "jack", "mary", ...);

   Define the following sets

   (a) The set of all even natural numbers.
   (b) The set of all prime numbers.
   (c) The extension of the addition function on natural numbers.
   (d) The extension of the negation function NOT on booleans.
   (e) The extension of the conjuction function AND on booleans.
   (f) The extension of the disjunction OR and implication $IMPLIES$ function on booleans, based on AND and NOT.
   (g) A contact card contains a name, a set of phone numbers, and a set of emails. Define the set of all contact cards.

(h) A function *FIND* that given a set of contact cards and a name, returns the corresponding phone numbers and emails, or `FALSE`, if the given name is not in any contact card.

To what set does *FIND* belong?

(i) A function that given two contact books gives the contact book obtained by merging the two sets of contacts.

(j) The set of all pairs of strings.

(k) The set M of all functions from the set of natural numbers to the set of all sets of natural numbers.

(l) The set M defined in the previous item contains many many elements. Define the element g of M that maps each natural n to the set of all naturals less than n. Check that g as you have defined indeed belongs to M.

2. Consider the following candidate definition of an "ordered pair"

$$(x, y) \triangleq \{x, \{y\}\}$$

Why does this not really work?

Hint: find sets a, a', b, b' where $a \neq a'$ and $b \neq b'$ such that $(a, b) = (a', b')$ according to the candidate definition.

3. Model the following data with sets

(a) the set of dates since 1 Jan 1904 until the end of the XXth century.

(b) the set of integers supported by the Sun Java Virtual Machine.

(c) the set of floating point numbers representable in IEEE 754-2008 half precision format.

4. Model the following system with a structure.

A lamp with two states `ON` and `OFF`.

(a) Model the set of states of a lamp with a set *SLAMP*.

(b) Define a function in $SLAMP \rightarrow SLAMP$ that models the "turn on" operation.

(c) Define a function in $SLAMP \rightarrow SLAMP$ that models the "turn off" operation.

(d) Define a function in $SLAMP \rightarrow BOOL$ that returns the current state of the lamp.

5. Model the following system with a structure.

   A counter keeps the count of cars inside a tunnel by keeping track if cars entering the tunnel and cars exiting the tunnel.

   (a) Model the set of states of a counter with a set $SCOUNTER$.

   (b) Define a function in $SCOUNTER \rightarrow SCOUNTER$ that models the "car enter" operation.

   (c) Define a partial function in $SCOUNTER \rightarrow SCOUNTER$ that models the "car exit" operation.

   (d) Define a function in $SCOUNTER \rightarrow NAT$ that yields the number of cars currently inside the tunnel.

6. Model the following data with sets

   (a) The set of all states of a bank account, including the owner, the account number, and the balance.

   (b) The set of all transactions of bank accounts, each transaction includes the account number, the type of operation (deposit or withdrawal, and the exchanged amount.

   (c) Define a function $JOIN$ that given a set of bank accounts, and two account numbers, yields a set of bank accounts identical to the given one, except that the two given accounts are merged in a new account, under the number of (and owner of) smallest account number.

   (d) To what set belongs the function $JOIN$?

   (e) The set of all states of a bank account with history. A bank account with history is like a bank account, but also keeps a set of performed transactions.

   (f) A bank account is correct if its set of transactions only refers to the given account. Define a function that given a set of bank accounts with history, returns its subset of correct bank accounts.

7. Model the following system with a structure.

   A stock list keeps a list of products, each one with its identifier (name), price, and stock available.

   (a) Model the set of states a stock list with a set $SLIST$.

(b) Model (with a function or relation) the operation that adds a product to a stock list.

(c) Model (with a function or relation) the operation on stock lists that removes a product.

(d) Model (with a function or relation) the operation on stock lists that given a product returns its price.

(e) Model (with a function or relation) the operation on stock lists that given a product returns its price and stock.

(f) Model (with a function or relation) the operation on stock lists that gives the set of products out of stock.

(g) Model (with a function or relation) the operation on stock lists that increases the stock of product by N items.

(h) Model (with a function or relation) the operation on stock lists that decreases the stock of product by N items.

8. Model the following data with sets

(a) A graph is a structure that comprises a set of nodes and a set of arcs, which are arrows connecting nodes. A simple example of a graph is a computer network, where the nodes are the computers and the arcs represent the connections between computers. Define a set modeling the graph in Figure 1.

(b) Define a function $GRAPHS$ that given a set of nodes (assume that nodes are simply represented by strings), yields the set of all graphs over that set of nodes.

(c) To what set belongs the function $GRAPHS$?

(d) Define a function that given a graph, returns the graph obtained by reversing the direction of all arcs.

(e) Consider a graph G, modeled as you have done above, and a node N of G. Define a sequence of sets $S_0, S_1, S_2, ...$ such that S_0 contains only N, and for all $k \geq 0$, S_{k+1} contains the set of notes reachable by transversing a single arc from a node in S_k.

9. Model the following system with a structure.

A weather station keeps measuring the temperature using a sensor, and reports on the values detected. Each measurement consists of the time of the sample, and the temperature.
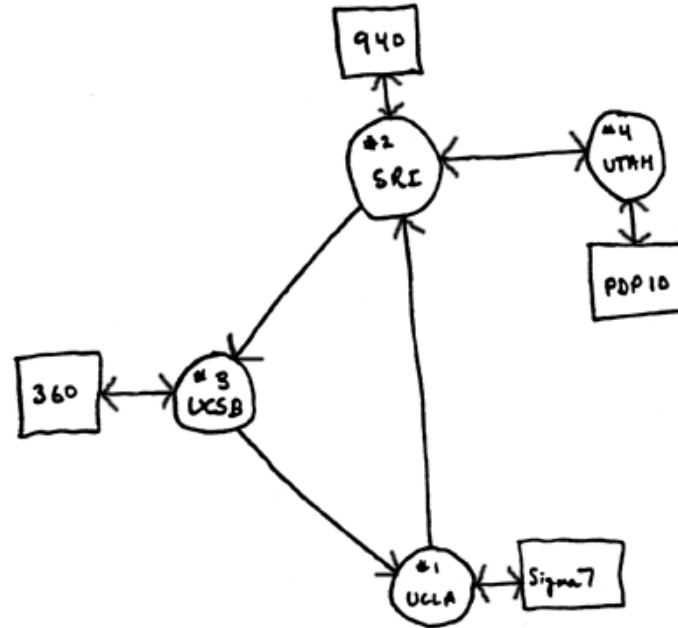
Figure 1: Arpanet (the embrionic Internet, 1969).

(a) Model the set of states a weather station with a set *SWEATHER*.

(b) Model (with a function or relation) the operation that performs a sample and registers the temperature value and measurement time.

(c) Model (with a function or relation) that clears all performed measurements.

(d) Model (with a function or relation) an operation that gives the number of recorded measurements.

(e) Model (with a function or relation) an operation that gives the maximum and minimal temperature measured.

(f) Model (with a function or relation) an operation that gives the maximum and minimal temperature measured in a given time interval within the recording period.

10. Model the following system with a structure.

Consider a system that keeps track of the access permissions regarding a set of printers within a network. Consider that for each user there is a set printers that s/he is allowed to use.

(a) Model the set of states of the access system with a set *APRINTER*.

(b) Model (with a function or relation) the operation that gives a user access to a printer.

(c) Model (with a function or relation) the operation that blocks access of an user to a printer.

(d) Model (with a function or relation) the operation that checks if a user has access to a specific printer.

(e) Model (with a function or relation) an operation that returns the users that can access more printers.

11. Model the following system with a structure.

A dictionary mapping portuguese words into (finite) sets of portuguese sentences. For example, a dictionary entry contains the following information,

"aluno"

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"aquele que recebe doutrem educação e instrução"; "discipulo"; "aprendiz"; "educando"

(a) Model the set of states of the dictionary with a set *SDICT*. Before defining this, look to the list of operations the dictionary is expected to support.

(b) Model (with a function or relation) the operation that adds an entry to the dictionary. If an entry for the given word already exists, the new synonyms should be added to the existing entry.

(c) Model (with a function or relation) the operation that checks if a given word is defined in the dictionary.

(d) Model (with a function or relation) the operation that adds replaces the definitions associated to an existing entry for a given word with a different set of sentences.

(e) Model (with a function or relation) the operation that removes an entry from the dictionary.

(f) Model (with a function or relation) the operation that lookups in the dictionary the meaning of a word.

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12. Model the following system with a structure.

    A stack of text notes. Each note may be represented by a string. As usual, only the top of the stack is accessible to add or remove elements.

    (a) Model the set of states of a stack of notes with a set $SSTACK$.

    (b) Model (with a function or relation) the operation that pushes a note in the stack.

    (c) Model (with a function or relation) the operation that pops and returns the top node in the stack.

    (d) Model (with a function or relation) an operation that observes if the stack is empty or not.

13. Model the following system with a structure.

    Consider a system that keeps track of playlists. A playlist includes a name and a set of songs. Each song is defined by a title, an artist or band name, a duration, and a rating (1 to 5).

    (a) Model the set of states of the system with a set $SPLAYLIST$.

    (b) Model (with a function or relation) the operation that adds a song to a playlist.

    (c) Model (with a function or relation) the operation that removes a song from a playlist.

    (d) Model (with a function or relation) the operation that removes a song from all playlists.

    (e) Model (with a function or relation) the operation that determines the name of the favorite song (or songs) of all playlists.

    (f) Model (with a function or relation) the operation that given a song name returns the playlist names that include the song.

14. Model the following system with a structure.

    An array of names.

    A name array consists of a collection names, each name being associated to an index.

    Indexes start at 0, are incremented by 1 for each name, and are consecutive (0, 1, 2, ...).

    (a) Model the set of states of the name array with a set $SARRAY$.

(b) Model (with a function or relation) the operation `add` that adds a name to the array (at the last free index).

(c) Model (with a function or relation) the operation `find` that returns the first index of a given name. If no such name exists, the operation returns `false` .

(d) Model (with a function or relation) the operation `remove` that removes the first occurrence of a name from an array (and decrements all indexes greater than the removed name's index). If no such name exists in the array, it is left unchanged.

15. Model the following system with a structure.

Consider a simplified version of $Google^+$ Circles where a user organizes friends using circles.

A circle is simply a collection of users identified by a name.

A user can be added to more than one circle.

Assume that each user is identified by an email.

(a) Model SCircle, the set of all states of the system.

(b) Define (with a function or relation) the operation on SCircle that adds a user to a set of circles

(c) Define (with a function or relation) the operation on SCircles that retrieves all people in a circle

(d) Define (with a function or relation) the operation on SCircle that blocks a user, i.e. deletes a user from all circles.

(e) Define (with a function or relation) the operation on SCircle that retrieves all the circles associated with a given user.

(f) Define (with a function or relation) the operation on SCircle that given a user, retrieves its larger circle

(g) Define (with a function or relation) the operation on SCircle that given two circles names, retrieves the users that belong to the intersection.

(h) An extended circle of an user are its circle's circles, i.e. people who are at one degree of distance.

Define (with a function or relation) the operation on SCircle that given a user retrieves its extended circle.

(i) Imagine a super extended circle that is defined by all people who are connected at any arbitrary degree of distance.

Define (with a function or relation) the operation on SCircle that given a user retrieves its super extended circle.

16. Provide an inductive definition for the following set

$$PowersOfTwo \triangleq \{n \in NAT \mid n = 2^p \wedge (p >= 0)\}$$

17. Provide an inductive definition for the function

$$exp2 \in NAT \rightarrow NAT$$

such that

$$exp2(n) = 2^n$$

Give a bottom up justification that $4 \mapsto 16 \in exp2$ (or equivalently, that $exp2(4) = 16$).

18. Provide an inductive definition for the following sets

$$EvenSequences \triangleq \{s \in SEQ \mid len(s)\%2 = 0\}$$

$$OddSequences \triangleq \{s \in SEQ \mid len(s)\%2 = 1\}$$

N.B. Recall the (indutive) definition of the set SEQ of all sequences of natural numbers, and the function len that gives the length of a sequence (both are defined in the lecture notes).

Write down a top-down justification that $(1, 2, 3) \in OddSequences$.

Write down a bottom-up justification that $(0, 0) \in EvenSequences$.

19. Provide an inductive definition for the set $SortedSequences$ of the increasingly sorted sequences of natural numbers. For example,

$$(2, 5, 6, 6, 7) \in SortedSequences$$

$$(1, 2, 1, 3, 4) \notin SortedSequences$$

20. Provide an inductive definition for the function

$$addLast \in SEQ \times NAT \rightarrow SEQ$$

such that $addLast(s_1, n) = s_2$ if and only if s_2 is obtained from s_1 by adding n as last element. For example,

$$addLast(((1, 3, 5), 9)) = (1, 3, 5, 9)$$

21. Provide an inductive definition for the relation

$$Reverse \subset SEQ \times SEQ$$

such that $(s_1, s_2) \in Reverse$ if and only if s_2 is the reverse sequence of s_1. For example,

$$((1, 3, 5), (5, 3, 1)) \in Reverse$$

$$((1, 3, 5), (2, 3)) \notin Reverse$$

22. Model the following system with a structure.

Consider a (very) simplified version of Facebook. The system maintains a set of users, and the information about who is friend of each user.

(a) Model SFace, the set of all states of the system.

(b) Define (with a function or relation) the operation on SFace that adds a user to the system.

(c) Define (with a function or relation) the operation on SFace that sets a user as friend of other user.

(d) Define (with a function or relation) the operation on SFace that says if a user if a friend of other user.

(e) Define (with a function or relation) the operation on SFace that returns the set of friends of some user.

(f) Define (with a function or relation) the operation on SFace that says if a user is connected to other user by a sequence of friends.

(g) Define (with a function or relation) the operation on SFace that returns the network of a user (the set of all users that are connected by a chain of friendship to the given user).

23. Justify whether the following sets are finite, countable, or uncountable.

(a) The set of all functions from $SLAMP$ to $SLAMP$.

(b) The set of all functions $SLAMP$ to NAT.

(c) The set of all real numbers which are roots of some polynomial with natural coeficients.

(d) The set of all real numbers which are roots of some polynomial with rational coeficients.

(e) The set of all functions $NAT \rightarrow NAT$.

(f) The set of all graphs with less that 128 nodes.

(g) The set of states of any stock list $S \in SLIST$.

(h) $SLIST$, the set of all stock lists.

(i) $SCOUNT$, the set of all counters.

(j) The set of all correctly written Java programs that fit in a 2GB flash pen drive.

(k) The set of all real numbers.

2   Machines and Specifications

1. Specify a Deterministic Finite Automaton (DFA) over the alphabet

$$\Sigma = \{\texttt{X}\}$$

   that checks if a sequence of $\texttt{X}$'s is of even length, in other words, that only accepts the words over Σ of even length.

2. Specify a DFA) over the alphabet

$$\Sigma = \{\texttt{X}\}$$

   that checks if a sequence of $\texttt{X}$'s is of odd length.

3. Consider the alphabet

$$DIGITS \triangleq \{\texttt{0}, \texttt{1}, \texttt{2}, \texttt{3}, \texttt{4}, \texttt{5}, \texttt{6}, \texttt{7}, \texttt{8}, \texttt{9}\}$$

   Specify a DFA over the alphabet $DIGITS$ that checks if a word over $DIGITS$ is a possible pin for a mobile phone.

4. Consider the alphabet

$$DNA \triangleq \{\texttt{A}, \texttt{T}, \texttt{C}, \texttt{G}\}$$

   Specify a DFA over the alphabet AB that only accepts the words over DNA that contain at least one occurrence of $\texttt{ACT}$ as a substring.

5. Specify a DFA over the alphabet $DIGITS$ that checks if a word over $DIGITS$ is the representation of an even natural number.

6. Specify a DFA over the alphabet

$$\Sigma = \{ \quad \texttt{insertcard,}$$
$$\texttt{pin,}$$
$$\texttt{checkbalance,}$$
$$\texttt{widthdraw,}$$
$$\texttt{moreops?,}$$
$$\texttt{yes,}$$
$$\texttt{no,}$$
$$\texttt{retrievecard} \quad \}$$

   that checks if a word (trace) over Σ describes a valid interaction between a user and an ATM.

7. Consider the alphabet
$$AB \triangleq \{\texttt{a}, \texttt{b}\}$$

Specify a DFA over the alphabet AB that only accepts the words over AB that contain an odd number of b's.

8. Specify a DFA over the alphabet $\Sigma \triangleq DIGITS \cup \{-, +\}$ that checks if a word over Σ is the representation of an integer literal in the Java language (such as $+\texttt{120}$, or $-\texttt{99}$).

9. Specify a DFA over the alphabet $\Sigma \triangleq DIGITS \cup \{-, +\}$ that checks if a word over Σ is the representation of an integer literal in the java language.

10. Consider the alphabet

$$LETTERS \triangleq \{\texttt{a}, \texttt{b}, \ldots \texttt{z}\}$$

Specify a DFA over the alphabet $\Sigma \triangleq DIGITS \cup LETTERS \cup \{., :, /\}$ that checks if a word over Σ can represent the URL of a web site.

11. Consider the alphabet
$$AB \triangleq \{\texttt{a}, \texttt{b}\}$$

Specify a DFA over the alphabet AB that only accepts the words over AB that contain an odd number of b's and an even number of a's.

12. Consider the alphabet

$$DNA \triangleq \{\texttt{A}, \texttt{T}, \texttt{C}, \texttt{G}\}$$

Specify a DFA over the alphabet AB that only accepts the words over DNA that contain at least one occurrence of ATACA as a substring.